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Boundary Behavior of Berezin Symbols and Related Results

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Abstract. For a given function $\varphi \in H^\infty$ with $\left|\varphi(z)\right| < 1$ $(z \in \mathbb{D})$, we associate some special operators subspace and study some properties of these operators including behavior of their Berezin symbols. It turns that such boundary behavior is closely related to the Blaschke condition of sequences in the unit disk $\mathbb{D}$ of the complex plane. In terms of Berezin symbols the trace of some nuclear truncated Toeplitz operator is also calculated.

1. Introduction

Let Ω be a subset of a topological space X such that boundary $\partial\Omega$ is nonempty. Let $\mathcal{H} = \mathcal{H}(\Omega)$ be an infinite-dimensional Hilbert space of functions defined on Ω. We say that $\mathcal{H}$ is a reproducing kernel Hilbert space (RKHS) if the following two conditions are satisfied:
(i) for any $\lambda \in \Omega$, the functionals $f \to f(\lambda)$ are continuous on $\mathcal{H}$;
(ii) for any $\lambda \in \Omega$, there exists $f_\lambda \in \mathcal{H}$ such that $f_\lambda(\lambda) \neq 0$.

According to the classical Riesz representation theorem, the assumption (i) implies that for any $\lambda \in \Omega$ there exists $k_{\mathcal{H},\lambda} \in \mathcal{H}$ such that

$$f(\lambda) = \left\langle f, k_{\mathcal{H},\lambda}\right\rangle_{\mathcal{H}},\ f \in \mathcal{H}.$$

The function $k_{\mathcal{H},\lambda}$ is called the reproducing kernel of $\mathcal{H}$ at point λ. Note that by (ii), we surely have $k_\lambda \neq 0$ and we denote by $\widehat{k}_{\mathcal{H},\lambda}$ the normalized reproducing kernel, that is $\widehat{k}_{\mathcal{H},\lambda} := k_{\mathcal{H},\lambda}/\left\|k_{\mathcal{H},\lambda}\right\|$.

Following the definition of [12], we say that a RKHS $\mathcal{H}$ is standard, if $\widehat{k}_{\mathcal{H},\lambda} \to 0$ weakly as $\lambda \to \xi$ for any point $\xi \in \partial\Omega$. For example, the Hardy Hilbert space is a standard RKHS.

Recall that if $\mathcal{B}(\mathcal{H})$ denotes the space of all bounded and linear operators on $\mathcal{H}$, then the Berezin symbol $\widetilde{A}$ of any operator $A \in \mathcal{B}(\mathcal{H})$ is the function defined on Ω by

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle,\ \lambda \in \Omega.$$

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Note that Nordgren and Rosenthal [12] established a characterization of compact operators acting on such spaces in terms of the Berezin symbols of their unitary orbits, which is the following.

Theorem 1.1 ([12, Corollary 2.8]). *Let $\mathcal{H}$ be a standard RKHS on Ω and let $A \in \mathcal{B}(\mathcal{H})$. Then A is compact if and only if*

$$\lim_{\lambda \to \xi} \widetilde{U^{-1}AU}(\lambda) = 0$$

for every unitary operator U on $\mathcal{H}$ and every point ξ in $\partial\Omega$.

In particular, it follows from Theorem 1.1 that if $\mathcal{K}$ is a compact operator on the standard RKHS $\mathcal{H} = \mathcal{H}(\Omega)$, then $\widetilde{\mathcal{K}}(\lambda) \to 0$ as $\lambda \to \xi \in \partial\Omega$ (For more information about Berezin symbols, see [8] and [12]).

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc of the complex plane $\mathbb{C}$. Recall that for any function $\varphi \in L^\infty(\partial\mathbb{D})$ the corresponding Toeplitz operator on the Hardy space $H^2 = H^2(\mathbb{D})$ is defined by $T_\varphi f = P_+\varphi f$, $f \in H^2$, where $P_+ : L^2(\partial\mathbb{D}) \to H^2$ is the Riesz projector.

In this article, we associate in terms of a given analytic Toeplitz operator some special operators subspace and study some properties of these operators including boundary behavior their Berezin symbols. It is shown that such boundary behavior is closely related to the Blaschke sequences. In terms of Berezin symbols the trace of some nuclear truncated Toeplitz operator is also calculated.

2. Blaschke condition and boundary behavior of Berezin symbols

A sequence $\{\lambda_n\}_{n=1}^\infty \subset \mathbb{D}$ of complex numbers is said to be a Blaschke sequence if $\sum\limits_{n=1}^{\infty}\left(1 - |\lambda_n|^2\right) < +\infty$.

In this section, we study the boundary behavior of Berezin symbols of operators via the Blaschke condition. We also prove in terms of the Berezin symbols some assertions concerning to the Blaschke sequences.

Our first result is the following elementary result. Recall that $H^\infty = H^\infty(\mathbb{D})$ is the Banach algebra of all bounded analytic functions in the unit disk $\mathbb{D}$ with the norm

$$\left\|f\right\|_\infty := \sup_{z\in\mathbb{D}}\left|f(z)\right| < +\infty,\ f \in H^\infty.$$

Proposition 2.1. *Let $\varphi \in H^\infty$ be a function and $\{\lambda_n\}_{n=1}^\infty \subset \mathbb{D}$ be a sequence. Then $\sum\limits_{n=1}^{\infty}\left(1 - \left|\varphi(\lambda_n)\right|^2\right) < +\infty$ if and only if $\sum\limits_{n=1}^{\infty}\left(I - \widetilde{T_\varphi T_\varphi^*}\right)(\lambda_n) < +\infty$.*

The proof of this proposition is immediate from the well-known fact that $T_\varphi^* k_{H^2,\lambda} = \overline{\varphi}(\lambda)\, k_{H^2,\lambda}$ for every $\lambda \in \mathbb{D}$, where $k_{H^2,\lambda} = \dfrac{1}{1-\overline{\lambda}z}$ is the reproducing kernel of the Hardy space H^2.

Before giving our next results, we need some notations, and also we give some motivation.

Recall that the Brown-Halmos theorem [4] says that a bounded linear operator X on H^2 is a Toeplitz operator if and only if $S^*XS = X$, where $Sf = zf$ is the well-known unilateral shift on H^2. This notion of "Toeplitzness" was extended in various ways. Barria and Halmos [4] examined the so-called asymptotically Toeplitz operators X on H^2 for which the sequence of operators $\{S^{*n}XS^n\}_{n\geq 1}$ converges strongly. This class certainly includes the Toeplitz operators, but also includes other operators such as those in the Hankel algebra. Feintuch [5] discovered that one need not restrict to strong convergence and uniform (or norm) limits of this sequence. Indeed, an operator X on H^2 is uniformly asymptotically Toeplitz, i.e., $S^{*n}XS^n$ converges in operator norm, if and only if

$$X = X_1 + K,$$

where X_1 is a Toeplitz operator, i.e., $S^*X_1S = X_1$, and K is a compact operator on H^2. Nazarov and Shapiro [11] examined other associated notions of "Toeplitzness" with regards to certain composition operators on H^2.

For any $\varphi \in H^\infty$ with $\left|\varphi(z)\right| < 1$, let us define the following operators subspace in $\mathcal{B}\left(H^2\right)$:

$$\mathcal{L}_\varphi\left(H^2\right) := \left\{X \in \mathcal{B}\left(H^2\right) : \sum_{n=0}^{\infty} T_\varphi^n X T_\varphi^{*n} \text{ is strongly convergent and } \sum_{n=0}^{\infty} T_\varphi^n X T_\varphi^{*n} \in \mathcal{B}\left(H^2\right)\right\}.$$

Here we study some properties of $\mathcal{L}_\varphi\left(H^2\right)$-class operators.

Proposition 2.2. *For any operator $A \in \mathcal{L}_z\left(H^2\right)$ such that $\lim_{\lambda\to\partial\mathbb{D}} \widetilde{A}(\lambda)$ exists, we have:*

(a) $\lim_{N\to+\infty} \lim_{\lambda\to\partial\mathbb{D}} \frac{\sum_{n=0}^{N} (S^nAS^{*n})^\sim(\lambda)}{N+1} = \lim_{\lambda\to\partial\mathbb{D}} \widetilde{A}(\lambda)$;

(b)

$$\lim_{N\to+\infty} \left(\sum_{n=0}^{N} S^nAS^{*n}\right)^\sim(\lambda) = \frac{\widetilde{A}(\lambda)}{1-|\lambda|^2} \ \ (\forall\lambda \in \mathbb{D}). \tag{1}$$

Proof. *(a)* Indeed, for any $A \in \mathcal{L}_z\left(H^2\right)$ and $N \in \mathbb{N}$, we have:

$$\begin{aligned}\left(\sum_{n=0}^{N} S^nAS^{*n}\right)^\sim(\lambda) &= \widetilde{A}(\lambda)\frac{1-|\lambda|^{2(N+1)}}{1-|\lambda|^2}\\ &= \widetilde{A}(\lambda)\left(1+|\lambda|^2+|\lambda|^4+...+|\lambda|^{2N}\right),\end{aligned}$$

or

$$\frac{\left(\sum_{n=0}^{N} S^nAS^{*n}\right)^\sim(\lambda)}{1+|\lambda|^2+|\lambda|^4+...+|\lambda|^{2N}} = \widetilde{A}(\lambda) \ \ (\lambda \in \mathbb{D}).$$

So

$$\lim_{\lambda\to\partial\mathbb{D}} \frac{\sum_{n=0}^{N} (S^nAS^{*n})^\sim(\lambda)}{1+|\lambda|^2+|\lambda|^4+...+|\lambda|^{2N}} = \lim_{\lambda\to\partial\mathbb{D}} \widetilde{A}(\lambda),$$

or

$$\frac{\lim_{\lambda\to\partial\mathbb{D}} \sum_{n=0}^{N} (S^nAS^{*n})^\sim(\lambda)}{N+1} = \lim_{\lambda\to\partial\mathbb{D}} \widetilde{A}(\lambda),$$

and hence

$$\lim_{N\to+\infty} \frac{\lim_{\lambda\to\partial\mathbb{D}} \sum_{n=0}^{N} (S^nAS^{*n})^\sim(\lambda)}{N+1} = \lim_{\lambda\to\partial\mathbb{D}} \widetilde{A}(\lambda),$$

which proves *(a)*.

(b) We have from the equality

$$\sum_{n=0}^{N}\left(S^{n}AS^{*n}\right)^{\sim}(\lambda)=\widetilde{A}(\lambda)\frac{1-|\lambda|^{2(N+1)}}{1-|\lambda|^{2}}$$

that $\lim_{N\to+\infty}\sum\limits_{n=0}^{N}\left(S^{n}AS^{*n}\right)^{\sim}(\lambda)=\frac{\widetilde{A}(\lambda)}{1-|\lambda|^{2}}\ (\forall\lambda\in\mathbb{D})$, which proves *(b)*. The corollary is proven. □

Let us define for any $\varphi\in H^{\infty}$ with $\left|\varphi(z)\right|<1$ $(z\in\mathbb{D})$ the following map from $\mathcal{L}_{\varphi}\left(H^{2}\right)$ into the Banach algebra $\mathcal{B}\left(H^{2}\right)$:

$$\Phi_{\varphi}(A):=\sum_{n=0}^{\infty}T_{\varphi}^{n}AT_{\varphi}^{*n},\ A\in\mathcal{L}_{z}\left(H^{2}\right).$$

Our next result gives an equivalent characterization of the Blaschke sequence $\{\varphi(\lambda_n)\}_{n=0}^{\infty}$.

Proposition 2.3. *Let $\varphi\in H^{\infty}$ be a function such that $\left|\varphi(z)\right|<1$ for all $z\in\mathbb{D}$, and let $\{\lambda_n\}_{n=0}^{\infty}\subset\mathbb{D}$ be a sequence of nonzero distinct complex numbers. Then $\{\varphi(\lambda_n)\}_{n=0}^{\infty}$ is a Blaschke sequence if and only if the series*

$$\sum_{n=0}^{\infty}\frac{\widetilde{A}(\lambda_n)}{\widetilde{\Phi_{\varphi}(A)}(\lambda_n)}$$

is convergent whenever A be an operator in $\mathcal{L}_{\varphi}\left(H^{2}\right)$ such that $\widetilde{A}(\lambda_n)\neq 0$ for every $n\geq 0$.

Proof. Indeed, for any operator $A\in\mathcal{L}_{\varphi}\left(H^{2}\right)$ with $\widetilde{A}(\lambda_n)\neq 0\ (\forall n\geq 0)$, we have:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\widetilde{A}(\lambda_n)}{\widetilde{\Phi_{\varphi}(A)}(\lambda_n)}&=\sum_{n=0}^{\infty}\frac{\widetilde{A}(\lambda_n)}{\left(\sum\limits_{m=0}^{\infty}T_{\varphi}^{m}AT_{\varphi}^{*m}\right)^{\sim}(\lambda_n)}\\&=\sum_{n=0}^{\infty}\frac{\widetilde{A}(\lambda_n)}{\left(\sum\limits_{m=0}^{\infty}T_{\varphi^{m}}AT_{\overline{\varphi}^{m}}\right)^{\sim}(\lambda_n)}\end{aligned}$$

$$\begin{aligned}&=\sum_{n=0}^{\infty}\frac{\widetilde{A}(\lambda_n)}{\sum\limits_{m=0}^{\infty}\left\langle T_{\varphi^{m}}AT_{\overline{\varphi}^{m}}\widehat{k}_{H^2,\lambda_n},\widehat{k}_{H^2,\lambda_n}\right\rangle}\\&=\sum_{n=0}^{\infty}\frac{\widetilde{A}(\lambda_n)}{\sum\limits_{m=0}^{\infty}\left\langle A\overline{\varphi(\lambda_n)}^{m}\widehat{k}_{H^2,\lambda_n},T_{\overline{\varphi}^{m}}\widehat{k}_{H^2,\lambda_n}\right\rangle}\\&=\sum_{n=0}^{\infty}\frac{\widetilde{A}(\lambda_n)}{\sum\limits_{m=0}^{\infty}\left|\varphi(\lambda_n)\right|^{2m}\widetilde{A}(\lambda_n)}=\sum_{n=0}^{\infty}\frac{1}{\sum\limits_{m=0}^{\infty}\left|\varphi(\lambda_n)\right|^{2m}}\\&=\sum_{n=0}^{\infty}\frac{1}{\frac{1}{1-\left|\varphi(\lambda_n)\right|^{2}}}=\sum_{n=0}^{\infty}\left(1-\left|\varphi(\lambda_n)\right|^{2}\right).\end{aligned}$$

Thus

$$\sum_{n=0}^{m} \frac{\widetilde{A}(\lambda_n)}{\widetilde{\Phi_\varphi (A)}(\lambda_n)} = \sum_{n=0}^{\infty} \left(1 - \left|\varphi(\lambda_n)\right|^2\right),$$

which implies the proof of the proposition. □

Corollary 2.4. *Let $\varphi \in H^\infty$ be a function such that $\left|\varphi(z)\right| < 1$ for every $z \in \mathbb{D}$, $\{\lambda_n\}_{n=0}^{\infty} \subset \mathbb{D}$ be a Blaschke sequence, and let $\mathcal{K} \in \mathcal{L}_\varphi\left(H^2\right)$ be a compact operator such that $\widetilde{\mathcal{K}}(\lambda_n) \neq 0$ $(\forall n \geq 0)$. Then $\{\varphi(\lambda_n)\}_{n=0}^{\infty}$ is a Blaschke sequence if and only if*

$$\sum_{n=0}^{\infty} \frac{\widetilde{\mathcal{K}}(\lambda_n)}{\widetilde{\Phi_\varphi (\mathcal{K})}(\lambda_n)}$$

is a convergent series.

Since H^2 is a standard RKHS, $\widetilde{\mathcal{K}}(\lambda) \to 0$ as $\lambda \to \xi \in \partial\mathbb{D}$ for any compact operator $\mathcal{K} \in \mathcal{B}\left(H^2\right)$. Thus, Corollary 2.4 shows that the Blaschke property of the sequence $\{\varphi(\lambda_n)\}_{n\geq 0}$ depends on the boundary behavior of the Berezin symbol of the compact operator $\Phi_\varphi(\mathcal{K})$.

Corollary 2.5. *If $\varphi \in H^\infty$ is a function such that $\left|\varphi(z)\right| < 1$ for all $z \in \mathbb{D}$, and $A \in \mathcal{L}_\varphi\left(H^2\right)$, then:*

$$\left|\left(\sum_{n=0}^{\infty} T_\varphi^n A T_\varphi^{*n}\right)^{\sim}(\lambda)\right| \leq \frac{\left\|A\widehat{k}_{H^2,\lambda}\right\|}{1 - \left|\varphi(\lambda)\right|^2} \; (\forall \lambda \in \mathbb{D}). \tag{2}$$

Proof. Indeed,

$$\left|\sum_{n=0}^{\infty} \left\langle T_\varphi^n A T_\varphi^{*n} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right| = \left|\sum_{n=0}^{\infty} \left|\varphi(\lambda)\right|^{2n} \widetilde{A}(\lambda)\right| = \frac{\left|\widetilde{A}(\lambda)\right|}{1 - \left|\varphi(\lambda)\right|^2} \leq \frac{\left\|A\widehat{k}_\lambda\right\|}{1 - \left|\varphi(\lambda)\right|^2},$$

which proves (2). □

Remark. Let φ be an analytic function such that $\varphi : \mathbb{D} \to \mathbb{D}$, i.e., $\left|\varphi(z)\right| < 1$ for all $z \in \mathbb{D}$. The authors do not know the answer to the following questions:

If $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{D}$ is a sequence, then under what conditions $\{\varphi(\lambda_n)\}_{n=1}^{\infty} \subset \mathbb{D}$ is a Blaschke sequence?

Notice that if $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{D}$ is a Blaschke sequence and $\left|\varphi(z)\right| > |z|$, $z \in \mathbb{D}$, then clearly $\{\varphi(\lambda_n)\}_{n=1}^{\infty} \subset \mathbb{D}$ is a Blaschke sequence.

So, in this case, it will be interesting to characterize the Blaschke sequences $\{\varphi(\lambda_n)\}_{n=1}^{\infty} \subset \mathbb{D}$ more transparantly in terms of functions $\varphi \in H^\infty$ with $\left|\varphi(z)\right| < 1$, $\forall z \in \mathbb{D}$.

3. Trace of nuclear truncated Toeplitz operator and Berezin symbol

Let $\varphi \in L^2(\partial\mathbb{D})$ and $\theta \in H^\infty$ be an inner function, i.e., $|\theta(\xi)| = 1$ for almost all $\xi \in \partial\mathbb{D}$. We consider the truncated Toeplitz operator

$$A_\varphi = P_\theta T_\varphi | K_\theta$$

on the model subspace $K_\theta = H^2 \Theta \theta H^2$, where $P_\theta : L^2(\partial\mathbb{D}) \to K_\theta$ is the orthogonal projector defined by

$$P_\theta f = P_+ f - P_+\left(\overline{\theta} f\right), \; f \in K_\theta \cap L^\infty(\partial\mathbb{D}).$$

The kernel function in K_θ for the evaluation functional at the point λ of $\mathbb{D}$ is the function

$$k_{\theta,\lambda}(z) := \frac{1-\overline{\theta(\lambda)}\theta(z)}{1-\overline{\lambda}z} \quad (z \in \mathbb{D}).$$

In this section, the trace of nuclear truncated Toeplitz operator with unbounded symbol $f \in L^2(\partial\mathbb{D})$ is calculated in terms of its Berezin symbol. Our next result is a slight improvement of a result of the paper [6], where a truncated Toeplitz operator with a bounded symbol $\varphi \in L^\infty(\partial\mathbb{D})$ is considered. Its proof uses the similar arguments as in [6], but only for the sake of completeness we provide it here.

Proposition 3.1. *Let $\varphi \in L^2(\partial\mathbb{D})$, and $B = B_{\{\lambda_k\}}$ be the Blaschke product with distinct zeros $\{\lambda_k\}_{k\geq 1}$. If $A_\varphi \in \mathcal{B}(K_B)$ is an operator of trace class (that is $A_\varphi \in \sigma_1(K_B)$), then*

$$trace\left(A_\varphi\right) = \sum_{n=1}^{\infty} \widetilde{A}_\varphi(\lambda_n) = \sum_{n=1}^{\infty} \widetilde{\varphi}(\lambda_n), \tag{3}$$

where $\widetilde{\varphi}$ is the harmonic continuation of the function φ into the unit disk $\mathbb{D}$.

Proof. By $B_n(z)$ we denote the Blaschke product with zeros $\lambda_1, \lambda_2, ..., \lambda_{n-1}$, that is

$$B_n(z) := \prod_{i=1}^{n-1} \frac{z-\lambda i}{1-\overline{\lambda}_i z};$$

for $n = 1$, we assume $B_1(z) = 1$. The functions

$$e_n(z) := \frac{\left(1-|\lambda_n|^2\right)^{1/2}}{1-\overline{\lambda}_n z} B_n(z), \ n \geq 1,$$

form an orthonormal basis in the subspace $K_B = H^2 \Theta B H^2$.

Note that the Toeplitz operator T_φ is densely defined. In fact, the domain of T_φ contains H^∞ which is dense in H^2. For any $z \in \mathbb{D}$, let $\widehat{k}_{H^2,\lambda}$ be the normalized kernel of H^2, that is,

$$\widehat{k}_{H^2,\lambda}(t) = \frac{\sqrt{1-|\lambda|^2}}{1-\overline{\lambda}e^{it}}.$$

Since each $k_{H^2,\lambda}$ is in the domain of T_φ, we can consider the inner product $\left\langle T_\varphi \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \right\rangle$ in H^2, and it is easy to see that

$$\left\langle T_\varphi \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \right\rangle = \widetilde{\varphi}(\lambda) \quad (\lambda \in \mathbb{D}),$$

where $\widetilde{\varphi}$ is the harmonic extension of φ to $\mathbb{D}$ (see Zhu [13, Chapter 6]). Then we obtain:

$$\begin{aligned}
\sum_{n=1}^{\infty} \left\langle A_\varphi e_n(z), e_n(z) \right\rangle &= \sum_{n=1}^{\infty} \left\langle P_\theta T_\varphi B_n(z) \frac{\sqrt{1-|\lambda_n|^2}}{1-\overline{\lambda}_n z}, B_n(z) \frac{\sqrt{1-|\lambda_n|^2}}{1-\overline{\lambda}_n z} \right\rangle \\
&= \sum_{n=1}^{\infty} \left\langle T_{\overline{B}_n} T_\varphi T_{B_n} \widehat{k}_{H^2,\lambda n}, \widehat{k}_{H^2,\lambda n} \right\rangle \\
&= \sum_{n=1}^{\infty} \left\langle T_{\overline{B}_n \varphi B_n} \widehat{k}_{H^2,\lambda n}, \widehat{k}_{H^2,\lambda n} \right\rangle \\
&= \sum_{n=1}^{\infty} \left\langle T_\varphi \widehat{k}_{H^2,\lambda n}, \widehat{k}_{H^2,\lambda n} \right\rangle \\
&= \sum_{n=1}^{\infty} \widetilde{T_\varphi}(\lambda_n) = \sum_{n=1}^{\infty} \widetilde{\varphi}(\lambda_n).
\end{aligned}$$

Hence

$$trace\left(A_{\varphi}\right)=\sum_{n=1}^{\infty}\widetilde{\varphi}(\lambda_n). \tag{4}$$

On the other hand,

$$\begin{aligned}\widetilde{A_{\varphi}}(\lambda_n) &= \left\langle A_{\varphi}\widehat{k}_{B,\lambda_n},\widehat{k}_{B,\lambda_n}\right\rangle = \left\langle P_B T_{\varphi}\widehat{k}_{B,\lambda_n},\widehat{k}_{B,\lambda_n}\right\rangle \\ &= \left\langle T_{\varphi}\widehat{k}_{B,\lambda_n},\widehat{k}_{B,\lambda_n}\right\rangle \\ &= \frac{1-|\lambda_n|^2}{1-|B(\lambda_n)|^2}\left\langle T_{\varphi}\frac{1-\overline{B(\lambda_n)}B(z)}{1-\overline{\lambda}_n z},\frac{1-\overline{B(\lambda_n)}B(z)}{1-\overline{\lambda}_n z}\right\rangle \\ &= \left(1-|\lambda_n|^2\right)\left\langle T_{\varphi}\frac{1}{1-\overline{\lambda}_n z},\frac{1}{1-\overline{\lambda}_n z}\right\rangle \\ &= \left\langle T_{\varphi}\widehat{k}_{H^2,\lambda n},\widehat{k}_{H^2,\lambda n}\right\rangle = \widetilde{T_{\varphi}}(\lambda_n) = \widetilde{\varphi}(\lambda_n).\end{aligned}$$

Thus,

$$\widetilde{A_{\varphi}}(\lambda_n)=\widetilde{\varphi}(\lambda_n),\ n\geq 1. \tag{5}$$

Formulas (4) and (5) imply the desired formulas (3). The proposition is proven. □

Other applications of Berezin symbols and Berezin numbers method can be found, for instance, in [2, 3, 7–10], and their references.

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