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Composite Steepest-Descent Method for the Triple Hierarchical Variational Inequalities

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Abstract. In this paper, we introduce and analyze a composite steepest-descent algorithm for solving the triple hierarchical variational inequality problem in a real Hilbert space. Under mild conditions, the strong convergence of the iteration sequences generated by the algorithm is established.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and P_C be the metric projection of H onto C. If $\{x_k\}$ is a sequence in H, then we denote by $x_k \to x$ (respectively, $x_k \rightharpoonup x$) the strong (respectively, weak) convergence of the sequence $\{x_k\}$ to x. Let $S : C \to H$ be a nonlinear mapping. We denote by $\text{Fix}(S)$ the set of fixed points of S. A mapping $S : C \to H$ is called L-Lipschitz if there exists a constant $L \geq 0$ such that

$$\|Sx - Sy\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

In particular, if $L = 1$ then S is called a nonexpansive mapping; if $L \in [0, 1)$ then S is called a contraction.

Let $\mathcal{A} : C \to H$ be a nonlinear mapping. The classical variational inequality problem (VIP) is to find $x \in C$ such that

$$\langle \mathcal{A}x, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1}$$

The solution set of VIP (1) is denoted by $\text{VI}(C, \mathcal{A})$.

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The VIP (1) was first discussed by Lions [23]. There are many applications of VIP (1) in various fields; see e.g., [9, 10, 13, 15, 16, 18, 34, 37, 39, 41, 43, 44, 47]. It is well known that, if A is a strongly monotone and Lipschitz-continuous mapping on C, then VIP (1) has a unique solution. In 1976, Korpelevich [22] proposed an iterative algorithm for solving the VIP (1) in Euclidean space $\mathbf{R}^n$:

$$\begin{cases} y_k = P_C(x_k - \tau A x_k), \\ x_{k+1} = P_C(x_k - \tau A y_k), \quad \forall k \geq 0, \end{cases}$$

with $\tau > 0$ a given number, which is known as the extragradient method. The literature on the VIP is vast and Korpelevich's extragradient method has received great attention given by many authors, who improved it in various ways; see e.g., [1, 7, 12, 14, 33, 35, 38, 40, 42] and references therein.

In 2001, Yamada [32] introduced the following hybrid steepest-descent method for solving the VIP (1) with $C = \mathrm{Fix}(S)$, $x_{n+1} = (I - \lambda_n \mu \mathcal{A}) S x_n$, $\forall n \geq 0$, where $S : H \to H$ is a nonexpansive mapping with $\mathrm{Fix}(S) \neq \emptyset$, $\mathcal{A} : H \to H$ is a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$ and $0 < \mu < \frac{2\eta}{\kappa^2}$, and then proved that under appropriate conditions, the sequence $\{x_n\}$ converges strongly to the unique solution of VIP (1). The problem of finding a point in $\mathrm{VI}(\mathrm{Fix}(S), \mathcal{A})$ is called a hierarchical VIP or a hierarchical fixed point problem. Yamada's hybrid steepest-descent method has received great attention given by many authors, see e.g., [3, 4, 6, 8, 11, 24, 27, 30, 36, 45] and references therein.

Let $A : C \to H$ and $B : H \to H$ be two mappings. Consider the following bilevel variational inequality problem (BVIP).

Problem 1.1. *Find $x^* \in \mathrm{VI}(C, B)$ such that*

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{VI}(C, B).$$

In particular, whenever $H = \mathbf{R}^n$, the BVIP was recently studied by Anh, Kim and Muu [1]. Bilevel variational inequalities are special classes of quasivariational inequalities ([2, 31]) and of equilibrium with equilibrium constraints. However it covers some classes of mathematical programs with equilibrium constraints, bilevel minimization problems ([26]), variational inequalities ([18, 45]) and complementarity problems.

In what follows, suppose that A and B satisfy the following conditions:

(C1) B is pseudomonotone on H and A is β-strongly monotone on C;
(C2) A is L_1-Lipschitz continuous on C;
(C3) B is L_2-Lipschitz continuous on H;
(C4) $\mathrm{VI}(C, B) \neq \emptyset$.

In 2012, Anh, Kim and Muu [1] introduced the following extragradient iterative algorithm for solving the above bilevel variational inequality.

Algorithm 1.2. *([1]). Initialization. Choose $u \in \mathbf{R}^n$, $x_0 \in C$, $k = 0$, $0 < \lambda \leq \frac{2\beta}{L_1^2}$, positive sequences $\{\delta_k\}, \{\lambda_k\}, \{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ and $\{\bar{\epsilon}_k\}$ such that*

$$\begin{cases} \lim\limits_{k\to\infty} \delta_k = 0, \ \sum\limits_{k=0}^{\infty} \bar{\epsilon}_k < \infty, \\ \alpha_k + \beta_k + \gamma_k = 1, \ \forall k \geq 0, \ \sum\limits_{k=0}^{\infty} \alpha_k = \infty, \\ \lim\limits_{k\to\infty} \alpha_k = 0, \ \lim\limits_{k\to\infty} \beta_k = \xi \in (0, \frac{1}{2}], \ \lim\limits_{k\to\infty} \lambda_k = 0, \ \lambda_k \leq \frac{1}{L_2}, \ \forall k \geq 0. \end{cases}$$

Step 1. Compute $y_k := P_C(x_k - \lambda_k B x_k)$ and $z_k := P_C(x_k - \lambda_k B y_k)$.

Step 2. Inner loop $j = 0, 1, \ldots$. Compute

$$\begin{cases} x_{k,0} := z_k - \lambda A z_k, \\ y_{k,j} := P_C(x_{k,j} - \delta_j B x_{k,j}), \\ x_{k,j+1} := \alpha_j x_{k,0} + \beta_j x_{k,j} + \gamma_j P_C(x_{k,j} - \delta_j B y_{k,j}). \\ \text{If } \|x_{k,j+1} - P_{\mathrm{VI}(C,B)} x_{k,0}\| \le \bar{\epsilon}_k \text{ then set } h_k := x_{k,j+1} \text{ and go to Step 3.} \\ \text{Otherwise, increase } j \text{ by 1 and repeat the inner loop Step 2.} \end{cases}$$

Step 3. Set $x_{k+1} := \alpha_k u + \beta_k x_k + \gamma_k h_k$. Then increase k by 1 and go to Step 1.

On the other hand, recall the variational inequality for a monotone operator $A_1 : H \to H$ over the fixed point set of a nonexpansive mapping $T : H \to H$:

$$\text{Find} \quad \bar{x} \in \mathrm{VI}(\mathrm{Fix}(T), A_1) := \{\bar{x} \in \mathrm{Fix}(T) : \langle A_1 \bar{x}, y - \bar{x} \rangle \ge 0, \forall y \in \mathrm{Fix}(T)\},$$

where $\mathrm{Fix}(T) := \{x \in H : Tx = x\} \neq \emptyset$. In [19, 20], Iiduka introduced the following three-stage variational inequality problem, that is, the following monotone variational inequality with variational inequality constraint over the fixed point set of a nonexpansive mapping.

Problem 1.3. *([20]). Assume that*

(i) $T : H \to H$ *is a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$*;*
(ii) $A_1 : H \to H$ *is α-inverse strongly monotone;*
(iii) $A_2 : H \to H$ *is β-strongly monotone and L-Lipschitz continuous;*
(iv) $\mathrm{VI}(\mathrm{Fix}(T), A_1) \neq \emptyset$.

Then the objective is to

$$\text{find} \quad x^* \in \mathrm{VI}(\mathrm{VI}(\mathrm{Fix}(T), A_1), A_2) := \{x^* \in \mathrm{VI}(\mathrm{Fix}(T), A_1) : \langle A_2 x^*, v - x^* \rangle \ge 0, \forall v \in \mathrm{VI}(\mathrm{Fix}(T), A_1)\}.$$

Very recently, some authors continued the study of Iiduka's THVIP (i.e., Problem I) and its variant and extension; see e.g., [5, 8, 46]. In 2012, Ceng, Ansari and Yao [6] proposed a relaxed hybrid steepest-descent algorithm for solving Problem 1.3.

Algorithm 1.4. *([6]). Assume that the operators $T : H \to H$ and $A_i : H \to H$ $(i = 1, 2)$ satisfy conditions (i)-(iv) in Problem 1.3.*

Step 0. Take $\{\alpha_k\}_{k=0}^{\infty} \subset (0, 1]$, $\{\lambda_k\}_{k=0}^{\infty} \subset (0, 2\alpha]$, $\{\mu_k\}_{k=0}^{\infty} \subset (0, \frac{2\beta}{L^2})$, choose $x_0 \in H$ arbitrarily, and let $k := 0$.

Step 1. Given $x_k \in H$, compute $x_{k+1} \in H$ as

$$y_k := T(x_k - \lambda_k A_1 x_k),$$
$$x_{k+1} := y_k - \mu_k \alpha_k A_2 y_k.$$

Update $k := k + 1$ and go to Step 1.

Moreover, the authors [6] also considered the following monotone variational inequality with the variational inequality constraint over the intersection of the fixed point sets of N nonexpansive mappings $T_i : H \to H$, where $N \ge 1$ an integer.

Problem 1.5. *([6]). Assume that*

(i) for $i = 1, \ldots, N$, $T_i : H \to H$ is a nonexpansive mapping with $\cap_{i=1}^{N} \mathrm{Fix}(T_i) \neq \emptyset$;
(ii) $A_1 : H \to H$ *is α-inverse strongly monotone;*
(iii) $A_2 : H \to H$ *is β-strongly monotone and L-Lipschitz continuous;*
(iv) $\mathrm{VI}(\cap_{i=1}^{N} \mathrm{Fix}(T_i), A_1) \neq \emptyset$.

Then the objective is to

$$\text{find } x^* \in \mathrm{VI}(\mathrm{VI}(\bigcap_{i=1}^{N}\mathrm{Fix}(T_i), A_1), A_2) := \{x^* \in \mathrm{VI}(\bigcap_{i=1}^{N}\mathrm{Fix}(T_i), A_1) : \langle A_2x^*, v - x^*\rangle \geq 0, \forall v \in \mathrm{VI}(\bigcap_{i=1}^{N}\mathrm{Fix}(T_i), A_1)\}.$$

The authors [6] also proposed another relaxed hybrid steepest-descent algorithm below for solving Problem 1.5, and established the strong convergence result for the proposed algorithm.

Algorithm 1.6. *([6]). Assume that the operators $T_i : H \to H$ $(i = 1, ..., N)$ and $A_j : H \to H$ $(j = 1, 2)$ satisfy assumptions (i)-(iv) in Problem 1.5.*

Step 0. Take $\{\alpha_k\}_{k=0}^{\infty} \subset (0, 1]$, $\{\lambda_k\}_{k=0}^{\infty} \subset (0, 2\alpha]$, $\{\mu_k\}_{k=0}^{\infty} \subset (0, \frac{2\beta}{L^2})$, choose $x_0 \in H$ arbitrarily and let $k := 0$.
Step 1. Given $x_k \in H$, compute $x_{k+1} \in H$ as

$$y_k := T_{[k+1]}(x_k - \lambda_k A_1 x_k),$$
$$x_{k+1} := y_k - \mu_k\alpha_k A_2 y_k.$$

Update $k := k + 1$ and go to Step 1.

In this paper, we introduce and analyze a composite steepest-descent algorithm for solving the triple hierarchical variational inequality problem (THVIP) with the constraints of the monotone variational inequality problem (VIP) and the common fixed point problem of finitely many nonexpansive mappings in a real Hilbert space. The proposed algorithm is based on Korpelevich's extragradient method [22], hybrid steepest-descent method [32] and Halpern's iteration method. Under mild conditions, the strong convergence of the iteration sequences generated by the algorithm is derived. Our results improve and extend the corresponding results announced by some others, e.g., Iiduka [20, Theorem 4.1], Ceng, Ansari and Yao [6, Theorem 3.2] and Anh, Kim and Muu [1, Theorem 3.1].

2. Preliminaries

Throughout this paper, we assume that C is a nonempty closed convex subset of a real Hilbert space H. We use $\omega_w(x_k)$ to denote the weak ω-limit set of the sequence $\{x_k\}$, i.e.,

$$\omega_w(x_k) := \{x \in H : x_{k_i} \rightharpoonup x \text{ for some subsequence } \{x_{k_i}\} \text{ of } \{x_k\}\}.$$

Recall that a mapping $A : C \to H$ is called

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C;$$

(iii) α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y\in C}\|x - y\| =: d(x, C).$$

Some important properties of projections are gathered in the following proposition.

Proposition 2.1. *For given $x \in H$ and $z \in C$:*

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z\rangle \le 0, \ \forall y \in C;$
(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \le \|x - y\|^2 - \|y - z\|^2, \ \forall y \in C;$
(iii) $\langle P_C x - P_C y, x - y\rangle \ge \|P_C x - P_C y\|^2, \ \forall y \in H.$

If A is an α-inverse-strongly monotone mapping of C into H, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. We also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\|(I - \lambda A)u - (I - \lambda A)v\|^2 \le \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2. \tag{2}$$

So, if $\lambda \le 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

Definition 2.2. *A mapping $T : H \to H$ is said to be firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently, if T is 1-inverse strongly monotone (1-ism),*

$$\langle x - y, Tx - Ty\rangle \ge \|Tx - Ty\|^2, \quad \forall x, y \in H;$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as $T = \frac{1}{2}(I + S)$, where $S : H \to H$ is nonexpansive.

It can be easily seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that a projection P_C is 1-ism. Inverse strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

Proposition 2.3. *([17]). Let $T : C \to C$ be a nonexpansive mapping. Then the following hold:*

(i) $\mathrm{Fix}(T)$ *is closed and convex;*
(ii) $\mathrm{Fix}(T) \ne \emptyset$ *when C is bounded.*

Recall that, a mapping $A : C \to H$ is called hemicontinuous if for all $x, y \in C$, the mapping $g : [0, 1] \to H$, defined by $g(t) := A(tx + (1 - t)y)$, is continuous. Some properties of the solution set of the monotone variational inequality are mentioned in the following result.

Lemma 2.4. *([21, 28]) Let $A : C \to H$ be a monotone and hemicontinuous mapping. Then the following hold:*

(i) $\mathrm{VI}(C, A)$ *is equivalent to* $\mathrm{MVI}(C, A) := \{x^* \in C : \langle Ay, y - x^*\rangle \ge 0, \forall y \in C\}$;
(ii) $\mathrm{VI}(C, A) \ne \emptyset$ *when C is bounded;*
(iii) $\mathrm{VI}(C, A) = \mathrm{Fix}(P_C(I - \lambda A))$ *for all $\lambda > 0$, where I is the identity mapping on H;*
(iv) $\mathrm{VI}(C, A)$ *consists of only one point, if A is strongly monotone and Lipschitz continuous.*

Lemma 2.5. *([16]). Let C be a nonempty closed convex subset of a real Hilbert space H. Let S be a nonexpansive self-mapping on C with $\mathrm{Fix}(S) \ne \emptyset$. Then $I - S$ is demiclosed. That is, whenever $\{x_k\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - S)x_k\}$ strongly converges to some y, it follows that $(I - S)x = y$. Here I is the identity operator of H.*

Recall that, a mapping $T : C \to C$ is called a ζ-strictly pseudocontractive mapping (or a ζ-strict pseudocontraction) if there exists a constant $\zeta \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \zeta\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

Note that the class of strictly pseudocontractive mappings strictly includes the class of nonexpansive mappings. It is clear that T is nonexpansive if and only if T is a 0-strict pseudocontraction.

Lemma 2.6. *([25]). Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a mapping.*

(i) If T is a ζ-strictly pseudocontractive mapping, then T satisfies the Lipschitzian condition

$$\|Tx - Ty\| \le \frac{1+\zeta}{1-\zeta}\|x - y\|, \quad \forall x, y \in C.$$

(ii) If T is a ζ-strictly pseudocontractive mapping, then the mapping $I - T$ is semiclosed at 0, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I - T)x_n \to 0$, then $(I - T)\tilde{x} = 0$.

(iii) If T is ζ-(quasi-)strict pseudocontraction, then the fixed-point set $\mathrm{Fix}(T)$ of T is closed and convex so that the projection $P_{\mathrm{Fix}(T)}$ is well defined.

Lemma 2.7. *([35]). Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a ζ-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers such that $(\gamma + \delta)\zeta \le \gamma$. Then*

$$\|\gamma(x - y) + \delta(Tx - Ty)\| \le (\gamma + \delta)\|x - y\|, \quad \forall x, y \in C.$$

Lemma 2.8. *([16]). Let $\{a_k\}_{k=0}^{\infty}$ be a bounded sequence of nonnegative real numbers and $\{b_k\}_{k=0}^{\infty}$ be a sequence of real numbers such that $\limsup_{k\to\infty} b_k \le 0$. Then, $\limsup_{k\to\infty} a_k b_k \le 0$.*

Let λ be a number in $(0, 1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $S : C \to H$, we define the mapping $S^{(\lambda,\mu)} : C \to H$ by $S^{(\lambda,\mu)}x := Sx - \lambda\mu F(Sx)$, $\forall x \in C$, where $F : H \to H$ is κ-Lipschitzian and η-strongly monotone.

Lemma 2.9. *([30]). $S^{(\lambda,\mu)}$ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$; that is,*

$$\|S^{(\lambda,\mu)}x - S^{(\lambda,\mu)}y\| \le (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in C,$$

where $\tau := 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

Lemma 2.10. *([29]). Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{k+1} \le (1 - s_k)a_k + s_k t_k + \delta_k, \quad \forall k \ge 0,$$

where $\{s_k\}, \{t_k\}$ and $\{\delta_k\}$ are sequences of real numbers such that

(i) $\{s_k\} \subset [0, 1]$ and $\sum_{k=0}^{\infty} s_k = \infty$;
(ii) either $\limsup_{k\to\infty} t_k \le 0$, or $\sum_{k=0}^{\infty} |s_k t_k| < \infty$;
(iii) $\sum_{k=0}^{\infty} \delta_k < \infty$ with $\delta_k \ge 0$, $\forall k \ge 0$.

Then, $\lim_{k\to\infty} a_k = 0$.

Lemma 2.11. *([16]). Let H be a real Hilbert space. Then the following hold:*

(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ for all $x, y \in H$;
(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x - y\|^2$ for all $x, y \in H$ and $\lambda, \mu \in [0, 1]$ with $\lambda + \mu = 1$;
(c) If $\{x_k\}$ is a sequence in H such that $x_k \rightharpoonup x$, it follows that

$$\limsup_{k\to\infty} \|x_k - y\|^2 = \limsup_{k\to\infty} \|x_k - x\|^2 + \|x - y\|^2, \quad \forall y \in H.$$

3. Iterative Algorithm and Convergence Criteria

Let H be a real Hilbert space. In this section, we always assume the following:

$T_i : H \to H$ is a nonexpansive mapping for each $i = 1, ..., N$ such that $\cap_{i=1}^{N} \mathrm{Fix}(T_i) \ne \emptyset$;

$\mathcal{A} : H \to H$ is η-strongly monotone and L_1-Lipschitz continuous, and $\mathcal{B} : H \to H$ is α-inverse strongly monotone;

$\{\alpha_k\}, \{\beta_k\} \subset (0, 1]$, $\{\lambda_k\} \subset (0, 2\alpha]$ and $\{\mu_k\} \subset (0, \frac{2\eta}{L_1^2})$ satisfy the conditions (A1)-(A3):

(A1) $|\mu_k - \frac{\eta}{L_1^2}| \le \frac{\sqrt{\eta^2 - cL_1^2}}{L_1^2}$ for some $c \in (0, \frac{\eta^2}{L_1^2})$,
(A2) $\lim_{k\to\infty}(\mu_{k+1} - \frac{\alpha_k}{\alpha_{k+1}}\mu_k) = 0$ and $\sum_{k=0}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty$,
(A3) $\sum_{k=0}^{\infty} |\beta_{k+1} - \beta_k| < \infty$ and $\sum_{k=0}^{\infty} \lambda_k < \infty$;

$A : H \to H$ and $B : H \to H$ are two mappings such that the hypotheses (H1)-(H4) hold:

(H1) B is monotone on H,
(H2) A is β-inverse-strongly monotone on H,
(H3) B is L_2-Lipschitz continuous on H,
(H4) $\mathrm{VI}(\mathrm{VI}(\Omega, B), A) \ne \emptyset$ where $\Omega = \mathrm{VI}(\cap_{i=1}^N \mathrm{Fix}(T_i), \mathcal{B})$.

Next, we introduce the following triple hierarchical variational inequality problem (THVIP) with the constraints of the monotone VIP and the common fixed point problem of finitely many nonexpansive mappings.

Problem 3.1. *The objective is to*

$$\text{find} \quad x^* \in \mathrm{VI}(\mathrm{VI}(\Omega, B), A) := \{x^* \in \mathrm{VI}(\Omega, B) : \langle Ax^*, x - x^*\rangle \ge 0, \forall x \in \mathrm{VI}(\Omega, B)\}.$$

That is, the objective is to find $x^* \in \mathrm{VI}(\Omega, B)$ *such that*

$$\langle Ax^*, x - x^*\rangle \ge 0, \quad \forall x \in \mathrm{VI}(\Omega, B), \tag{3}$$

where $\mathrm{VI}(\Omega, B)$ *denotes the set of solutions of the VIP: Find* $y^* \in \Omega$ *such that*

$$\langle By^*, y - y^*\rangle \ge 0, \quad \forall y \in \Omega. \tag{4}$$

Algorithm 3.2. *Initialization. Choose* $u \in H,\ x_0 \in H,\ k = 0,\ 0 < \lambda \le 2\beta$, *positive sequences* $\{\delta_k\}, \{\lambda_k\}, \{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ *and* $\{\bar{\epsilon}_k\}$ *such that*

$$\begin{cases} \lim\limits_{k\to\infty} \delta_k = 0,\ \sum\limits_{k=0}^{\infty} \bar{\epsilon}_k < \infty, \\ \alpha_k + \beta_k + \gamma_k = 1,\ \forall k \ge 0,\ \lim\limits_{k\to\infty} \beta_k = \xi \in (0, \frac{1}{2}], \\ \sum\limits_{k=0}^{\infty} \alpha_k = \infty,\ \lim\limits_{k\to\infty} \alpha_k = 0,\ \lambda_k = o(\alpha_k),\ \lambda_k \le \frac{1}{L_2},\ \forall k \ge 0. \end{cases}$$

Step 1. Compute

$$\begin{cases} u_k = T_{[k+1]}(x_k - \lambda_k \mathcal{B} x_k), \\ v_k = u_k - \mu_k \alpha_k \mathcal{A} u_k, \\ y_k := P_\Omega(v_k - \lambda_k B v_k), \\ z_k := P_\Omega(v_k - \lambda_k B y_k). \end{cases}$$

Step 2. Inner loop $j = 0, 1, \ldots$. *Compute*

$$\begin{cases} x_{k,0} := z_k - \lambda A z_k, \\ y_{k,j} := P_\Omega(x_{k,j} - \delta_j B x_{k,j}), \\ x_{k,j+1} := \alpha_j x_{k,0} + \beta_j x_{k,j} + \gamma_j P_\Omega(x_{k,j} - \delta_j B y_{k,j}). \\ \text{If } \|x_{k,j+1} - P_{\mathrm{VI}(\Omega,B)} x_{k,0}\| \le \bar{\epsilon}_k \text{ then set } h_k := x_{k,j+1} \text{ and go to Step 3.} \\ \text{Otherwise, increase } j \text{ by 1 and repeat the inner loop Step 2.} \end{cases}$$

Step 3. Set $x_{k+1} := \alpha_k u + \beta_k x_k + \gamma_k h_k$. *Then increase k by* 1 *and go to Step 1.*

In the sequel we always suppose that the inner loop in the Algorithm 3.2 terminates after a finite number of steps.

Lemma 3.3. *Let sequences $\{v_k\}, \{y_k\}$ and $\{z_k\}$ be generated by Algorithm 3.2, B be L_2-Lipschitzian and monotone on H, and $p \in \mathrm{VI}(\Omega, B)$. Then, we have*

$$\|z_k - p\|^2 \le \|v_k - p\|^2 - (1 - \lambda_k L_2)\|v_k - y_k\|^2 - (1 - \lambda_k L_2)\|y_k - z_k\|^2. \tag{5}$$

Proof. Let $p \in \mathrm{VI}(\Omega, B)$. That means $\langle Bp, x - p\rangle \ge 0,\ \forall x \in \Omega$. Then, for each $\lambda_k > 0$, p satisfies the fixed point equation $p = P_\Omega(p - \lambda_k Bp)$. Since B is monotone on H and $p \in \mathrm{VI}(\Omega, B)$, we have $\langle By_k, y_k - p\rangle \ge \langle Bp, y_k - p\rangle \ge 0$. Then, applying Proposition 2.1 (ii) with $v_k - \lambda_k By_k$ and p, we obtain

$$\begin{aligned}
\|z_k - p\|^2 &\le \|v_k - \lambda_k By_k - p\|^2 - \|v_k - \lambda_k By_k - z_k\|^2\\
&= \|v_k - p\|^2 - \|v_k - z_k\|^2 + 2\lambda_k\langle By_k, p - z_k\rangle\\
&= \|v_k - p\|^2 - \|v_k - z_k\|^2 + 2\lambda_k\langle By_k, p - y_k\rangle + 2\lambda_k\langle By_k, y_k - z_k\rangle\\
&\le \|v_k - p\|^2 - \|v_k - z_k\|^2 + 2\lambda_k\langle By_k, y_k - z_k\rangle.
\end{aligned} \tag{6}$$

Applying Proposition 2.1 (i) with $v_k - \lambda_k Bv_k$ and z_k, we also have $\langle v_k - \lambda_k Bv_k - y_k, z_k - y_k\rangle \le 0$. Combining this inequality with (6) and observing that B is L_2-Lipschitz continuous on H, we obtain

$$\begin{aligned}
\|z_k - p\|^2 &\le \|v_k - p\|^2 - \|(v_k - y_k) + (y_k - z_k)\|^2 + 2\lambda_k\langle By_k, y_k - z_k\rangle\\
&= \|v_k - p\|^2 - \|v_k - y_k\|^2 - \|y_k - z_k\|^2 - 2\langle v_k - \lambda_k Bv_k - y_k, y_k - z_k\rangle + 2\lambda_k\langle Bv_k - By_k, z_k - y_k\rangle\\
&\le \|v_k - p\|^2 - \|v_k - y_k\|^2 - \|y_k - z_k\|^2 + 2\lambda_k\langle Bv_k - By_k, z_k - y_k\rangle\\
&\le \|v_k - p\|^2 - \|v_k - y_k\|^2 - \|y_k - z_k\|^2 + 2\lambda_k\|Bv_k - By_k\|\|z_k - y_k\|\\
&\le \|v_k - p\|^2 - \|v_k - y_k\|^2 - \|y_k - z_k\|^2 + 2\lambda_k L_2\|v_k - y_k\|\|z_k - y_k\|\\
&\le \|v_k - p\|^2 - \|v_k - y_k\|^2 - \|y_k - z_k\|^2 + \lambda_k L_2(\|v_k - y_k\|^2 + \|z_k - y_k\|^2)\\
&\le \|v_k - p\|^2 - (1 - \lambda_k L_2)\|v_k - y_k\|^2 - (1 - \lambda_k L_2)\|y_k - z_k\|^2.
\end{aligned} \tag{7}$$

□

Lemma 3.4. *Suppose that the conditions (A1)-(A3) and (H1)-(H4) hold. Then the sequence $\{x_k\}$ generated by Algorithm 3.2 is bounded.*

Proof. Take an arbitrary $p \in \mathrm{VI}(\mathrm{VI}(\Omega, B), A)$. Since $S^{(\alpha_k,\mu_k)}p = p - \alpha_k\mu_k\mathcal{A}p$, by Lemma 2.9 we have

$$\begin{aligned}
\|v_k - p\| &\le \|S^{(\alpha_k,\mu_k)}u_k - S^{(\alpha_k,\mu_k)}p\| + \|S^{(\alpha_k,\mu_k)}p - p\|\\
&\le (1 - \alpha_k\tau_k)\|u_k - p\| + \alpha_k\mu_k\|\mathcal{A}p\|,
\end{aligned} \tag{8}$$

where $\tau_k := 1 - \sqrt{1 - \mu_k(2\eta - \mu_k L_1^2)}$.

We claim that $\tau_k \ge \tau$, where $\tau = 1 - \sqrt{1 - c}$. Indeed, it follows from condition (A1) that

$$\frac{\eta - \sqrt{\eta^2 - cL_1^2}}{L_1^2} \le \mu_k \le \frac{\eta + \sqrt{\eta^2 - cL_1^2}}{L_1^2} < \frac{2\eta}{L_1^2} =: \mu,$$

and hence

$$(\mu_k - \frac{\eta}{L_1^2} + \frac{\sqrt{\eta^2 - cL_1^2}}{L_1^2}) \cdot (\mu_k - \frac{\eta}{L_1^2} - \frac{\sqrt{\eta^2 - cL_1^2}}{L_1^2}) \le 0.$$

This implies that $L_1^2\mu_k^2 - 2\eta\mu_k + c \le 0$. Observe that $\mu_k(2\eta - \mu_k L_1^2) \ge c = 1 - [1 - (1 - \sqrt{1-c})]^2 = 1 - (1-\tau)^2$, where $\tau := 1 - \sqrt{1-c}$. Hence, we have $\tau_k = 1 - \sqrt{1 - \mu_k(2\eta - \mu_k L_1^2)} \ge \tau,\ \forall k \ge 0$. Thus, from (8) we get

$$\begin{aligned}\|v_k - p\| &\le (1-\alpha_k\tau)\|u_k - p\| + \alpha_k\mu\|\mathcal{A}p\| \\ &\le \max\{\|u_k - p\|, \frac{\mu}{\tau}\|\mathcal{A}p\|\}.\end{aligned} \tag{9}$$

Furthermore, utilizing the nonexpansivity of T and the α-inverse strong monotonicity of $\mathcal{B}$, we deduce from $\{\lambda_k\} \subset (0, 2\alpha]$ that

$$\begin{aligned}\|u_k - p\| &= \|T_{[k+1]}(x_k - \lambda_k\mathcal{B}x_k) - p\| \\ &\le \|T_{[k+1]}(x_k - \lambda_k\mathcal{B}x_k) - T_{[k+1]}(p - \lambda_k\mathcal{B}p)\| + \|T_{[k+1]}(p - \lambda_k\mathcal{B}p) - T_{[k+1]}p\| \\ &\le \|x_k - \lambda_k\mathcal{B}x_k - (p - \lambda_k\mathcal{B}p)\| + \|p - \lambda_k\mathcal{B}p - p\| \\ &= \|x_k - p - \lambda_k(\mathcal{B}x_k - \mathcal{B}p)\| + \lambda_k\|\mathcal{B}p\| \\ &\le \|x_k - p\| + \lambda_k\|\mathcal{B}p\|.\end{aligned} \tag{10}$$

Combining (9) and (10), we obtain

$$\begin{aligned}\|v_k - p\| &\le \max\{\|u_k - p\|, \frac{\mu}{\tau}\|\mathcal{A}p\|\} \\ &\le \max\{\|x_k - p\| + \lambda_k\|\mathcal{B}p\|, \frac{\mu}{\tau}\|\mathcal{A}p\|\} \\ &\le \max\{\|x_k - p\|, \frac{\mu}{\tau}\|\mathcal{A}p\|\} + \lambda_k\|\mathcal{B}p\|.\end{aligned} \tag{11}$$

Taking into account $p \in \mathrm{VI}(\mathrm{VI}(B,\Omega), A)$. Then we have $\langle Ap, x - p\rangle \ge 0,\ \forall x \in \mathrm{VI}(\Omega, B)$, which implies $p = P_{\mathrm{VI}(\Omega,B)}(p - \lambda Ap)$. Then, it follows from (2), Proposition 2.1 (iii), β-inverse strong monotonicity of A, and $0 < \lambda \le 2\beta$ that

$$\begin{aligned}\|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k) - p\|^2 &= \|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k) - P_{\mathrm{VI}(\Omega,B)}(p - \lambda Ap)\|^2 \\ &\le \|(I - \lambda A)z_k - (I - \lambda A)p\|^2 \\ &\le \|z_k - p\|^2 + \lambda(\lambda - 2\beta)\|Az_k - Ap\|^2 \\ &\le \|z_k - p\|^2.\end{aligned} \tag{12}$$

Utilizing (7), (11), (12) and the assumptions $0 < \lambda \le 2\beta,\ \sum_{k=0}^{\infty}\bar{\epsilon}_k < \infty$ we obtain that

$$\begin{aligned}\|x_{k+1} - p\| &= \|\alpha_k u + \beta_k x_k + \gamma_k h_k - p\| \\ &\le \alpha_k\|u - p\| + \beta_k\|x_k - p\| + \gamma_k\|h_k - p\| \\ &\le \alpha_k\|u - p\| + (\beta_k + \gamma_k)\max\{\|x_k - p\|, \frac{\mu}{\tau}\|\mathcal{A}p\|\} + \bar{\epsilon}_k + \lambda_k\|\mathcal{B}p\| \\ &= \alpha_k\|u - p\| + (1 - \alpha_k)\max\{\|x_k - p\|, \frac{\mu}{\tau}\|\mathcal{A}p\|\} + \bar{\epsilon}_k + \lambda_k\|\mathcal{B}p\| \\ &\le \max\{\|x_k - p\|, \|u - p\|, \frac{\mu}{\tau}\|\mathcal{A}p\|\} + \bar{\epsilon}_k + \lambda_k\|\mathcal{B}p\| \\ &\le \max\{\|x_0 - p\|, \|u - p\|, \frac{\mu}{\tau}\|\mathcal{A}p\|\} + \sum_{j=0}^{k}(\bar{\epsilon}_k + \lambda_k\|\mathcal{B}p\|) \\ &< \infty,\end{aligned}$$

which shows that the sequence $\{x_k\}$ is bounded, and so are the sequences $\{u_k\}, \{v_k\}, \{y_k\}$ and $\{z_k\}$. □

Lemma 3.5. *Suppose that the conditions (H1)-(H4) hold. Assume that the sequences $\{v_k\}$ and $\{z_k\}$ are generated by Algorithm 3.2. Then, we have*

$$\|z_{k+1} - z_k\| \le (1 + \lambda_{k+1}L_2)\|v_{k+1} - v_k\| + \lambda_k\|By_k\| + \lambda_{k+1}(\|Bv_{k+1}\| + \|By_{k+1}\| + \|Bv_k\|). \tag{13}$$

Proof. Since $\lim_{k\to\infty}\beta_k = \xi \in (0, \frac{1}{2}]$, we may assume, without loss of generality, that $\{\beta_k\} \subset [a, b] \subset (0, 1)$ for all $k \ge 0$. Taking into account the L_2-Lipschitzian property of B, for each $x, y \in H$ we have

$$\begin{aligned}\|(I - \lambda_k B)x - (I - \lambda_k B)y\| &\le \|x - y\| + \lambda_k\|Bx - By\| \\ &\le (1 + \lambda_k L_2)\|x - y\|.\end{aligned}$$

Combining this inequality with Proposition 2.1 (iii), we have

$$\begin{aligned}\|z_{k+1} - z_k\| &\le \|(v_{k+1} - \lambda_{k+1}By_{k+1}) - v_k + \lambda_k By_k\| \\ &= \|(v_{k+1} - \lambda_{k+1}Bv_{k+1}) - (v_k - \lambda_{k+1}Bv_k) + \lambda_{k+1}(Bv_{k+1} - By_{k+1} - Bv_k) + \lambda_k By_k\| \\ &\le (1 + \lambda_{k+1}L_2)\|v_{k+1} - v_k\| + \lambda_k\|By_k\| + \lambda_{k+1}(\|Bv_{k+1}\| + \|By_{k+1}\| + \|Bv_k\|).\end{aligned} \tag{14}$$

This is the desired result (13). □

Lemma 3.6. *Suppose that the conditions (A1)-(A3) and (H1)-(H4) hold. Assume that the sequence $\{x_k\}$ is generated by Algorithm 3.2. Then, $\lim_{k\to\infty}\|x_{k+1} - x_k\| = 0$.*

Proof. We write $x_{k+1} = (1 - \beta_k)w_k + \beta_k x_k$ for all $k \ge 0$. Then, we have

$$\begin{aligned}w_{k+1} - w_k &= \frac{\alpha_{k+1}u + \gamma_{k+1}h_{k+1}}{1 - \beta_{k+1}} - \frac{\alpha_k u + \gamma_k h_k}{1 - \beta_k} \\ &= (\frac{\alpha_{k+1}}{1 - \beta_{k+1}} - \frac{\alpha_k}{1 - \beta_k})u + (\frac{\gamma_{k+1}}{1 - \beta_{k+1}} - \frac{\gamma_k}{1 - \beta_k})h_k + \frac{\gamma_{k+1}}{1 - \beta_{k+1}}(h_{k+1} - h_k).\end{aligned} \tag{15}$$

Note that, for $0 < \lambda \le 2\beta$, we have from (2) that

$$\begin{aligned}\|P_{\mathrm{VI}(\Omega,B)}(z_{k+1} - \lambda Az_{k+1}) - P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k)\|^2 &\le \|(I - \lambda A)z_{k+1} - (I - \lambda A)z_k\|^2 \\ &\le \|z_{k+1} - z_k\|^2 + \lambda(\lambda - 2\beta)\|Az_{k+1} - Az_k\|^2 \\ &\le \|z_{k+1} - z_k\|^2.\end{aligned}$$

Then, utilizing (13) and (15) we get

$$\begin{aligned}\|w_{k+1} - w_k\| &\le |\frac{\alpha_{k+1}}{1 - \beta_{k+1}} - \frac{\alpha_k}{1 - \beta_k}|\|u\| + |\frac{\gamma_{k+1}}{1 - \beta_{k+1}} - \frac{\gamma_k}{1 - \beta_k}|\|h_k\| + \frac{\gamma_{k+1}}{1 - \beta_{k+1}}\|h_{k+1} - h_k\| \\ &\le |\frac{\alpha_{k+1}}{1 - \beta_{k+1}} - \frac{\alpha_k}{1 - \beta_k}|\|u\| + |\frac{\gamma_{k+1}}{1 - \beta_{k+1}} - \frac{\gamma_k}{1 - \beta_k}|(\|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k)\| + \bar{\epsilon}_k) \\ &\quad + \frac{\gamma_{k+1}(1 + \lambda_{k+1}L_2)}{1 - \beta_{k+1}}\|v_{k+1} - v_k\| + \frac{\gamma_{k+1}}{1 - \beta_{k+1}}(\bar{\epsilon}_{k+1} + \bar{\epsilon}_k) \\ &\quad + \frac{\gamma_{k+1}}{1 - \beta_{k+1}}(\lambda_{k+1}(\|Bv_{k+1}\| + \|By_{k+1}\| + \|Bv_k\|) + \lambda_k\|By_k\|) \\ &= |\frac{\alpha_{k+1}}{1 - \beta_{k+1}} - \frac{\alpha_k}{1 - \beta_k}|\|u\| + |\frac{\alpha_{k+1}}{1 - \beta_{k+1}} - \frac{\alpha_k}{1 - \beta_k}|(\|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k)\| + \bar{\epsilon}_k) \\ &\quad + \frac{\gamma_{k+1}(1 + \lambda_{k+1}L_2)}{\alpha_{k+1} + \gamma_{k+1}}\|v_{k+1} - v_k\| + \frac{\gamma_{k+1}}{\alpha_{k+1} + \gamma_{k+1}}(\bar{\epsilon}_{k+1} + \bar{\epsilon}_k) \\ &\quad + \frac{\gamma_{k+1}}{\alpha_{k+1} + \gamma_{k+1}}(\lambda_{k+1}(\|Bv_{k+1}\| + \|By_{k+1}\| + \|Bv_k\|) + \lambda_k\|By_k\|) \\ &\le \|v_{k+1} - v_k\| + (|\alpha_{k+1} - \alpha_k| + |\beta_{k+1} - \beta_k|)\frac{\|u\| + \|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k)\| + \bar{\epsilon}_k}{1 - b} + \bar{\epsilon}_{k+1} \\ &\quad + \bar{\epsilon}_k + \lambda_{k+1}(L_2\|v_{k+1} - v_k\| + \|Bv_{k+1}\| + \|By_{k+1}\| + \|Bv_k\|) + \lambda_k\|By_k\|.\end{aligned} \tag{16}$$

On the other hand, from (2), the nonexpansivity of T and the α-inverse strong monotonicity of $\mathcal{B}$ with $\{\lambda_k\} \subset (0, 2\alpha]$, we conclude that for all $k \geq 0$,

$$\begin{aligned}\|u_{k+1} - u_k\| &\leq \|(x_{k+1} - \lambda_{k+1}\mathcal{B}x_{k+1}) - (x_k - \lambda_k\mathcal{B}x_k)\| \\ &\leq \|(x_{k+1} - \lambda_{k+1}\mathcal{B}x_{k+1}) - (x_k - \lambda_{k+1}\mathcal{B}x_k)\| + |\lambda_k - \lambda_{k+1}|\|\mathcal{B}x_k\| \\ &\leq \|x_{k+1} - x_k\| + |\lambda_k - \lambda_{k+1}|\|\mathcal{B}x_k\|.\end{aligned} \tag{17}$$

From (17) and Lemma 2.9, we have

$$\begin{aligned}\|v_{k+1} - v_k\| &\leq \|S^{(\alpha_{k+1},\mu_{k+1})}u_{k+1} - S^{(\alpha_{k+1},\mu_{k+1})}u_k\| + \|S^{(\alpha_{k+1},\mu_{k+1})}u_k - S^{(\alpha_k,\mu_k)}u_k\| \\ &\leq (1 - \alpha_{k+1}\tau_{k+1})\|u_{k+1} - u_k\| + |\alpha_{k+1}\mu_{k+1} - \alpha_k\mu_k|\|\mathcal{A}u_k\| \\ &\leq (1 - \alpha_{k+1}\tau_{k+1})[\|x_{k+1} - x_k\| + |\lambda_k - \lambda_{k+1}|\|\mathcal{B}x_k\|] + |\alpha_{k+1}\mu_{k+1} - \alpha_k\mu_k|\|\mathcal{A}u_k\| \\ &\leq (1 - \alpha_{k+1}\tau)\|x_{k+1} - x_k\| + \alpha_{k+1}|\mu_{k+1} - \frac{\alpha_k}{\alpha_{k+1}}\mu_k|\|\mathcal{A}u_k\| + (\lambda_{k+1} + \lambda_k)\|\mathcal{B}x_k\|.\end{aligned} \tag{18}$$

Utilizing the relation $x_{k+1} = \beta_k x_k + (1 - \beta_k)w_k$, we obtain from (16) and (18) that

$$\begin{aligned}\|x_{k+2} - x_{k+1}\| &= \|\beta_{k+1}x_{k+1} + (1 - \beta_{k+1})w_{k+1} - \beta_k x_k - (1 - \beta_k)w_k\| \\ &\leq \beta_k\|x_{k+1} - x_k\| + (1 - \beta_k)\|w_{k+1} - w_k\| + |\beta_{k+1} - \beta_k|\|x_{k+1} - w_{k+1}\| \\ &\leq \beta_k\|x_{k+1} - x_k\| + (1 - \beta_k)\{(1 - \alpha_{k+1}\tau)\|x_{k+1} - x_k\| + \alpha_{k+1}|\mu_{k+1} - \frac{\alpha_k}{\alpha_{k+1}}\mu_k|\|\mathcal{A}u_k\| \\ &\quad + (\lambda_{k+1} + \lambda_k)\|\mathcal{B}x_k\| + (|\alpha_{k+1} - \alpha_k| + |\beta_{k+1} - \beta_k|)\frac{\|u\| + \|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k)\| + \bar{\epsilon}_k}{1 - b} \\ &\quad + \bar{\epsilon}_{k+1} + \bar{\epsilon}_k + \lambda_{k+1}(L_2\|v_{k+1} - v_k\| + \|Bv_{k+1}\| \\ &\quad + \|By_{k+1}\| + \|Bv_k\|) + \lambda_k\|By_k\|\} + |\beta_{k+1} - \beta_k|\|x_{k+1} - w_{k+1}\| \\ &\leq (1 - \alpha_{k+1}(1 - \beta_k)\tau)\|x_{k+1} - x_k\| + \alpha_{k+1}(1 - \beta_k)|\mu_{k+1} - \frac{\alpha_k}{\alpha_{k+1}}\mu_k|\|\mathcal{A}u_k\| \\ &\quad + (\lambda_{k+1} + \lambda_k)\|\mathcal{B}x_k\| + (|\alpha_{k+1} - \alpha_k| + |\beta_{k+1} - \beta_k|)\frac{\|u\| + \|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k)\| + \bar{\epsilon}_k}{1 - b} \\ &\quad + \bar{\epsilon}_{k+1} + \bar{\epsilon}_k + \lambda_{k+1}(L_2\|v_{k+1} - v_k\| + \|Bv_{k+1}\| + \|By_{k+1}\| + \|Bv_k\|) \\ &\quad + \lambda_k\|By_k\| + |\beta_{k+1} - \beta_k|\|x_{k+1} - w_{k+1}\| \\ &\leq (1 - \alpha_{k+1}(1 - \beta_k)\tau)\|x_{k+1} - x_k\| + \alpha_{k+1}(1 - \beta_k)|\mu_{k+1} - \frac{\alpha_k}{\alpha_{k+1}}\mu_k|M \\ &\quad + (\lambda_{k+1} + \lambda_k)M + (|\alpha_{k+1} - \alpha_k| + |\beta_{k+1} - \beta_k|)M + \bar{\epsilon}_{k+1} + \bar{\epsilon}_k + \lambda_{k+1}M \\ &\quad + \lambda_k M + |\beta_{k+1} - \beta_k|M \\ &\leq (1 - \alpha_{k+1}(1 - \beta_k)\tau)\|x_{k+1} - x_k\| + \alpha_{k+1}(1 - \beta_k)\tau|\mu_{k+1} - \frac{\alpha_k}{\alpha_{k+1}}\mu_k|\frac{M}{\tau} \\ &\quad + (\lambda_{k+1} + \lambda_k + |\alpha_{k+1} - \alpha_k| + |\beta_{k+1} - \beta_k|)2M + \bar{\epsilon}_{k+1} + \bar{\epsilon}_k,\end{aligned} \tag{19}$$

where there is a constant $M > 0$ such that

$$\begin{aligned}\sup_{k\geq 0}\{&\|\mathcal{A}u_k\| + \|\mathcal{B}x_k\| + \|By_k\| + \frac{\|u\| + \|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k)\| + \bar{\epsilon}_k}{1 - b} \\ &+ \|x_{k+1} - w_{k+1}\| + L_2\|v_{k+1} - v_k\| + \|Bv_{k+1}\| + \|By_{k+1}\| + \|Bv_k\|\} \leq M.\end{aligned}$$

Since $\sum_{k=0}^{\infty} \bar{\epsilon}_k < \infty$, $\sum_{k=0}^{\infty} \alpha_k = \infty$, $\sum_{k=0}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty$ and $\sum_{k=0}^{\infty} |\beta_{k+1} - \beta_k| < \infty$, it is easy to see from conditions (A2) and (A3) that $\sum_{k=0}^{\infty} \alpha_{k+1}(1 - \beta_k)\tau \geq \sum_{k=0}^{\infty} \alpha_{k+1}(1 - b)\tau = \infty$, $\lim_{k\to\infty} |\mu_{k+1} - \frac{\alpha_k}{\alpha_{k+1}}\mu_k|\frac{M}{\tau} = 0$ and

$$\sum_{k=0}^{\infty}[(\lambda_{k+1} + \lambda_k + |\alpha_{k+1} - \alpha_k| + |\beta_{k+1} - \beta_k|)2M + \bar{\epsilon}_{k+1} + \bar{\epsilon}_k] < \infty.$$

Therefore, applying Lemma 2.10 to (19), we know that

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{20}$$

□

Lemma 3.7. *Suppose that the conditions (A1)-(A3) and (H1)-(H4) hold. Then for any $p \in \mathrm{VI}(\mathrm{VI}(\Omega, B), A)$ we have*

$$\begin{aligned}\|x_{k+1} - p\|^2 \le{}& \alpha_k\|u - p\|^2 + \beta_k\|x_k - p\|^2 + \gamma_k\|v_k - p\|^2 + 2\gamma_k\bar{\epsilon}_k\|z_k - p\| \\ &+ \gamma_k\bar{\epsilon}_k^2 - \gamma_k(1 - \lambda_k L_2)(\|v_k - y_k\|^2 + \|y_k - z_k\|^2).\end{aligned} \tag{21}$$

Moreover, if $\lim_{k\to\infty} \|x_k - v_k\| = 0$, then

$$\lim_{k\to\infty} \|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda_k A z_k) - z_k\| = \lim_{k\to\infty} \|P_{\mathrm{VI}(\Omega,B)}(y_k - \lambda_k A y_k) - y_k\| = 0.$$

Proof. By [5], we know that $\lim_{j\to\infty} x_{k,j} = P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda A z_k)$, which together with $0 < \lambda \le 2\beta$, inequality (5), $\lim_{k\to\infty} \beta_k = \xi \in (0, \frac{1}{2}]$ and $p \in \mathrm{VI}(\mathrm{VI}(\Omega, B), A)$, implies that

$$\begin{aligned}\|x_{k+1} - p\|^2 &= \|\alpha_k u + \beta_k x_k + \gamma_k h_k - p\|^2 \\ &\le \alpha_k\|u - p\|^2 + \beta_k\|x_k - p\|^2 + \gamma_k\|h_k - p\|^2 \\ &\le \alpha_k\|u - p\|^2 + \beta_k\|x_k - p\|^2 + \gamma_k(\|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda A z_k) - p\| + \bar{\epsilon}_k)^2 \\ &\le \alpha_k\|u - p\|^2 + \beta_k\|x_k - p\|^2 + \gamma_k(\|(I - \lambda A)z_k - (I - \lambda A)p\| + \bar{\epsilon}_k)^2 \\ &\le \alpha_k\|u - p\|^2 + \beta_k\|x_k - p\|^2 + \gamma_k(\|z_k - p\| + \bar{\epsilon}_k)^2 \\ &\le \alpha_k\|u - p\|^2 + \beta_k\|x_k - p\|^2 + 2\gamma_k\bar{\epsilon}_k\|z_k - p\| + \gamma_k\bar{\epsilon}_k^2 \\ &\quad + \gamma_k(\|v_k - p\|^2 - (1 - \lambda_k L_2)\|v_k - y_k\|^2 - (1 - \lambda_k L_2)\|y_k - z_k\|^2) \\ &= \alpha_k\|u - p\|^2 + \beta_k\|x_k - p\|^2 + \gamma_k\|v_k - p\|^2 + 2\gamma_k\bar{\epsilon}_k\|z_k - p\| + \gamma_k\bar{\epsilon}_k^2 \\ &\quad - \gamma_k(1 - \lambda_k L_2)(\|v_k - y_k\|^2 + \|y_k - z_k\|^2).\end{aligned} \tag{22}$$

On the other hand, note that $\|v_k - u_k\| = \mu_k\alpha_k\|\mathcal{A}u_k\| \le \mu\alpha_k\|\mathcal{A}u_k\|$. Since $\mathcal{A}$ is L_1-Lipschitz continuous and $\{u_k\}$ is bounded, we know that $\{\mathcal{A}u_k\}$ is bounded. Hence, it follows that

$$\lim_{k\to\infty} \|v_k - u_k\| = 0, \tag{23}$$

which together with the assumption $\lim_{k\to\infty} \|x_k - v_k\| = 0$, yields

$$\lim_{k\to\infty} \|x_k - u_k\| = 0. \tag{24}$$

Also, from (22) it is found that

$$\begin{aligned}&\gamma_k(1 - \lambda_k L_2)(\|v_k - y_k\|^2 + \|y_k - z_k\|^2) \\ &\le \alpha_k\|u - p\|^2 + \beta_k\|x_k - p\|^2 + \gamma_k\|v_k - p\|^2 - \|x_{k+1} - p\|^2 + 2\gamma_k\bar{\epsilon}_k\|z_k - p\| + \gamma_k\bar{\epsilon}_k^2 \\ &= \alpha_k(\|u - p\|^2 - \|x_{k+1} - p\|^2) + \beta_k(\|x_k - p\|^2 - \|x_{k+1} - p\|^2) \\ &\quad + \gamma_k(\|v_k - p\|^2 - \|x_{k+1} - p\|^2) + 2\gamma_k\bar{\epsilon}_k\|z_k - p\| + \gamma_k\bar{\epsilon}_k^2 \\ &\le \alpha_k\|u - p\|^2 + \gamma_k(\|v_k - x_k\| + \|x_k - x_{k+1}\|)(\|v_k - p\| + \|x_{k+1} - p\|) \\ &\quad + \beta_k\|x_k - x_{k+1}\|(\|x_k - p\| + \|x_{k+1} - p\|) + 2\gamma_k\bar{\epsilon}_k\|z_k - p\| + \gamma_k\bar{\epsilon}_k^2.\end{aligned}$$

Since $\alpha_k + \beta_k + \gamma_k = 1,\ \alpha_k \to 0,\ \beta_k \to \xi \in (0, \frac{1}{2}],\ \bar{\epsilon}_k \to 0,\ \lambda_k \to 0,\ \|x_k - v_k\| \to 0$ and $\|x_{k+1} - x_k\| \to 0$ (due to (20)), we deduce from the boundedness of $\{x_k\}, \{v_k\}$ and $\{z_k\}$ that

$$\lim_{k\to\infty} \|v_k - y_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|y_k - z_k\| = 0, \tag{25}$$

which together with $\lim_{k\to\infty}\|x_k - v_k\| = 0$, imply that

$$\lim_{k\to\infty}\|x_k - y_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty}\|x_k - z_k\| = 0. \tag{26}$$

Again by Proposition 2.1 (iii), we have

$$\begin{aligned}\|P_{\mathrm{VI}(\Omega,B)}(y_k - \lambda Ay_k) - x_{k+1}\| &\le \|P_{\mathrm{VI}(\Omega,B)}(y_k - \lambda Ay_k) - P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k)\| + \|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k) - x_{k+1}\| \\ &\le (1+\lambda L_1)\|y_k - z_k\| + \alpha_k\|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k) - u\| \\ &\quad + \beta_k\|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k) - x_k\| + \bar{\epsilon}_k \\ &\le (1+\lambda L_1)\|y_k - z_k\| + \alpha_k\|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k) - u\| + \bar{\epsilon}_k \\ &\quad + \beta_k\|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k) - P_{\mathrm{VI}(\Omega,B)}(y_k - \lambda Ay_k)\| \\ &\quad + \beta_k\|P_{\mathrm{VI}(\Omega,B)}(y_k - \lambda Ay_k) - y_k\| + \beta_k\|y_k - x_k\| \\ &\le (1+\lambda L_1)\|y_k - z_k\| + \alpha_k\|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k) - u\| + \bar{\epsilon}_k \\ &\quad + \beta_k(1+\lambda L_1)\|z_k - y_k\| + \beta_k\|P_{\mathrm{VI}(\Omega,B)}(y_k - \lambda Ay_k) - y_k\| + \beta_k\|y_k - x_k\|.\end{aligned} \tag{27}$$

Consequently, from (27), we have

$$\begin{aligned}\|P_{\mathrm{VI}(\Omega,B)}(y_k - \lambda Ay_k) - y_k\| &\le \|P_{\mathrm{VI}(\Omega,B)}(y_k - \lambda Ay_k) - x_{k+1}\| + \|x_{k+1} - x_k\| + \|x_k - y_k\| \\ &\le (1+\lambda L_1)\|y_k - z_k\| + \alpha_k\|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k) - u\| + \bar{\epsilon}_k \\ &\quad + \beta_k(1+\lambda L_1)\|z_k - y_k\| + \beta_k\|P_{\mathrm{VI}(\Omega,B)}(y_k - \lambda Ay_k) - y_k\| + \beta_k\|y_k - x_k\| \\ &\quad + \|x_{k+1} - x_k\| + \|x_k - y_k\| \\ &= (1+\beta_k)(1+\lambda L_1)\|y_k - z_k\| + \alpha_k\|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k) - u\| + \bar{\epsilon}_k \\ &\quad + \beta_k\|P_{\mathrm{VI}(\Omega,B)}(y_k - \lambda Ay_k) - y_k\| + (1+\beta_k)\|y_k - x_k\| + \|x_{k+1} - x_k\|,\end{aligned}$$

which immediately yields

$$\begin{aligned}\|P_{\mathrm{VI}(\Omega,B)}(y_k - \lambda Ay_k) - y_k\| &\le \frac{1+\beta_k}{1-\beta_k}(1+\lambda L_1)\|y_k - z_k\| + \frac{\alpha_k}{1-\beta_k}\|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k) - u\| + \frac{\bar{\epsilon}_k}{1-\beta_k} \\ &\quad + \frac{1+\beta_k}{1-\beta_k}\|y_k - x_k\| + \frac{1}{1-\beta_k}\|x_{k+1} - x_k\|.\end{aligned}$$

Since $\alpha_k + \beta_k + \gamma_k = 1,\ \alpha_k \to 0,\ \beta_k \to \xi \in (0, \frac{1}{2}],\ \bar{\epsilon}_k \to 0,\ \|y_k - z_k\| \to 0,\ \|x_k - y_k\| \to 0$ and $\|x_{k+1} - x_k\| \to 0$ (due to (20), (25) and (26)), we conclude that

$$\lim_{k\to\infty}\|P_{\mathrm{VI}(\Omega,B)}(y_k - \lambda Ay_k) - y_k\| = 0. \tag{28}$$

From Proposition 2.1 (iii), it follows that

$$\begin{aligned}\|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k) - z_k\| &\le \|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k) - P_{\mathrm{VI}(\Omega,B)}(y_k - \lambda Ay_k)\| + \|P_{\mathrm{VI}(\Omega,B)}(y_k - \lambda Ay_k) - y_k\| + \|y_k - z_k\| \\ &\le (1+\lambda L_1)\|z_k - y_k\| + \|P_{\mathrm{VI}(\Omega,B)}(y_k - \lambda Ay_k) - y_k\| + \|y_k - z_k\| \\ &\le \|P_{\mathrm{VI}(\Omega,B)}(y_k - \lambda Ay_k) - y_k\| + (2+\lambda L_1)\|y_k - z_k\|.\end{aligned}$$

Utilizing the last inequality we obtain from (25) and (28) that

$$\lim_{k\to\infty}\|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda Az_k) - z_k\| = 0. \tag{29}$$

□

Theorem 3.8. *Suppose that the conditions (A1)-(A3) and (H1)-(H4) hold. Then the two sequences $\{x_k\}$ and $\{z_k\}$ in Algorithm 3.2 converge strongly to the same point $x^* \in \mathrm{VI}(\mathrm{VI}(\Omega, B), A)$ provided $\|x_k - v_k\| = o(\alpha_k)$, which is a unique solution to the VIP*

$$\langle \mathcal{A}x^*, p - x^*\rangle \ge 0, \quad \forall p \in \mathrm{VI}(\mathrm{VI}(\Omega, B), A).$$

Proof. Note that Lemma 3.4 shows the boundedness of $\{x_k\}$. Since H is reflexive, there is at least a weak convergence subsequence of $\{x_k\}$. First, let us assert that $\omega_w(x_k) \subset \mathrm{VI}(\mathrm{VI}(\Omega, B), A)$. As a matter of fact, take an arbitrary $w \in \omega_w(x_k)$. Then there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i} \rightharpoonup w$. From (26), we know that $y_{k_i} \rightharpoonup w$. It is easy to see that the mapping $P_{\mathrm{VI}(\Omega,B)}(I - \lambda A) : H \to \mathrm{VI}(\Omega, B) \subset H$ is nonexpansive because $P_{\mathrm{VI}(\Omega,B)}$ is nonexpansive and $I - \lambda A$ is nonexpansive for β-inverse-strongly monotone mapping A with $0 < \lambda \leq 2\beta$. So, utilizing Lemma 2.5 and (28), we obtain $w = P_{\mathrm{VI}(\Omega,B)}(w - \lambda A w)$, which leads to $w \in \mathrm{VI}(\mathrm{VI}(\Omega, B), A)$. Thus, the assertion is valid.

Also, note that $\mathcal{A}$ is η-strongly monotone and L_1-Lipschitz continuous on H. Thus, by Lemma 2.4 (iv), we know that there exists a unique solution $x^* \in \mathrm{VI}(\mathrm{VI}(\Omega, B), A)$ to the VIP

$$\langle \mathcal{A}x^*, p - x^* \rangle \geq 0, \quad \forall p \in \mathrm{VI}(\mathrm{VI}(\Omega, B), A). \tag{30}$$

Next, let us show that $x_k \rightharpoonup x^*$. Indeed, take an arbitrary $p \in \mathrm{VI}(\mathrm{VI}(\Omega, B), A)$. Utilizing the monotonicity of $\mathcal{A}$, we obtain from Algorithm 3.2 that for all $k \geq 0$,

$$\begin{aligned}
\|v_k - p\|^2 &= \|u_k - \mu_k\alpha_k\mathcal{A}u_k - p\|^2 \\
&\leq \|(u_k - p) - \mu_k\alpha_k\mathcal{A}u_k\|^2 \\
&= \|u_k - p\|^2 + 2\mu_k\alpha_k\langle \mathcal{A}u_k, p - u_k\rangle + \mu_k^2\alpha_k^2\|\mathcal{A}u_k\|^2 \\
&\leq \|u_k - p\|^2 + 2\mu_k\alpha_k\langle \mathcal{A}p, p - u_k\rangle + \mu_k^2\alpha_k^2\|\mathcal{A}u_k\|^2 \\
&= \|u_k - p\|^2 + 2\mu_k\alpha_k\langle \mathcal{A}p, p - x_k + x_k - u_k\rangle + \mu_k^2\alpha_k^2\|\mathcal{A}u_k\|^2 \\
&\leq \|u_k - p\|^2 + 2\mu_k\alpha_k(\langle \mathcal{A}p, p - x_k\rangle + \|\mathcal{A}p\|\|x_k - u_k\|) + \mu_k^2\alpha_k^2\|\mathcal{A}u_k\|^2.
\end{aligned} \tag{31}$$

Also, utilizing the nonexpansivity of each T_i $(i = 1, ..., N)$ and monotonicity of $\mathcal{B}$, we have that for all $k \geq 0$,

$$\begin{aligned}
\|u_k - p\|^2 &= \|T_{[k+1]}(x_k - \lambda_k\mathcal{B}x_k) - T_{[k+1]}p\|^2 \\
&\leq \|(x_k - p) - \lambda_k\mathcal{B}x_k\|^2 \\
&= \|x_k - p\|^2 + 2\lambda_k\langle \mathcal{B}x_k, p - x_k\rangle + \lambda_k^2\|\mathcal{B}x_k\|^2 \\
&\leq \|x_k - p\|^2 + 2\lambda_k\langle \mathcal{B}p, p - x_k\rangle + \lambda_k^2\|\mathcal{B}x_k\|^2 \\
&\leq \|x_k - p\|^2 + 2\lambda_k\|\mathcal{B}p\|\|p - x_k\| + \lambda_k^2\|\mathcal{B}x_k\|^2.
\end{aligned} \tag{32}$$

Combining (31) and (32), we get

$$\begin{aligned}
\|v_k - p\|^2 &\leq \|u_k - p\|^2 + 2\mu_k\alpha_k(\langle \mathcal{A}p, p - x_k\rangle + \|\mathcal{A}p\|\|x_k - u_k\|) + \mu_k^2\alpha_k^2\|\mathcal{A}u_k\|^2 \\
&\leq \|x_k - p\|^2 + 2\lambda_k\|\mathcal{B}p\|\|p - x_k\| + \lambda_k^2\|\mathcal{B}x_k\|^2 + \mu_k^2\alpha_k^2\|\mathcal{A}u_k\|^2 \\
&\quad + 2\mu_k\alpha_k(\langle \mathcal{A}p, p - x_k\rangle + \|\mathcal{A}p\|\|x_k - u_k\|),
\end{aligned} \tag{33}$$

which immediately leads to

$$\begin{aligned}
0 &\leq \|x_k - p\|^2 - \|v_k - p\|^2 + 2\lambda_k\|\mathcal{B}p\|\|p - x_k\| + \lambda_k^2\|\mathcal{B}x_k\|^2 + 2\mu_k\alpha_k(\langle \mathcal{A}p, p - x_k\rangle + \|\mathcal{A}p\|\|x_k - u_k\|) + \mu_k^2\alpha_k^2\|\mathcal{A}u_k\|^2 \\
&\leq \|x_k - v_k\|(\|x_k - p\| + \|v_k - p\|) + 2\lambda_k\|\mathcal{B}p\|\|p - x_k\| + \lambda_k^2\|\mathcal{B}x_k\|^2 \\
&\quad + 2\mu_k\alpha_k(\langle \mathcal{A}p, p - x_k\rangle + \|\mathcal{A}p\|\|x_k - u_k\|) + \mu_k^2\alpha_k^2\|\mathcal{A}u_k\|^2.
\end{aligned}$$

That is,

$$\begin{aligned}
0 \leq\ & \frac{\|x_k - v_k\|}{2\mu_k\alpha_k}(\|x_k - p\| + \|v_k - p\|) + \frac{\lambda_k}{\mu_k\alpha_k}\|\mathcal{B}p\|\|p - x_k\| + \frac{\lambda_k^2}{2\mu_k\alpha_k}\|\mathcal{B}x_k\|^2 \\
& + \langle \mathcal{A}p, p - x_k\rangle + \|\mathcal{A}p\|\|x_k - u_k\| + \frac{\mu_k\alpha_k}{2}\|\mathcal{A}u_k\|^2.
\end{aligned} \tag{34}$$

Since for any $w \in \omega_w(x_k)$ there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i} \rightharpoonup w$, we deduce from (34), $\alpha_k \to 0$, $\|x_k - u_k\| \to 0$, $\lambda_k = o(\alpha_k)$ and $\|x_k - v_k\| = o(\alpha_k)$ that for all $p \in \mathrm{VI}(\mathrm{VI}(\Omega, B), A)$

$$\begin{aligned}0 &\le \lim_{i\to\infty}\{\frac{\|x_{k_i} - v_{k_i}\|}{2\mu_{k_i}\alpha_{k_i}}(\|x_{k_i} - p\| + \|v_{k_i} - p\|) + \frac{\lambda_{k_i}}{\mu_{k_i}\alpha_{k_i}}\|\mathcal{B}p\|\|p - x_{k_i}\| + \frac{\lambda_{k_i}^2}{2\mu_{k_i}\alpha_{k_i}}\|\mathcal{B}x_{k_i}\|^2 \\ &\quad + \langle \mathcal{A}p, p - x_{k_i}\rangle + \|\mathcal{A}p\|\|x_{k_i} - u_{k_i}\| + \frac{\mu_{k_i}\alpha_{k_i}}{2}\|\mathcal{A}u_{k_i}\|^2\} \\ &= \lim_{i\to\infty}\langle \mathcal{A}p, p - x_{k_i}\rangle \\ &= \langle \mathcal{A}p, p - w\rangle.\end{aligned}$$

Thus, by Lemma 2.4 (i), we know that

$$\langle \mathcal{A}w, p - w\rangle \ge 0, \quad \forall p \in \mathrm{VI}(\mathrm{VI}(\Omega, B), A);$$

that is, w is a solution of VIP (30). By the uniqueness of solutions of VIP (30), we get $w = x^*$, which hence implies that $\omega_w(x_k) = \{x^*\}$. Therefore, it is known that $\{x_k\}$ converges weakly to the unique solution $x^* \in \mathrm{VI}(\mathrm{VI}(\Omega, B), A)$ of VIP (30).

Finally, let us show that $\|x_k - x^*\| \to 0$ as $k \to \infty$. Indeed, in terms of Algorithm 3.2, we conclude from (7) and the β-inverse-strong monotonicity of A with $0 < \lambda \le 2\beta$, that

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &= \|\alpha_k u + \beta_k x_k + \gamma_k h_k - x^*\|^2 \\ &\le \|\beta_k(x_k - x^*) + \gamma_k(h_k - x^*)\|^2 + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\ &\le \beta_k\|x_k - x^*\|^2 + \gamma_k\|h_k - x^*\|^2 + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\ &\le \beta_k\|x_k - x^*\|^2 + \gamma_k(\|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda A z_k) - x^*\| + \bar{\epsilon}_k)^2 + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\ &= \beta_k\|x_k - x^*\|^2 + \gamma_k(\|P_{\mathrm{VI}(\Omega,B)}(z_k - \lambda A z_k) - P_{\mathrm{VI}(\Omega,B)}(x^* - \lambda A x^*)\| + \bar{\epsilon}_k)^2 + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &\le \beta_k\|x_k - x^*\|^2 + \gamma_k(\|(I - \lambda A)z_k - (I - \lambda A)x^*\| + \bar{\epsilon}_k)^2 + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\ &\le \beta_k\|x_k - x^*\|^2 + \gamma_k(\|z_k - x^*\| + \bar{\epsilon}_k)^2 + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\ &= \beta_k\|x_k - x^*\|^2 + \gamma_k\|z_k - x^*\|^2 + \gamma_k\bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\ &\le \beta_k\|x_k - x^*\|^2 + \gamma_k\|v_k - x^*\|^2 + \gamma_k\bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\ &= \beta_k\|x_k - x^*\|^2 + \gamma_k\|x_k - x^* + v_k - x_k\|^2 + \gamma_k\bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\ &= \beta_k\|x_k - x^*\|^2 + \gamma_k(\|x_k - x^*\|^2 + 2\langle x_k - x^*, v_k - x_k\rangle + \|v_k - x_k\|^2) \\ &\quad + \gamma_k\bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\ &\le \beta_k\|x_k - x^*\|^2 + \gamma_k(\|x_k - x^*\|^2 + \|v_k - x_k\|(2\|x_k - x^*\| + \|v_k - x_k\|) \\ &\quad + \gamma_k\bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\ &\le (1 - \alpha_k)\|x_k - x^*\|^2 + \|v_k - x_k\|(2\|x_k - x^*\| + \|v_k - x_k\|) \\ &\quad + \bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\ &= (1 - \alpha_k)\|x_k - x^*\|^2 + \alpha_k[\frac{\|v_k - x_k\|}{\alpha_k}(2\|x_k - x^*\| + \|v_k - x_k\|) + 2\langle u - x^*, x_{k+1} - x^*\rangle] \\ &\quad + \bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k).\end{aligned} \tag{35}$$

Since $\alpha_k \to 0$, $\|v_k - x_k\| = o(\alpha_k)$, $\sum_{k=0}^{\infty} \bar{\epsilon}_k < \infty$ and $x_k \rightharpoonup x^*$, we deduce from the boundedness of $\{x_k\}, \{v_k\}, \{z_k\}$ that $\sum_{k=0}^{\infty} \bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) < \infty$ and

$$\limsup_{k\to\infty}[\frac{\|v_k - x_k\|}{\alpha_k}(2\|x_k - x^*\| + \|v_k - x_k\|) + 2\langle u - x^*, x_{k+1} - x^*\rangle] \le 0.$$

Therefore, applying Lemma 2.10 to (35), we infer from $\sum_{k=0}^{\infty} \alpha_k = \infty$ that $\|x_k - x^*\| \to 0$ as $k \to \infty$. Utilizing (26) we also obtain that $\|z_k - x^*\| \to 0$ as $k \to \infty$. This completes the proof. □

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