



Two $\{4, n - 3\}$ -Isomorphic n -Vertex Digraphs are Hereditarily Isomorphic

Youssef Boudabbous^a

^aDepartment of Mathematics, College of Science, King Saud University, P.O.Box 2455, Riyadh 11451, Saudi Arabia.

Abstract. Let D and D' be two digraphs with the same vertex set V , and let F be a set of positive integers. The digraphs D and D' are *hereditarily isomorphic* whenever the (induced) subdigraphs $D[X]$ and $D'[X]$ are isomorphic for each nonempty vertex subset X . They are *F-isomorphic* if the subdigraphs $D[X]$ and $D'[X]$ are isomorphic for each vertex subset X with $|X| \in F$.

In this paper, we prove that if D and D' are two $\{4, n - 3\}$ -isomorphic n -vertex digraphs, where $n \geq 9$, then D and D' are hereditarily isomorphic. As a corollary, we obtain that given integers k and n with $4 \leq k \leq n - 6$, if D and D' are two $\{n - k\}$ -isomorphic n -vertex digraphs, then D and D' are hereditarily isomorphic.

To the memory of my dear master Gérard LOPEZ who taught me and gave me the passion of reconstruction. With all my gratitude and admiration.

1. Introduction

All digraphs mentioned here are finite, and have no loops and no multiple edges. Thus a digraph (or directed graph) D consists of a nonempty and finite set $V(D)$ of *vertices* with a collection $E(D)$ of ordered pairs of distinct vertices, called the set of *edges* of D . Such a digraph is denoted by $(V(D), E(D))$.

We recall the basic notions of the reconstruction problem in the theory of relations what we apply to the case of digraphs.

Consider two digraphs D and D' on the same vertex set V with $|V| = n \geq 1$, and let k be a positive integer. The digraphs D and D' are *hereditarily isomorphic* if the subdigraphs $D[X]$ and $D'[X]$, induced on X , are isomorphic for each nonempty subset X of V . They are *k-isomorphic* whenever for every k -element vertex subset X , the subdigraphs $D[X]$ and $D'[X]$ are isomorphic. They are $(\leq k)$ -isomorphic if they are k' -isomorphic for every positive integer k' with $k' \leq k$. The digraphs D and D' are $(-k)$ -isomorphic whenever either $k \geq n$ or D and D' are $(n - k)$ -isomorphic with $k < n$.

Let F be a set of non zero integers. The digraphs D and D' are *F-isomorphic* whenever D and D' are p -isomorphic for every $p \in F$. The digraph D is *F-reconstructible* if every digraph F -isomorphic to D is

2010 *Mathematics Subject Classification.* 05C20;05C60

Keywords. Reconstruction, isomorphism, hereditarily isomorphic, difference class, prechain, modular decomposition.

Received: 18 February 2019; Accepted: 03 July 2019

Communicated by Paola Bonacini

Research supported by the Deanship of Scientific Research at King Saud University.

Email address: yboudabbous@ksu.edu.sa; youssef_boudabbous@yahoo.fr (Youssef Boudabbous)

isomorphic to D . Notice that D and D' are $(\leq k)$ -isomorphic if and only if they are $\{1, \dots, k\}$ -isomorphic. Thus the " $\{1, \dots, k\}$ -reconstruction" is also denoted by " $(\leq k)$ -reconstruction". Recall that the *converse* D^* of D is obtained from D by reversing all its edges. The digraph D is *F-self converse* if D is F -isomorphic to its converse. It is *hereditarily self converse* if it is hereditarily isomorphic to its converse.

Notice that the concept of "hereditarily isomorphic tournaments" and that of "hereditarily self converse tournaments" were introduced by K. B. Reid and C. Thomassen in [37].

S. M. Ulam [41] conjectured that the symmetric digraphs with at least three vertices are (-1) -reconstructible. Arbitrary large counter-examples were given by P. K. Stockmeyer [40] in the case of digraphs. Using one of these counter-examples, Y. Boudabbous and J. Dammak [12] showed that there is an arbitrary large non self converse tournament which is (-1) -self converse. But this conjecture is still open in the case of symmetric digraphs. For this problem of reconstruction, we cite [3, 4, 38]. As for R. Fraïssé [22], he conjectured the $(\leq k)$ -reconstruction of relations (of any arity m), k is a sufficiently large integer. In 1972, G. Lopez [27, 28] gave a positive answer to the conjecture of Fraïssé for the binary relations and hence for the digraphs by proving the following well-known result.

Theorem 1.1. [27–29] *Any digraph is (≤ 6) -reconstructible.*

The next corollary is immediately deduced.

Corollary 1.2. *If D and D' are two (≤ 6) -isomorphic digraphs, then D and D' are hereditarily isomorphic.*

The conjecture of R. Fraïssé is then true for the binary relations. However, in 1979, M. Pouzet [34] showed that there is no analogue to this result for ternary relations or for relations with arity greater than 3. Building on Kelly's lemma he also established a combinatorial lemma (see Section 3) whose immediate consequence is: *Given positive integers k and n with $n \geq 2k$, if two n -vertex digraphs are $(-k)$ -isomorphic, then they are $(\leq k)$ -isomorphic.*

Thus, from the (≤ 6) -reconstruction follows that each digraph with at least 12 vertices is (-6) -reconstructible. This positive result of reconstruction led M. Pouzet (see [3, 4]) to reformulate the problem of Ulam as follows.

Problem 1.3. *What is the smallest positive integer k such that any n -vertex digraph, n large enough, is $(-k)$ -reconstructible?*

This problem, still unresolved, initiated many works. The two following results were obtained respectively by P. Ille [26], and by G. Lopez and C. Rauzy [31].

Theorem 1.4. [26] *Any digraph with at least 11 vertices is (-5) -reconstructible.*

Theorem 1.5. [31] *Any digraph with at least 10 vertices is (-4) -reconstructible.*

By the above consequence of the combinatorial lemma of M. Pouzet, Theorem 1.1 and Corollary 1.2 imply immediately the following.

Corollary 1.6. *Given integers k and n with $6 \leq k \leq n - 6$, if D and D' are two $(-k)$ -isomorphic n -vertex digraphs, then D and D' are hereditarily isomorphic.*

Following Corollary 1.6, we are interesting to the following problem.

Problem 1.7. *What is the smallest positive integer k such that if D and D' are two $(-k)$ -isomorphic n -vertex digraphs, n large enough, then D and D' are hereditarily isomorphic?*

From the counter-examples of P. K. Stockmeyer and Corollary 1.6, the integer k of Problem 1.7 satisfies the condition: $2 \leq k \leq 6$.

Our main result is:

Theorem 1.8. *If D and D' are two $\{4, -3\}$ -isomorphic n -vertex digraphs, where $n \geq 9$, then D and D' are hereditarily isomorphic.*

To prove that the value (-3) in Theorem 1.8 is sharp, we obtain the following result.

Proposition 1.9. *Given an integer n with $n \geq 8$, there are two $\{4, -2, -1\}$ -isomorphic n -vertex digraphs D and D' such that D and D' are not hereditarily isomorphic.*

From Theorem 1.8, we deduce easily the following corollary which is an improvement of Theorem 1.4, Theorem 1.5, and Corollary 1.6.

Corollary 1.10. *Given integers k and n with $4 \leq k \leq n - 6$, if D and D' are two $(-k)$ -isomorphic n -vertex digraphs, then D and D' are hereditarily isomorphic.*

Notice that the result similar to Corollary 1.10 for the special case of tournaments was obtained in [11].

By Proposition 1.9 and Corollary 1.10, we improve, in particular, the condition " $2 \leq k \leq 6$ " concerning the integer k of Problem 1.7 by: " $3 \leq k \leq 4$ ". Other than this, we do not know the exact value of such integer k .

For other problems of reconstruction, we cite: the half-reconstruction of digraphs [16, 25], and the reconstruction up to complementation of graphs [18, 19, 35, 36].

This paper is organized as follows. In Section 2, we fix our conventions about digraphs. In Section 3, we recall the tools of our proofs: Gallai's decomposition of digraphs, the notion of difference classes and their basic properties, the definition and some properties of a prechain, the morphology of the difference classes of two (≤ 4) -isomorphic digraphs, the description of the hereditarily self converse digraphs, the combinatorial lemma of Pouzet, and the balanced lemma of Boussairi. The main result is proved in Section 4. In the last section, we prove Proposition 1.9 and Corollary 1.10, we present two examples to show that the lower bounds of the orders of the digraphs given in Theorem 1.8 and Corollary 1.10 are the best possible, and we give two open problems.

2. Preliminaries

A directed graph (or digraph) D consists of a nonempty and finite set $V(D)$ of vertices with a collection $E(D)$ of ordered pairs of distinct vertices, called the set of edges of D . Such a digraph is denoted by $(V(D), E(D))$. Given two digraphs D and D' , a bijection f from $V(D)$ onto $V(D')$ is an *isomorphism* from D onto D' provided that for any vertices x and y of D , $xy \in E(D) \iff f(x)f(y) \in E(D')$. The digraphs D and D' are *isomorphic*, which is denoted by $D \cong D'$, if there exists an isomorphism from one onto the other, otherwise we denote $D \not\cong D'$. For elementary definitions and notations in digraphs we follow [42]. In particular, we recall the following notations. The subdigraph of a digraph D induced by a nonempty vertex subset X is denoted by $D[X]$. Given a proper subset X of $V(D)$, the subdigraph $D[V(D) \setminus X]$ is also denoted by $D - X$, and by $D - x$ whenever $X = \{x\}$.

2.1. Module, Modular partition, Dilatation

In a digraph D , a vertex subset M is a *module* in D if for any vertices a and b in M and each vertex v outside M , $va \in E(D) \iff vb \in E(D)$, and $av \in E(D) \iff bv \in E(D)$. Thus, a module of D is a set M of vertices indistinguishable by the vertices outside M . This concept was introduced in [2, 39] and independently under the name *interval* in [23], *autonomous set* in [24], and *clan* in [20, 21]. A module distinct from $V(D)$ is a *proper module* of D . The empty set, the singleton sets, and the full set of vertices are *trivial modules*. A digraph is *indecomposable* if all its modules are trivial; indecomposable digraphs with at least three vertices are *prime digraphs*.

The following properties of the modules of a digraph are well known.

Proposition 2.1. [20] Let D be a digraph.

1. Given a nonempty vertex subset W , if X is a module of D , then $W \cap X$ is a module of $D[W]$.
2. If X, Y are modules of D , then $X \cap Y$ is a module of D .
3. If X, Y are modules of D such that $X \cap Y \neq \emptyset$, then $X \cup Y$ is a module of D .
4. If X, Y are modules of D such that $X \setminus Y \neq \emptyset$, then $Y \setminus X$ is a module of D .

A partition \mathcal{P} of the vertex set $V(D)$ of a digraph D is a *modular partition* of D if all its elements are modules of D . It follows that the elements of \mathcal{P} may be considered as the vertices of a new digraph, the *quotient* of D by \mathcal{P} , denoted by D/\mathcal{P} , and defined on \mathcal{P} as follows: for any distinct elements X and Y of \mathcal{P} , $XY \in E(D/\mathcal{P})$ if $xy \in E(D)$ for any x and y with $x \in X$ and $y \in Y$.

Given two digraphs H and D with a unique common vertex v , as in [1] for the special case of tournaments, we say that we *dilate* D on the vertex v by H if we transform the digraph D to the digraph D' defined on $V(D) \cup V(H)$ such that $D'[V(D)] = D$, $D'[V(H)] = H$, and $V(H)$ is a module of D' . This new digraph D' is denoted by $D(v, H)$.

2.2. Types of pairs

Consider a digraph D . Let x and y be two distinct vertices. We say that $\{x, y\}$ is an *oriented pair* if $|\{xy, yx\} \cap E(D)| = 1$, otherwise we say that $\{x, y\}$ is a *neutral pair*. The type of a neutral pair $\{x, y\}$ is *full* if x and y are adjacent, or *void* otherwise. Thus, every pair of distinct vertices is a pair of one of the three types: oriented or full or void. Two neutral pairs having exactly one common vertex are said *adjacent neutral pairs*. Thus the digraph D is complete, respectively empty, if all its pairs are full, respectively void. Moreover, the digraph D is a tournament if all its pairs are oriented. Notice that, all the pairs of adjacent vertices of the digraph $(V(D), E(D) \setminus E(D^*))$ are oriented, and hence $(V(D), E(D) \setminus E(D^*))$ is an oriented graph.

2.3. Special digraphs

- If T is the acyclic tournament on $\{a_1, \dots, a_n\}$ such that $a_i a_j \in E(T)$ if and only if $i < j$, then T is denoted by $(a_1 < \dots < a_n)$, and the vertices a_1 and a_n are the *extremities* of T . As in [14], a *pot* is an acyclic tournament or a digraph obtained from an acyclic tournament with at least two vertices by modifying the adjacency type of its two extremities.
- A *peak* is any digraph isomorphic to a digraph on 3 vertices a, b and c for which $\{a, b\}$ is a module and it is the unique neutral pair. Notice that up to isomorphism, there are four peaks.
- A *3-cycle* is a tournament isomorphic to $(\{0, 1, 2\}, \{01, 12, 20\})$.
- A *diamond* is any 4-vertex tournament having a unique 3-cycle. Notice that up to isomorphism, there are two diamonds.
- Let D be a digraph. The *arc-connected components* of D are the connected components of the underlying graph of the oriented graph $(V(D), E(D) \setminus E(D^*))$. The digraph D is *arc-connected* when it admits one arc-connected component.
- Given a property that a digraph may enjoy (such as "being a path", "being a cycle", "being arc-connected", "being self converse", and so on), we say that a vertex subset X of a digraph D has that property if the subdigraph $D[X]$ enjoys it.

3. Tools of our proofs

In this section we recall the tools of our proofs.

3.1. Gallai's decomposition

Let D be a digraph on a set V . A vertex subset X is a *strong module* of D provided that X is a module of D , and for every module Y of D , if $X \cap Y \neq \emptyset$, then either $X \subseteq Y$ or $Y \subseteq X$. If $|V| \geq 2$, then $\mathcal{P}(D)$ denotes the family of maximal, strong modules of D , under the inclusion, amongst the strong modules of D distinct from V .

The following theorem gives Gallai's decomposition result.

Theorem 3.1. [24, 32] *Let D be a digraph with at least two vertices. The class $\mathcal{P}(D)$ is a modular partition of D , and the quotient $D/\mathcal{P}(D)$ is prime, or an acyclic tournament, or a complete digraph, or an empty digraph.*

Definition 3.2. *Given a digraph D with at least two vertices, the elements of $\mathcal{P}(D)$ are the modular components of D , $\mathcal{P}(D)$ is its canonical partition, and the quotient $D/\mathcal{P}(D)$ is its frame.*

The next two lemmas are basic and their proofs are easy.

Lemma 3.3. *Given a digraph D with at least 2 vertices, the frame of D is prime if and only if D has a modular partition for which the corresponding quotient is prime. Moreover, if the frame of D is prime, then the following assertions hold.*

1. $\mathcal{P}(D)$ is the unique modular partition of D for which the corresponding quotient is prime.
2. $\mathcal{P}(D)$ is the class of the maximal proper modules of D .
3. Given a vertex subset B , if B does not include any element of $\mathcal{P}(D)$, then $\mathcal{P}(D - B) = \{X \setminus B : X \in \mathcal{P}(D)\}$, and $D - B$ has a prime frame which is isomorphic to the frame of D .
4. D has no modular partition in exactly two modules.

Lemma 3.4. *Let D be a digraph with at least 2 vertices.*

1. (a) *The frame of D is an acyclic tournament if and only if D has a modular partition \mathcal{Q} such that $|\mathcal{Q}| = 2$ and D/\mathcal{Q} is an acyclic tournament.*
 (b) *The frame of D is an empty digraph, respectively a complete digraph, if and only if D has a modular partition \mathcal{Q} such that $|\mathcal{Q}| = 2$ and D/\mathcal{Q} is an empty digraph, respectively a complete digraph.*
2. *If the frame of D is an acyclic tournament, then the following assertions hold.*
 - (a) *For each element X of $\mathcal{P}(D)$ with $|X| \geq 2$, the frame of the subdigraph $D[X]$ is not an acyclic tournament.*
 - (b) *A vertex subset M is a module of D if and only if either M is a union of some elements of $\mathcal{P}(D)$ which are consecutive vertices of the acyclic tournament $D/\mathcal{P}(D)$, or there is an element X of $\mathcal{P}(D)$ such that M is a module of $D[X]$.*
3. *If the frame of D is an empty digraph, respectively a complete digraph, then the following assertions hold.*
 - (a) *For each element X of $\mathcal{P}(D)$ with $|X| \geq 2$, the frame of the subdigraph $D[X]$ is not an empty digraph, respectively a complete digraph.*
 - (b) *A vertex subset M is a module of D if and only if either M is a union of some elements of $\mathcal{P}(D)$, or there is an element X of $\mathcal{P}(D)$ such that M is a module of $D[X]$.*

In our proofs, we use the following notations.

Notation 3.5. *Let D be a digraph defined on a vertex set V with $|V| \geq 2$.*

1. *Given a positive integer k , we denote by $\mathcal{P}_k(D)$ the set $\{X \in \mathcal{P}(D) : |X| = k\}$.*
2. *If the frame of D is an acyclic tournament, then we let $\widetilde{\mathcal{P}}(D)$ be defined as follows.*
A subset X of V belongs to $\widetilde{\mathcal{P}}(D)$ if and only if either X is an element of $\mathcal{P}(D)$ with $|X| \geq 2$, or X is a maximal union of consecutive vertices of the acyclic tournament $D/\mathcal{P}(D)$ which are singletons.
Thus, by the second assertion of Lemma 3.4, $\widetilde{\mathcal{P}}(D)$ is a modular partition of D . Moreover, if $M \in \widetilde{\mathcal{P}}(D)$, then either $M \in \mathcal{P}(D)$ with $|M| \geq 2$ or M is a union of some singleton elements of $\mathcal{P}(D)$ and M is a maximal module of D such that $D[M]$ is an acyclic tournament.

Notation 3.6. Let D and D' be two digraphs defined on the same vertex set V , and let X be a subset of V such that $|V \setminus X| \geq 2$. If the subdigraphs $D - X$ and $D' - X$ are isomorphic, then we denote by f_X an isomorphism from $D - X$ onto $D' - X$. Thus $f_X(\mathcal{P}_k(D - X)) = \mathcal{P}_k(D' - X)$ for every positive integer k .

The following lemma is a simple consequence of Lemma 3.4.

Lemma 3.7. Let D be a digraph with at least 2 vertices, and \mathcal{Q} be a modular partition of D with $|\mathcal{Q}| \geq 2$.

1. If the frame of D is an acyclic tournament, then $\mathcal{Q} = \widetilde{\mathcal{P}}(D)$ if and only if D and \mathcal{Q} satisfy the following conditions.
 - (a) The quotient D/\mathcal{Q} is an acyclic tournament.
 - (b) Given an element X of \mathcal{Q} with $|X| \geq 2$, if the frame of the subdigraph $D[X]$ is an acyclic tournament, then $D[X]$ is an acyclic tournament.
 - (c) If X and Y are two consecutive vertices of the acyclic tournament D/\mathcal{Q} , then the subdigraphs $D[X]$ and $D[Y]$ are not both acyclic tournaments.
2. If the frame of D is an empty digraph, respectively a complete digraph, then $\mathcal{Q} = \mathcal{P}(D)$ if and only if D and \mathcal{Q} satisfy the following conditions.
 - (a) The quotient D/\mathcal{Q} is an empty digraph, respectively a complete digraph.
 - (b) For each element X of \mathcal{Q} with $|X| \geq 2$, the frame of the subdigraph $D[X]$ is not an empty digraph, respectively a complete digraph.

3.2. Difference graph and basic properties of difference classes

The following concept of *difference classes* was introduced by G. Lopez in 1972 [27, 28]. This concept plays an important role in many reconstruction problems.

Definition 3.8. [27, 28] Let D and D' be two 2-isomorphic digraphs on the same vertex set V . Thus D and D' have the same full pairs, the same void pairs, and the same oriented pairs. It follows that $(V(D), E(D) \setminus E(D'))$ is an oriented graph and its converse is $(V(D), E(D') \setminus E(D))$, and hence the digraph $(V(D), E(D) \Delta E(D'))$, denoted by $\mathcal{D}_{D,D'}$, is a symmetric digraph, called the *difference graph* of D and D' . The connected components of $\mathcal{D}_{D,D'}$ are called the *difference classes* of D and D' . The set of difference classes of D and D' is denoted by $cl(\mathcal{D}_{D,D'})$.

Lemma 3.9. [29] Let D and D' be two (≤ 3) -isomorphic digraphs, and C be a difference class of D and D' . The following assertions hold.

1. The class C is a module of D and D' .
2. The subdigraphs $D[C]$ and $D'[C]$ are arc-connected.
3. The set $cl(\mathcal{D}_{D,D'})$ is a modular partition of D and D' with $D/cl(\mathcal{D}_{D,D'}) = D'/cl(\mathcal{D}_{D,D'})$.

Using Lemma 3.9, we obtain the following corollary.

Corollary 3.10. Let D and D' be two (≤ 3) -isomorphic digraphs. If \mathcal{Q} is a modular partition of D such that each element of \mathcal{Q} is a union of some elements of $cl(\mathcal{D}_{D,D'})$, then \mathcal{Q} is a modular partition of D' with $D'/\mathcal{Q} = D/\mathcal{Q}$.

Proof. Let X be an element of \mathcal{Q} . The vertex subset X is a module of D' because X is a module of D , and $D'[[t, x]] = D[[t, x]]$ for any vertices x and t with $x \in X$ and $t \notin X$. Thus \mathcal{Q} is a modular partition of D' . Moreover, $D'/\mathcal{Q} = D/\mathcal{Q}$ because $D'/cl(\mathcal{D}_{D,D'}) = D/cl(\mathcal{D}_{D,D'})$ by Lemma 3.9. \square

3.3. Prechains

In this subsection, we give the definition and some properties of prechains. The prechains were introduced by Y. Boudabbous and C. Delhommé in [13], where they studied the $(\leq k)$ -self converse (finite or infinite) digraphs where $k \geq 4$. Notice that the prechains were motivated by the morphology of the difference classes of two (≤ 4) -isomorphic digraphs which was obtained by G. Lopez and C. Rauzy in 1992 [30], and that we will recall in the next subsection.

Definition 3.11. [13] *A digraph is a prechain if it has neither adjacent neutral pairs, nor any induced subdigraph which is a peak or a diamond.*

Remark 3.12. *Any digraph with at most two vertices is a prechain. Moreover, if D is a prechain, with at least 3 vertices, which is not a tournament, then given a neutral pair $\{u, v\}$ of D and a vertex x of D with $x \notin \{u, v\}$, the pairs $\{u, x\}$ and $\{v, x\}$ are two oriented pairs of D because D has no adjacent neutral pairs. It follows that any prechain with at least 3 vertices is arc-connected.*

The following proposition gives some properties of prechains.

Proposition 3.13. [13] *Given a prechain D with at least 3 vertices, the following assertions hold.*

1. *The digraph D is arc-connected and (≤ 4) -self converse.*
2. *The subdigraph $D[X]$ is a prechain for each nonempty vertex subset X .*
3. *The frame of D is a prime prechain.*
4. *The subdigraph $D[M]$ is an acyclic tournament for each proper module M of D .*

3.4. Morphology of the difference classes of two (≤ 4) -isomorphic digraphs

The following result of G. Lopez and C. Rauzy [30] gives the morphology of difference classes of two (≤ 4) -isomorphic digraphs.

Theorem 3.14. [30] *Let D and D' be two (≤ 4) -isomorphic digraphs, and C be a difference class of D and D' . The following assertions hold.*

1. (a) *If $D[C]$ has no adjacent neutral pairs, then $D[C]$ is a prechain.*
 (b) *If $D[C]$ has adjacent neutral pairs, then $D[C]$ or its complement is either a path or a cycle.*
2. *$D[C]$ is (≤ 4) -self converse.*
3. *$D'[C]$ and $D^*[C]$ are hereditarily isomorphic.*

3.5. Hereditarily self converse digraphs

In [37], K. B. Reid and C. Thomassen described the hereditarily self converse tournaments. Then in [9], using the morphology of difference classes of two (≤ 6) -isomorphic digraphs obtained by G. Lopez in 1972 ([27, 28]), we obtained a complete description of the hereditarily self converse digraphs. The following proposition is a consequence of the above description.

Proposition 3.15. [9, 13]

1. *A digraph is hereditarily self converse if and only if all its arc-connected components are hereditarily self converse modules.*
2. *Given an arc-connected digraph D with at least 8 vertices, D is hereditarily self converse if and only if D is a pot, or D or its complement is either a path or a cycle.*

The complete list of small arc-connected and hereditarily self converse digraphs is given in [9, 13].

Recall the below result, due to H. Bouchaala, Y. Boudabbous and G. Lopez [8] (see [6] for a detailed proof), which extends the description of the $(-k)$ -self converse prechain tournaments, due to H. Bouchaala and Y. Boudabbous [7].

Proposition 3.16. [8] *Given a prechain D with at least 9 vertices, if D is (-3) -self converse, then D is hereditarily self converse.*

The following corollary is an easy consequence of Proposition 3.16 and Theorem 3.14.

Corollary 3.17. *Let D and D' be two $\{4, -3\}$ -isomorphic digraphs on the same vertex set V with $|V| \geq 9$. If the difference graph $\mathcal{D}_{D,D'}$ is connected, then D and D' are hereditarily isomorphic.*

3.6. Combinatorial lemma of Pouzet

The below result, called "combinatorial lemma of Pouzet", makes a link between the problems of $(\leq k)$ -reconstruction and those of $(-k)$ -reconstruction.

Lemma 3.18. [33] *Let p and r be positive integers, X be a finite set with $|X| \geq p + r$, and \mathcal{U} and \mathcal{V} be two sets of p -element subsets of X . If $|\{Y \in \mathcal{U} : Y \subseteq Q\}| = |\{Y \in \mathcal{V} : Y \subseteq Q\}|$ for every $(p + r)$ -element subset Q of X , then $|\{Y \in \mathcal{U} : P' \subseteq Y \subseteq Q'\}| = |\{Y \in \mathcal{V} : P' \subseteq Y \subseteq Q'\}|$ for any subsets P' and Q' of X such that $P' \subseteq Q'$ and $|Q' \setminus P'| \geq p + r$. In particular, if $|X| \geq 2p + r$, then $\mathcal{U} = \mathcal{V}$.*

The following corollary is an easy consequence of Lemma 3.18.

Corollary 3.19. [33] *Let D and D' be two n -vertex digraphs with the same vertex set V . For each integer p with $0 < p < n$, if D and D' are p -isomorphic, then D and D' are q -isomorphic for each integer q with $1 \leq q \leq \min(p, n - p)$.*

The following notation is needed to state an important second consequence of Lemma 3.18.

Notation 3.20. *Given a digraph D , a subset F of $V(D)$, and a digraph H , we denote by $S(D, H; F)$ the set $\{Y \subseteq V(D) : F \subseteq Y \text{ and } D[Y] \cong H\}$, and by $n(D, H; F)$ the cardinality of the set $S(D, H; F)$.*

Corollary 3.21. [34] *Let n, p and h be integers such that $0 < p < n$ and $1 \leq h \leq n - p$, and let H be a h -vertex digraph. If D and D' are two $(-p)$ -isomorphic n -vertex digraphs, then $n(D', H; X) = n(D, H; X)$ for each vertex subset X with at most p elements.*

This corollary plays an important role in our proofs.

3.7. Balanced lemma of Boussairi and applications

First, we recall the following lemma [15].

Lemma 3.22. [15] *Let p be an integer with $p \geq 2$, i be an element of $\{1, \dots, p\}$, and R be a digraph defined on $\{1, \dots, p\}$. If H and H' are digraphs such that $\{1, \dots, p\} \cap V(H) = \{1, \dots, p\} \cap V(H') = \{i\}$, then the digraphs $R(i, H)$ and $R(i, H')$ are isomorphic if and only if H and H' are isomorphic.*

Notice that Lemma 3.22 was firstly communicated by A. Boussairi, and a detailed proof of this lemma is presented by J. Dammak in [17].

Second, we obtain the following corollary, which is an easy consequence of Lemma 3.22.

Corollary 3.23. *Let p be an integer with $p \geq 2$, i be an element of $\{1, \dots, p\}$, R and R' be two isomorphic digraphs defined on $\{1, \dots, p\}$, and f be an isomorphism from R onto R' . If H and H' are digraphs such that $\{1, \dots, p\} \cap V(H) = \{i\}$ and $\{1, \dots, p\} \cap V(H') = \{f(i)\}$, then the digraphs $R(i, H)$ and $R'(f(i), H')$ are isomorphic if and only if H and H' are isomorphic.*

Proof. Sufficiency of the condition is immediate. For necessity, assume that $R(i, H)$ and $R'(f(i), H')$ are isomorphic. Consider the digraph $R(i, K)$, where K is a digraph isomorphic to H' with $\{1, \dots, p\} \cap V(K) = \{i\}$. It is easy to see that the digraphs $R(i, K)$ and $R'(f(i), H')$ are isomorphic. Since, $R(i, H)$ and $R'(f(i), H')$ are isomorphic, it follows that $R(i, H)$ and $R(i, K)$ are isomorphic. Therefore, Lemma 3.22 implies that H and K are isomorphic. Thus H and H' are isomorphic. \square

Notice that, in the case of tournaments, the above corollary was obtained by M. Bouaziz and Y. Boudabbous in [5].

Finally, we obtain the following consequence of Corollary 3.23.

Corollary 3.24. *Let D and D' be two isomorphic digraphs on the same vertex set V , \mathcal{Q} be a common modular partition of D and D' such that $D'/\mathcal{Q} = D/\mathcal{Q}$, and X_0 be an element of \mathcal{Q} . If $D'[X] \cong D[X]$ for every element X of $\mathcal{Q} \setminus \{X_0\}$, then $D'[X_0] \cong D[X_0]$.*

Proof. Denote by R the subdigraph $D[(V \setminus X_0) \cup \{x_0\}]$ and by R' the subdigraph $D'[(V \setminus X_0) \cup \{x_0\}]$, where $x_0 \in X_0$. Since $D'/\mathcal{Q} = D/\mathcal{Q}$ and $D'[X] \cong D[X]$ for every element X of $\mathcal{Q} \setminus \{X_0\}$, there exists an isomorphism g from R onto R' such that $g(x_0) = x_0$. Furthermore, it is not difficult to see that $D = R(x_0, D[X_0])$ and $D' = R'(x_0, D'[X_0])$. Therefore, since D and D' are isomorphic, Corollary 3.23 implies that the subdigraphs $D[X_0]$ and $D'[X_0]$ are isomorphic. \square

4. Proof of Theorem 1.8

4.1. *Sketch of the proof of Theorem 1.8*

To prove Theorem 1.8, we will proceed by induction on the number n of vertices of the digraphs. First, we prove the following lemma.

Lemma 4.1. *Given two (≤ 4) -isomorphic digraphs D and D' with $|cl(\mathcal{D}_{D,D'})| \geq 2$, the following assertions hold.*

1. *If the frame of D is not an acyclic tournament, then $\mathcal{P}(D') = \mathcal{P}(D)$ and $D' / \mathcal{P}(D) = D / \mathcal{P}(D)$.*
2. *If the frame of D is an acyclic tournament, then $\tilde{\mathcal{P}}(D') = \tilde{\mathcal{P}}(D)$ and $D' / \tilde{\mathcal{P}}(D) = D / \tilde{\mathcal{P}}(D)$.*

Second, we prove the initialization of the induction by the following lemma.

Lemma 4.2. *If D and D' are two $\{4, -3\}$ -isomorphic 9-vertex digraphs, then D and D' are hereditarily isomorphic.*

Third, we consider two $\{4, -3\}$ -isomorphic n -vertex digraphs D and D' on the same vertex set V where $n \geq 10$, and we assume that for each integer p with $9 \leq p < n$, if two p -vertex digraphs are $\{4, -3\}$ -isomorphic, then they are hereditarily isomorphic.

Since $|V| \geq 10$, Corollary 3.19 implies that D and D' are (≤ 4) -isomorphic. We have to prove that the digraphs D and D' are hereditarily isomorphic. By Corollary 3.17, we may assume that the difference graph $\mathcal{D}_{D,D'}$ is disconnected, and hence $|cl(\mathcal{D}_{D,D'})| \geq 2$.

According to Theorem 3.1 and Lemma 4.1, to prove that the digraphs D and D' are hereditarily isomorphic, we obtain the following three results.

Lemma 4.3. *If the frame of D is prime, then the subdigraphs $D[X]$ and $D'[X]$ are hereditary isomorphic for each element X of $\mathcal{P}(D)$.*

Lemma 4.4. *If the frame of D is an acyclic tournament, then the subdigraphs $D[X]$ and $D'[X]$ are hereditary isomorphic for each element X of $\tilde{\mathcal{P}}(D)$.*

Lemma 4.5. *If the frame of D is an empty or a complete digraph, then the subdigraphs $D[X]$ and $D'[X]$ are hereditary isomorphic for each element X of $\mathcal{P}(D)$.*

4.2. *Proof of Lemma 4.1*

Consider two (≤ 4) -isomorphic digraphs D and D' with $|cl(\mathcal{D}_{D,D'})| \geq 2$.

1. First, assume that the frame of D is an empty digraph, respectively a complete digraph. Set $\mathcal{Q} = \mathcal{P}(D)$. Given an element C of $cl(\mathcal{D}_{D,D'})$, since the subdigraph $D[C]$ is arc-connected, C is contained in some element of \mathcal{Q} . It follows that each element of the partition \mathcal{Q} is a union of some elements of $cl(\mathcal{D}_{D,D'})$. Thus Corollary 3.10 implies that \mathcal{Q} is a modular partition of D' with $D' / \mathcal{Q} = D / \mathcal{Q}$, and hence D' / \mathcal{Q} is an empty digraph, respectively a complete digraph. Moreover the digraph D' and the partition \mathcal{Q} satisfy the second condition of the second assertion of Lemma 3.7 because the digraph D and the partition \mathcal{Q} satisfy this condition, and the digraphs D and D' are (≤ 2) -isomorphic. Therefore, $\mathcal{Q} = \mathcal{P}(D')$ by the second assertion of Lemma 3.7.

Second, assume that the frame of D is prime. By Lemma 3.9, each element of $cl(\mathcal{D}_{D,D'})$ is a common proper module of D and D' . Moreover, by Lemma 3.3, $\mathcal{P}(D)$ is the class of the maximal proper modules of D . Therefore, each element of $\mathcal{P}(D)$ is a union of some elements of $cl(\mathcal{D}_{D,D'})$. Thus Corollary 3.10 implies that $\mathcal{P}(D)$ is a modular partition of D' with $D' / \mathcal{P}(D) = D / \mathcal{P}(D)$. Hence $\mathcal{P}(D)$ is a modular partition of D' , and $D' / \mathcal{P}(D)$ is prime. Thus Lemma 3.3 implies that $\mathcal{P}(D) = \mathcal{P}(D')$.

Therefore, the first assertion holds.

2. Since the result is obvious when D is an acyclic tournament, we may assume that $|\tilde{\mathcal{P}}(D)| \geq 2$. The second assertion is a consequence of the following three claims.

Claim 4.6. Each element of $\widetilde{\mathcal{P}}(D)$ is a union of some elements of $cl(\mathcal{D}_{D,D'})$.

Proof. Consider an element X of $\widetilde{\mathcal{P}}(D)$.

First assume that $X \in \mathcal{P}(D)$ with $|X| \geq 2$. By Lemma 3.4, the frame of the subdigraph $D[X]$ is not an acyclic tournament, and hence $D[X]$ is not an acyclic tournament. Thus there is a subset Y of X such that $D[Y]$ is a 3-cycle or Y is a neutral pair of D . It follows that for every vertex z outside X , the subdigraph $D[Y \cup \{z\}]$ is a diamond or a peak, and hence it is not (≤ 4) -self-converse. Therefore, by the second assertion of Theorem 3.14, there is no element of $cl(\mathcal{D}_{D,D'})$ including strictly X . On the other hand, by Lemma 3.4, for each element C of $cl(\mathcal{D}_{D,D'})$, either C is a union of some elements of $\mathcal{P}(D)$ or C is included in some element of $\mathcal{P}(D)$, because C is a module of D by Lemma 3.9. Therefore, X is a union of some elements of $cl(\mathcal{D}_{D,D'})$.

Second assume that $X \in \mathcal{P}(D)$ with $|X| = 1$ or $X \in (\widetilde{\mathcal{P}}(D) \setminus \mathcal{P}(D))$ with $|X| \geq 2$. Thus X is a union of some singleton elements of $\mathcal{P}(D)$ and X is a maximal module of D such that $D[X]$ is an acyclic tournament. Moreover, by Lemma 3.9, the elements of $cl(\mathcal{D}_{D,D'})$ are modules of D . It follows that X is a union of some elements of $cl(\mathcal{D}_{D,D'})$. \square

Claim 4.7. For each element X of $\mathcal{P}(D)$ with $|X| \geq 2$, the frame of the subdigraph $D'[X]$ is not an acyclic tournament.

Proof. Consider an element X of $\mathcal{P}(D)$ with $|X| \geq 2$. By Claim 4.6, there is a subset \mathcal{F} of $cl(\mathcal{D}_{D,D'})$ such that X is the union of the elements of \mathcal{F} . Clearly, the fact that $D'[\{x, y\}] = D[\{x, y\}]$, for each $(x, y) \in X \times (V \setminus X)$, implies that $\mathcal{F} = cl(\mathcal{D}_{D[X],D'[X]})$.

First assume that $|\mathcal{F}| = 1$. By Theorem 3.14, the subdigraphs $D'[X]$ and $D^*[X]$ are hereditarily isomorphic. On the other hand, by Lemma 3.4, the frame of the subdigraph $D[X]$ is not an acyclic tournament. It follows that the frame of the subdigraph $D'[X]$ is not an acyclic tournament.

Second assume that $|\mathcal{F}| \geq 2$. Since the frame of $D[X]$ is not an acyclic tournament, $|cl(\mathcal{D}_{D[X],D'[X]})| \geq 2$, and the subdigraphs $D[X]$ and $D'[X]$ are (≤ 4) -isomorphic, the first assertion applied to $D[X]$ and $D'[X]$ implies that $\mathcal{P}(D'[X]) = \mathcal{P}(D[X])$ and $D'[X]/\mathcal{P}(D[X]) = D[X]/\mathcal{P}(D[X])$, and hence the frame of $D'[X]$ is not an acyclic tournament. \square

Claim 4.8. $\widetilde{\mathcal{P}}(D') = \widetilde{\mathcal{P}}(D)$ and $D'/\widetilde{\mathcal{P}}(D) = D/\widetilde{\mathcal{P}}(D)$.

Proof. By Claim 4.6, each element of $\widetilde{\mathcal{P}}(D)$ is a union of some elements of $cl(\mathcal{D}_{D,D'})$. Thus Corollary 3.10 implies that $\widetilde{\mathcal{P}}(D)$ is a modular partition of D' with $D'/\widetilde{\mathcal{P}}(D) = D/\widetilde{\mathcal{P}}(D)$. Set $Q = \widetilde{\mathcal{P}}(D)$. Thus Q is modular partition of the digraph D' such that the quotient D'/Q is an acyclic tournament. Moreover, given an element X of Q with $|X| \geq 2$, if the frame of $D'[X]$ is an acyclic tournament, then Claim 4.7 implies that $X \in \widetilde{\mathcal{P}}(D) \setminus \mathcal{P}(D)$, and hence $D[X]$ is an acyclic tournament; which implies that $D'[X]$ is also an acyclic tournament because it is (≤ 4) -isomorphic to $D[X]$. Thus, the digraph D' and the partition Q satisfy the first and the second conditions of the first assertion of Lemma 3.7. On the other hand, the digraph D' and the partition Q satisfy the third condition of this assertion because the digraph D and the partition Q satisfy this condition, and the digraphs D and D' are (≤ 4) -isomorphic. Therefore, $Q = \widetilde{\mathcal{P}}(D')$ by the first assertion of Lemma 3.7. Thus $\widetilde{\mathcal{P}}(D') = \widetilde{\mathcal{P}}(D)$ and $D'/\widetilde{\mathcal{P}}(D) = D/\widetilde{\mathcal{P}}(D)$. \square

4.3. Proof of Lemma 4.2

Consider two $\{4, -3\}$ -isomorphic digraphs D and D' defined on the same vertex set V with $|V| = 9$. Since $|V| - 3 = 6$, D and D' are 6-isomorphic. Moreover, by Corollary 3.19, D and D' are $\{2, 3\}$ -isomorphic. Thus D and D' are $\{2, 3, 4, 6\}$ -isomorphic. To the contrary, suppose that D and D' are not hereditarily isomorphic. By Corollaries 1.2 and 3.17, the equivalence $\mathcal{D}_{D,D'}$ has at least two classes, and D and D' are not 5-isomorphic. By Lemma 3.9, the set $cl(\mathcal{D}_{D,D'})$ is a modular partition of D and D' with $D/cl(\mathcal{D}_{D,D'}) = D'/cl(\mathcal{D}_{D,D'})$. Since D

and D' are $\{2, 3, 4\}$ -isomorphic, it follows that there is a 5-element subset Y of some element X of $cl(\mathcal{D}_{D,D'})$ such that the subdigraphs $D'[Y]$ and $D[Y]$ are not isomorphic.

Consider an element z of $V \setminus X$, and let Z be the element of $cl(\mathcal{D}_{D,D'})$ containing z . Denote by H the 2-vertex digraph $D[\{y, z\}]$, where $y \in Y$. Since $D/cl(\mathcal{D}_{D,D'}) = D'/cl(\mathcal{D}_{D,D'})$, $D'[\{y, z\}] = D[\{y, z\}] = H$. It follows that $D[Y \cup \{z\}] = H(y, D[Y])$ and $D'[Y \cup \{z\}] = H(y, D'[Y])$. Furthermore, by Lemma 3.22, $H(y, D[Y])$ and $H(y, D'[Y])$ are not isomorphic because $D[Y]$ and $D'[Y]$ are not isomorphic. Therefore, the subdigraphs $D[Y \cup \{z\}]$ and $D'[Y \cup \{z\}]$ are not isomorphic; which contradicts the fact that the digraphs D and D' are 6-isomorphic. \square

4.4. Proof of Lemma 4.3

Lemma 4.3 is immediately deduced from the following eight claims, where we assume that the frame of D is prime. Thus $|\mathcal{P}(D)| \geq 3$, and by Lemma 4.1, $\mathcal{P}(D') = \mathcal{P}(D)$ and $D'/\mathcal{P}(D) = D/\mathcal{P}(D)$.

Claim 4.9. *The subdigraphs $D'[X]$ and $D[X]$ are hereditarily isomorphic for each element X of $\mathcal{P}(D)$ such that $|X| > n - |\mathcal{P}(D)| - 2$.*

Proof. Consider an element X of $\mathcal{P}(D)$ such that $|X| > n - |\mathcal{P}(D)| - 2$. Thus $|Y| \leq 3$ for every $Y \in \mathcal{P}(D) \setminus \{X\}$. Moreover, since $D'[X]$ and $D[X]$ are (≤ 4) -isomorphic, we may assume that $|X| \geq 5$.

First, we will prove that $D'[X]$ and $D[X]$ are (-3) -isomorphic. We may assume that $|X| \geq 8$. Let B_1 be a 3-element subset of X . By Lemma 3.3, $\mathcal{P}(D - B_1) = \mathcal{P}(D' - B_1) = (\mathcal{P}(D) \setminus \{X\}) \cup \{X \setminus B_1\}$. It follows that $\mathcal{P}_{|X|-3}(D - B_1) = \mathcal{P}_{|X|-3}(D' - B_1) = \{X \setminus B_1\}$, and hence $f_{B_1}(X \setminus B_1) = X \setminus B_1$. Thus $D'[X \setminus B_1]$ and $D[X \setminus B_1]$ are isomorphic, and hence $D'[X]$ and $D[X]$ are (-3) -isomorphic.

Second, we will prove that $D'[X]$ and $D[X]$ are hereditarily isomorphic. Since $D'[X]$ and $D[X]$ are (-3) -isomorphic with $|X| < |V|$, from the induction hypothesis, we may assume that $5 \leq |X| \leq 8$. Therefore, by Corollary 1.2, it suffices to prove that $D'[X]$ and $D[X]$ are isomorphic and $\{-1, -2\}$ -isomorphic. To do so, consider a p -element subset B_2 of X with $p \in \{0, 1, 2\}$, and let us prove that $D'[X \setminus B_2]$ and $D[X \setminus B_2]$ are isomorphic. Let B_3 be a $(3 - p)$ -element subset of $V \setminus X$, and let $Q = Q_1 \cup \{X \setminus B_2\}$, where Q_1 is the set of the nonempty elements of the set $\{Y \setminus B_3 : Y \in \mathcal{P}(D) \setminus \{X\}\}$. Since $D'/\mathcal{P}(D) = D/\mathcal{P}(D)$ and $|Y| \leq 3$ for every $Y \in \mathcal{P}(D) \setminus \{X\}$, Q is a common modular partition of the two isomorphic subdigraphs $D - (B_2 \cup B_3)$ and $D' - (B_2 \cup B_3)$ such that $(D - (B_2 \cup B_3))/Q = (D' - (B_2 \cup B_3))/Q$, and $D'[Z] \cong D[Z]$ for every $Z \in Q \setminus \{X \setminus B_2\}$. Thus Corollary 3.24 implies that $D'[X \setminus B_2]$ and $D[X \setminus B_2]$ are isomorphic. \square

Claim 4.10. *The subdigraphs $D'[X]$ and $D[X]$ are isomorphic for each element X of $\mathcal{P}(D)$.*

Proof. First, assume that there exists $X_0 \in \mathcal{P}(D)$ such that $|X_0| > n - |\mathcal{P}(D)| - 2$. By Claim 4.9, $D'[X_0]$ and $D[X_0]$ are hereditarily isomorphic. Furthermore, $|Y| \leq 3$ for every $Y \in \mathcal{P}(D) \setminus \{X_0\}$. Since D and D' are (≤ 4) -isomorphic, it follows that $D'[X]$ and $D[X]$ are isomorphic for each element X of $\mathcal{P}(D)$.

Second, assume that $|X| \leq n - |\mathcal{P}(D)| - 2$ for every $X \in \mathcal{P}(D)$. Consider an element X of $\mathcal{P}(D)$, a 2-element subset B of X , and a subset C of V including X such that $|C \cap Y| = 1$ for every $Y \in \mathcal{P}(D) \setminus \{X\}$. Denote by H the subdigraph $D[C]$ and by Q the set $\{X\} \cup \{y : y \in C \setminus X\}$. Clearly, Q is a modular partition of H such that the quotient H/Q is isomorphic to $D/\mathcal{P}(D)$, and hence it is prime. Thus Lemma 3.3 implies that the frame of H is prime with $\mathcal{P}(H) = Q$. Furthermore, since $C \in S(D, H; B)$, Corollary 3.21 implies that $n(D', H; B) \neq 0$. Consider an element K of $S(D', H; B)$. Since $D'[K]$ is isomorphic to H , the frame of K is prime, and K has a unique modular component J with $|J| > 1$. Moreover, $D'[J]$ is isomorphic to $D[X]$, and hence $|J| = |X|$.

Let $\mathcal{P}_1 = \{Y \in \mathcal{P}(D) : |Y \cap K| \geq 2\}$ and $\mathcal{P}_2 = \{Y \in \mathcal{P}(D) : |Y \cap K| = 1\}$. The set K is the union of the two disjoint sets K_1 and K_2 , where $K_1 = \bigcup_{Y \in \mathcal{P}_1} (K \cap Y)$ and $K_2 = \bigcup_{Y \in \mathcal{P}_2} (K \cap Y)$. Since $B \subseteq X \cap K$, $X \in \mathcal{P}_1$, and hence $\mathcal{P}_2 \subseteq (\mathcal{P}(D) \setminus \{X\})$. Consider an element Y of \mathcal{P}_1 . Since $\mathcal{P}(D') = \mathcal{P}(D)$, Y is a module of D' . Hence Proposition 2.1 implies that $Y \cap K$ is a non trivial module of $D'[K]$. Furthermore, by Lemma 3.3, the elements of $\mathcal{P}(D'[K])$ are the maximal proper modules of $D'[K]$. Thus $Y \cap K \subseteq J$. It follows

that $K_1 \subseteq J$, and hence $(K \setminus J) \subseteq K_2$. Therefore, $|\mathcal{P}(D) \setminus -1| = |K \setminus J| \leq \sum_{Y \in \mathcal{P}_2} |K \cap Y| = |\mathcal{P}_2| \leq |\mathcal{P}(D) \setminus -1|$. Thus $|\mathcal{P}_2| = |\mathcal{P}(D) \setminus -1|$, and hence $\mathcal{P}_2 = \mathcal{P}(D) \setminus \{X\}$ and $\mathcal{P}_1 = \{X\}$. Thus $|X \cap K| = |K| - |K_2| = |K| - |P_2| = |K| - |\mathcal{P}(D) \setminus +1| = |J|$; which implies that $X \cap K = J$ because $X \cap K \subseteq J$. Thus $J = X$ because $|J| = |X|$. Consequently, $D'[X]$ and $D[X]$ are isomorphic. \square

Claim 4.11. Consider distinct elements X_1 and X_2 of $\mathcal{P}(D)$, a proper subset B_1 of X_1 , and a proper subset B_2 of X_2 such that $|B_1 \cup B_2| = 3$. If $D'[X_1 - B_1] \cong D[X_1 - B_1]$, then $D'[X_2 - B_2] \cong D[X_2 - B_2]$.

Proof. Denote by Q the set $(\mathcal{P}(D) \setminus \{X_1, X_2\}) \cup \{X_1 - B_1, X_2 - B_2\}$. By Lemma 3.3, $Q = \mathcal{P}(D - (B_1 \cup B_2)) = \mathcal{P}(D' - (B_1 \cup B_2))$, and $(D - (B_1 \cup B_2))/Q = (D' - (B_1 \cup B_2))/Q$. Thus the fact that $D'[X_1 - B_1] \cong D[X_1 - B_1]$ and Claim 4.10 imply that $D'[Y] \cong D[Y]$ for every $Y \in Q \setminus \{X_2 - B_2\}$. Furthermore, the subdigraphs $D - (B_1 \cup B_2)$ and $D' - (B_1 \cup B_2)$ are isomorphic because the digraphs D and D' are (-3) -isomorphic. Thus Corollary 3.24 implies that $D'[X_2 - B_2] \cong D[X_2 - B_2]$. \square

Claim 4.12. The subdigraphs $D'[X]$ and $D[X]$ are (-3) -isomorphic for each element X of $\mathcal{P}(D)$.

Proof. We may assume that $|X| \geq 4$. Consider an element x of X , and a 3-element subset B of X . Denote by Q the set $(\mathcal{P}(D) \setminus \{X\}) \cup \{X \setminus B\}$. By Lemma 3.3, $Q = \mathcal{P}(D - B) = \mathcal{P}(D' - B)$, and $(D - B)/Q = (D' - B)/Q$. Thus Claim 4.10 implies that $D'[Y] \cong D[Y]$ for every $Y \in Q \setminus \{X \setminus B\}$. Furthermore, the subdigraphs $D - B$ and $D' - B$ are isomorphic because the digraphs D and D' are (-3) -isomorphic. Thus Corollary 3.24 implies that $D'[X \setminus B] \cong D[X \setminus B]$. Consequently, $D'[X]$ and $D[X]$ are (-3) -isomorphic. \square

Claim 4.13. If there exists an element Y of $\mathcal{P}(D)$ such that $|Y| \geq 9$, then $D'[X]$ and $D[X]$ are hereditarily isomorphic for each element X of $\mathcal{P}(D)$.

Proof. Consider an element X of $\mathcal{P}(D)$. Since D' and D are (≤ 4) -isomorphic, Claim 4.10 implies that if $|X| \leq 5$, then $D'[X]$ and $D[X]$ are hereditarily isomorphic. Furthermore, by Claim 4.12, $D'[X]$ and $D[X]$ are (-3) -isomorphic. Therefore, by the induction hypothesis, we may assume that $6 \leq |X| \leq 8$. By Corollary 1.2, it suffices to prove that $D'[X]$ and $D[X]$ are $\{5, 6\}$ -isomorphic. To do so, by Claims 4.10 and 4.12, it suffices to prove that $D'[X]$ and $D[X]$ are $\{-1, -2\}$ -isomorphic. Notice that $X \neq Y$ because $|Y| \geq 9$. Consider a p -element subset B_1 of X with $p \in \{1, 2\}$, a $(3 - p)$ -element subset B_2 of Y , and let $x \in X$. By Claim 4.12, the induction hypothesis implies that $D'[Y \setminus B_2] \cong D[Y \setminus B_2]$. Thus by Claim 4.11, $D'[X \setminus B_1] \cong D[X \setminus B_1]$, and hence $D'[X]$ and $D[X]$ are $\{-1, -2\}$ -isomorphic. \square

Claim 4.14. The subdigraphs $D'[X]$ and $D[X]$ are hereditarily isomorphic for each element X of $\mathcal{P}(D)$ such that $|X| = 8$.

Proof. From Claim 4.13, we may assume that $|Y| \leq 8$ for every $Y \in \mathcal{P}(D)$. Consider an element X of $\mathcal{P}(D)$ such that $|X| = 8$. Since D' and D are (≤ 4) -isomorphic, by Corollary 1.2, it is sufficient to prove that $D'[X]$ and $D[X]$ are $\{-3, -2\}$ -isomorphic. To do so, consider a p -element subset B_1 of X with $p \in \{2, 3\}$, and showing that $D'[X \setminus B_1]$ and $D[X \setminus B_1]$ are isomorphic.

First assume that there exists $X_0 \in \mathcal{P}(D) \setminus \{X\}$ such that $|X_0| \leq 5$, and let B_2 be a $(3 - p)$ -element subset of X_0 . Consider the set Q defined by: $Q = (\mathcal{P}(D) \setminus \{X, X_0\}) \cup \{X \setminus B_1, X_0 \setminus B_2\}$ when $X_0 \neq B_2$, and $Q = (\mathcal{P}(D) \setminus \{X, X_0\}) \cup \{X \setminus B_1\}$ when $X_0 = B_2$. Clearly, Q is a common modular partition of the two isomorphic subdigraphs $D' - (B_1 \cup B_2)$ and $D - (B_1 \cup B_2)$ such that $(D - (B_1 \cup B_2))/Q = (D' - (B_1 \cup B_2))/Q$. Furthermore, since D' and D are (≤ 4) -isomorphic, and $|X_0| \leq 5$, Claim 4.10 implies that $D'[Y]$ and $D[Y]$ are isomorphic for every $Y \in Q \setminus \{X \setminus B_1\}$. Thus Corollary 3.24 implies that $D'[X \setminus B_1] \cong D[X \setminus B_1]$.

Therefore, in the sequel we may assume that $6 \leq |Y| \leq 8$ for every $Y \in \mathcal{P}(D) \setminus \{X\}$.

Second assume that there exists $X_0 \in \mathcal{P}(D) \setminus \{X\}$ such that $|X_0| = 6$. Consider a $(3 - p)$ -element subset B_2 of X_0 , and let $Q = (\mathcal{P}(D) \setminus \{X, X_0\}) \cup \{X \setminus B_1, X_0 \setminus B_2\}$. If $p = 3$, then $B_2 = \emptyset$, and hence $D'[X_0 \setminus B_2] \cong D[X_0 \setminus B_2]$ by Claim 4.10. Otherwise, $p = 2$ and Lemma 3.3 implies that $\mathcal{P}(D - (B_1 \cup B_2)) = \mathcal{P}(D' - (B_1 \cup B_2)) = Q$. Thus

$\mathcal{P}_5(D - (B_1 \cup B_2)) = \mathcal{P}_5(D' - (B_1 \cup B_2)) = \{X_0 \setminus B_2\}$. It follows that $f_{B_1 \cup B_2}(X_0 \setminus B_2) = X_0 \setminus B_2$, and hence the subdigraphs $D'[X_0 \setminus B_2]$ and $D[X_0 \setminus B_2]$ are isomorphic. Therefore, Claim 4.11 implies that $D[X \setminus B_1]$ and $D'[X \setminus B_1]$ are isomorphic.

Thus in the sequel we may assume that $|Y| \in \{7, 8\}$ for every $Y \in \mathcal{P}(D) \setminus \{X\}$.

Third, assume that there exists $X_0 \in \mathcal{P}(D) \setminus \{X\}$ such that $|X_0| = 8$. Consider a $(3 - p)$ -element subset B_2 of X_0 , and let $Q = (\mathcal{P}(D) \setminus \{X, X_0\}) \cup \{X \setminus B_1, X_0 \setminus B_2\}$. By Lemma 3.3, $\mathcal{P}(D - (B_1 \cup B_2)) = \mathcal{P}(D' - (B_1 \cup B_2)) = Q$. Therefore, $\mathcal{P}_{|X|-p}(D - (B_1 \cup B_2)) = \mathcal{P}_{|X|-p}(D' - (B_1 \cup B_2)) = \{X \setminus B_1\}$. Thus $f_{B_1 \cup B_2}(X \setminus B_1) = X \setminus B_1$, and hence $D'[X \setminus B_1]$ and $D[X \setminus B_1]$ are isomorphic.

Finally, assume that $|Y| = 7$ for every $Y \in \mathcal{P}(D) \setminus \{X\}$.

Consider two distinct elements X_1 and X_2 of $\mathcal{P}(D) \setminus \{X\}$, an element x_1 of X_1 , and a 2-element subset A_2 of X_2 .

By Lemma 3.3, $\mathcal{P}(D - (A_2 \cup \{x_1\})) = \mathcal{P}(D' - (A_2 \cup \{x_1\})) = (\mathcal{P}(D) \setminus \{X_1, X_2\}) \cup \{X_1 \setminus \{x_1\}, X_2 \setminus A_2\}$. Therefore, $\mathcal{P}_6(D - (B_1 \cup B_2)) = \mathcal{P}_6(D' - (B_1 \cup B_2)) = \{X_1 \setminus \{x_1\}\}$, and $\mathcal{P}_5(D - (B_1 \cup B_2)) = \mathcal{P}_5(D' - (B_1 \cup B_2)) = \{X_2 \setminus A_2\}$. It follows that $f_{A_2 \cup \{x_1\}}(X_1 \setminus \{x_1\}) = X_1 \setminus \{x_1\}$ and $f_{A_2 \cup \{x_1\}}(X_2 \setminus A_2) = X_2 \setminus A_2$. Thus $D'[X_1 \setminus \{x_1\}] \cong D[X_1 \setminus \{x_1\}]$ and $D'[X_2 \setminus A_2] \cong D[X_2 \setminus A_2]$. It follows that $D'[Y]$ and $D[Y]$ are $\{-2, -1\}$ -isomorphic for every $Y \in \mathcal{P}(D) \setminus \{X\}$. At present assume an element X_0 of $\mathcal{P}(D) \setminus \{X\}$, and a $(3 - p)$ -element subset B_2 of X_0 . Since $D'[X_0]$ and $D[X_0]$ are $\{-2, -1, 1, 2, 3, 4\}$ -isomorphic with $|X_0| = 7$, Corollary 1.2 implies that $D'[X_0]$ and $D[X_0]$ are hereditarily isomorphic. Thus $D'[X_0 \setminus B_2] \cong D[X_0 \setminus B_2]$. Therefore, Claim 4.11 implies that $D'[X \setminus B_1]$ and $D[X \setminus B_1]$ are isomorphic. \square

Claim 4.15. *The subdigraphs $D'[X]$ and $D[X]$ are hereditarily isomorphic for each element X of $\mathcal{P}(D)$ such that $|X| = 7$.*

Proof. From Claim 4.13, we may assume that $|Y| \leq 8$ for every $Y \in \mathcal{P}(D)$. Consider an element X of $\mathcal{P}(D)$ such that $|X| = 7$, and let $x \in X$. Since $D'[X]$ and $D[X]$ are (≤ 4) -isomorphic, Corollary 1.2 implies that it is sufficient to prove that $D'[X]$ and $D[X]$ are $\{-2, -1\}$ -isomorphic. Consider a p -element subset B_1 of X with $p \in \{1, 2\}$, and showing that $D'[X \setminus B_1]$ and $D[X \setminus B_1]$ are isomorphic.

First, notice that if there exists $X_0 \in \mathcal{P}(D) \setminus \{X\}$ such that $|X_0| \geq 3$ and $D'[X_0]$ and $D[X_0]$ are hereditarily isomorphic, then by considering a $(3 - p)$ -element subset B_2 of X_0 , Claim 4.11 implies that $D'[X \setminus B_1]$ and $D[X \setminus B_1]$ are isomorphic. Moreover D' and D are (≤ 4) -isomorphic. From Claims 4.10 and 4.14, it follows that if there is $Y \in (\mathcal{P}(D) \setminus \{X\})$ such that $|Y| \in \{3, 4, 5, 8\}$, then $D'[X \setminus B_1] \cong D[X \setminus B_1]$.

Consequently, in the sequel we may assume that $|Y| \in \{1, 2, 6, 7\}$ for every $Y \in \mathcal{P}(D) \setminus \{X\}$.

Second assume that $|Y| \in \{1, 2\}$ for every $Y \in \mathcal{P}(D) \setminus \{X\}$. Consider a $(3 - p)$ -element subset B_2 of $V \setminus X$. Let $Q = Q_1 \cup \{X \setminus B_1\}$, where Q_1 is the set of the nonempty elements of the set $\{Y \setminus B_2 : Y \in \mathcal{P}(D) \setminus \{X\}\}$. Since $D'/\mathcal{P}(D) = D/\mathcal{P}(D)$, Q is a common modular partition of the two isomorphic subdigraphs $D' - (B_1 \cup B_2)$ and $D - (B_1 \cup B_2)$ such that $X \setminus B_1 \in Q$, $(D' - (B_1 \cup B_2))/Q = (D - (B_1 \cup B_2))/Q$ and $D'[Y] \cong D[Y]$ for every $Y \in Q \setminus \{X \setminus B_1\}$. Thus Corollary 3.24 implies that $D'[X \setminus B_1]$ and $D[X \setminus B_1]$ are isomorphic.

Consequently, in the sequel we may assume that there is $X_0 \in \mathcal{P}(D) \setminus \{X\}$ such that $|X_0| \in \{6, 7\}$, and that $|Y| \in \{1, 2, 6, 7\}$ for every $Y \in \mathcal{P}(D) \setminus \{X\}$.

Third, consider two distinct elements X_1 and X_2 of $\mathcal{P}(D)$ such that $|X_1| = 7$ and $|X_2| \in \{6, 7\}$, an element x_1 of X_1 and a 2-element subset A_2 of X_2 . By Lemma 3.3, $\mathcal{P}(D - (A_2 \cup \{x_1\})) = \mathcal{P}(D' - (A_2 \cup \{x_1\})) = (\mathcal{P}(D) \setminus \{X_1, X_2\}) \cup \{X_1 \setminus \{x_1\}, X_2 \setminus A_2\}$. Thus $\mathcal{P}_{|X_2|-2}(D - (A_2 \cup \{x_1\})) = \mathcal{P}_{|X_2|-2}(D' - (A_2 \cup \{x_1\})) = \{X_2 \setminus A_2\}$. It follows that $f_{A_2 \cup \{x_1\}}(X_2 \setminus A_2) = X_2 \setminus A_2$, and hence $D'[X_2 \setminus A_2] \cong D[X_2 \setminus A_2]$. Furthermore, $\mathcal{P}_6(D') = \mathcal{P}_6(D)$, $\mathcal{P}_6(D - (A_2 \cup \{x_1\})) = \mathcal{P}_6(D' - (A_2 \cup \{x_1\})) = \mathcal{P}_6(D) \cup \{X_1 \setminus \{x_1\}\}$, and by Claim 4.10, $D'[Y] \cong D[Y]$ for every $Y \in \mathcal{P}_6(D)$. Since $f_{A_2 \cup \{x_1\}}(\mathcal{P}_6(D - (A_2 \cup \{x_1\}))) = \mathcal{P}_6(D' - (A_2 \cup \{x_1\}))$, it follows that $D[X_1 \setminus \{x_1\}] \cong D'[X_1 \setminus \{x_1\}]$. By the foregoing, it follows that $D[Y]$ and $D'[Y]$ are (-1) -isomorphic for every $Y \in \mathcal{P}_7(D)$. Moreover, it follows that if $|\mathcal{P}_7(D)| \geq 2$, then $D[Y]$ and $D'[Y]$ are $\{-1, -2\}$ -isomorphic for every $Y \in \mathcal{P}_7(D)$, and hence $D'[X \setminus B_1] \cong D[X \setminus B_1]$. Therefore, in the sequel we assume that $\mathcal{P}_6(D) \neq \emptyset$, $\mathcal{P}_7(D) = \{X\}$, and $|Y| \in \{1, 2, 6\}$ for every $Y \in \mathcal{P}(D) \setminus \{X\}$. Since $D[X]$ and $D'[X]$ are (-1) -isomorphic, we may assume that $p = 2$. At the beginning, assume that $\mathcal{P}_1(D) \cup \mathcal{P}_2(D) \neq \emptyset$, and let $y \in Y$ where $Y \in \mathcal{P}_1(D) \cup \mathcal{P}_2(D)$. Let $Q = Q_1 \cup \{X \setminus B_1\}$, where Q_1 is the set of the nonempty elements of the set $\{Z \setminus \{y\} : Z \in \mathcal{P}(D) \setminus \{X\}\}$. Since $D'/\mathcal{P}(D) = D/\mathcal{P}(D)$, Q is a common modular partition of the two isomorphic subdigraphs $D' - (B_1 \cup \{y\})$ and $D - (B_1 \cup \{y\})$ such that

$X \setminus B_1 \in \mathcal{Q}$, $(D' - (B_1 \cup \{y\})) / \mathcal{Q} = (D - (B_1 \cup \{y\})) / \mathcal{Q}$, and $D'[Y] \cong D[Y]$ for every $Y \in \mathcal{Q} \setminus \{X \setminus B_1\}$. Thus Corollary 3.24 implies that $D'[X \setminus B_1]$ and $D[X \setminus B_1]$ are isomorphic. At the end, assume that $\mathcal{P}(D) \setminus \{X\} = \mathcal{P}_6(D)$, and consider two distinct elements X_1 and X_2 of $\mathcal{P}_6(D)$, an element x_1 of X_1 , and a 2-element subset A_2 of X_2 . Clearly, $\mathcal{P}_5(D - (A_2 \cup \{x_1\})) = \mathcal{P}_5(D' - (A_2 \cup \{x_1\})) = \{X_1 \setminus \{x_1\}\}$. Thus $f_{A_2 \cup \{x_1\}}(X_1 \setminus \{x_1\}) = X_1 \setminus \{x_1\}$, and hence $D[X_1 \setminus \{x_1\}] \cong D'[X_1 \setminus \{x_1\}]$. Moreover $|B_1 \cup \{x_1\}| = 3$. Thus Claim 4.11 implies that $D'[X \setminus B_1]$ and $D[X \setminus B_1]$ are isomorphic. \square

Claim 4.16. *The subdigraphs $D'[X]$ and $D[X]$ are hereditarily isomorphic for each element X of $\mathcal{P}(D)$ such that $|X| = 6$.*

Proof. From Claim 4.13, we may assume that $|Y| \leq 8$ for every $Y \in \mathcal{P}(D)$. Consider an element X of $\mathcal{P}(D)$ such that $|X| = 6$. By Claim 4.10, $D'[X] \cong D[X]$. Moreover, D and D' are (≤ 4) -isomorphic. Thus, by Corollary 1.2, it is sufficient to prove that $D'[X]$ and $D[X]$ are (-1) -isomorphic. Consider an element x_1 of X and showing that $D'[X \setminus \{x_1\}]$ and $D[X \setminus \{x_1\}]$ are isomorphic.

First assume that there is $Y \in \mathcal{P}(D) \setminus \{X\}$ such that $|Y| \geq 3$, and consider a 2-element subset B of Y . If $|Y| = 6$, then the subdigraphs $D'[Y \setminus B]$ and $D[Y \setminus B]$ are isomorphic because D and D' are (≤ 4) -isomorphic. Otherwise, $D'[Y \setminus B]$ and $D[Y \setminus B]$ are isomorphic because $D'[Y]$ and $D[Y]$ are hereditarily isomorphic by the preceding lemmas. Therefore, $D'[Y \setminus B] \cong D[Y \setminus B]$ and $|B \cup \{x_1\}| = 3$. Thus Claim 4.11 implies that $D[X \setminus \{x_1\}]$ and $D'[X \setminus \{x_1\}]$ are isomorphic.

Second assume that $|Y| \in \{1, 2\}$ for every $Y \in \mathcal{P}(D) \setminus \{X\}$. Consider a 2-element subset B of $V \setminus X$. Let $\mathcal{Q} = \mathcal{Q}_1 \cup \{X \setminus \{x_1\}\}$, where \mathcal{Q}_1 is the set of the nonempty elements of the set $\{Y \setminus B : Y \in \mathcal{P}(D) \setminus \{X\}\}$. Since $D'/\mathcal{P}(D) = D/\mathcal{P}(D)$, \mathcal{Q} is a common modular partition of the two isomorphic subdigraphs $D' - (B \cup \{x_1\})$ and $D - (B \cup \{x_1\})$ such that $X \setminus \{x_1\} \in \mathcal{Q}$, $(D' - (B \cup \{x_1\})) / \mathcal{Q} = (D - (B \cup \{x_1\})) / \mathcal{Q}$ and $D'[Y] \cong D[Y]$ for every $Y \in \mathcal{Q} \setminus \{X \setminus \{x_1\}\}$. Thus Corollary 3.24 implies that $D'[X \setminus \{x_1\}]$ and $D[X \setminus \{x_1\}]$ are isomorphic. \square

Now we continue the proof of Lemma 4.3.

From Claim 4.13, we may assume that $|Y| \leq 8$ for every $Y \in \mathcal{P}(D)$. Consider an element X of $\mathcal{P}(D)$. If $|X| \in \{6, 7, 8\}$, then $D'[X]$ and $D[X]$ are hereditarily isomorphic by Claims 4.14, 4.15, and 4.16. Otherwise, $|X| \leq 5$. Moreover, $D'[X]$ and $D[X]$ are isomorphic by Claim 4.10. Since, D and D' are (≤ 4) -isomorphic, it follows that $D'[X]$ and $D[X]$ are hereditarily isomorphic. \square

4.5. Proof of Lemma 4.4

We start with the following claim.

Claim 4.17. *Let R and R' be two digraphs on the same vertex set W , and \mathcal{Q} be a common modular partition of R and R' such that the quotients R/\mathcal{Q} and R'/\mathcal{Q} are two equal acyclic tournaments. If f is an isomorphism from R onto R' , then $f(X) = X$ for every element X of \mathcal{Q} .*

Proof. Let $\mathcal{Q} = \{Y_1, \dots, Y_p\}$ where the quotient R/\mathcal{Q} is the acyclic tournament: $Y_1 < \dots < Y_p$. Since f is an isomorphism from R onto R' , the set $f(\mathcal{Q})$ is a modular partition of R' , and the quotient $R'/f(\mathcal{Q})$ is the acyclic tournament: $f(Y_1) < \dots < f(Y_p)$. Furthermore, R'/\mathcal{Q} is the acyclic tournament: $Y_1 < \dots < Y_p$, and $|f(Y_i)| = |Y_i|$ for every $i \in \{1, \dots, p\}$. To the contrary, suppose that there is $X \in \mathcal{Q}$ such that $f(X) \neq X$, and let i_0 be the smallest element of $\{1, \dots, p\}$ such that $f(Y_{i_0}) \neq Y_{i_0}$. Clearly, $i_0 < p$ and there is $x \in Y_{i_0}$ such that $f(x) \notin Y_{i_0}$. Thus there is $j > i_0$ such that $f(x) \in Y_j$. It follows that the outdegrees $d_R^+(x)$ and $d_{R'}^+(f(x))$ satisfy $d_R^+(x) \geq \sum_{j \geq i_0+1} |Y_j|$ and $d_{R'}^+(f(x)) < \sum_{j \geq i_0+1} |Y_j|$. Thus $d_R^+(x) \neq d_{R'}^+(f(x))$; which contradicts the fact that f is an isomorphism from R onto R' . \square

Notice that since the result is obvious when D is an acyclic tournament, Lemma 4.4 is immediately deduced from the following three claims, where we assume that the frame of D is an acyclic tournament, and that $|\tilde{\mathcal{P}}(D)| \geq 2$. Recall that $\tilde{\mathcal{P}}(D') = \tilde{\mathcal{P}}(D)$ and $D'/\tilde{\mathcal{P}}(D) = D/\tilde{\mathcal{P}}(D)$ by Lemma 4.1.

Claim 4.18. *The subdigraphs $D'[X]$ and $D[X]$ are $\{-3, -2\}$ -isomorphic for each element X of $\tilde{\mathcal{P}}(D)$.*

Proof. Let X be an element of $\widetilde{\mathcal{P}}(D)$. Since D and D' are (≤ 4) -isomorphic, we may assume that $|X| \geq 7$. Consider a p -element subset B_1 of X with $p \in \{2, 3\}$, a $(3 - p)$ -element subset B_2 of $V \setminus X$, and showing that $D'[X \setminus B_1] \cong D[X \setminus B_1]$. Let $Q = Q_1 \cup \{X \setminus B_1\}$, where Q_1 is the set of the nonempty elements of the set $\{Y \setminus B_2 : Y \in (\widetilde{\mathcal{P}}(D) \setminus \{X\})\}$. It is clear that Q is a common modular partition of the two isomorphic subdigraphs $D - (B_1 \cup B_2)$ and $D' - (B_1 \cup B_2)$ such that $(D - (B_1 \cup B_2))/Q$ and $(D' - (B_1 \cup B_2))/Q$ are two equal acyclic tournaments. Thus Claim 4.17 implies that $f_{B_1 \cup B_2}(X \setminus B_1) = X \setminus B_1$, and hence $D'[X]$ and $D[X]$ are $\{-3, -2\}$ -isomorphic. \square

Claim 4.19. *The subdigraphs $D'[X]$ and $D[X]$ are hereditarily isomorphic for each element X of $\widetilde{\mathcal{P}}(D)$ such that $|X| \geq n - 2$.*

Proof. Let X be an element of $\widetilde{\mathcal{P}}(D)$ such that $|X| \geq n - 2$. By Claim 4.18, $D'[X]$ and $D[X]$ are $\{-3, -2\}$ -isomorphic, and hence by the induction hypothesis, we may assume that $|X| \leq 8$. Since $n \geq 10$, it follows that $n = 10$, $|X| = 8$, and $D'[X]$ and $D[X]$ are $\{-3, -2, 1, 2, 3, 4\}$ -isomorphic. Thus $D'[X]$ and $D[X]$ are (≤ 6) -isomorphic, and hence they are hereditarily isomorphic by Corollary 1.2. \square

Claim 4.20. *The subdigraphs $D'[X]$ and $D[X]$ are isomorphic for each element X of $\widetilde{\mathcal{P}}(D)$.*

Proof. First, assume that there exists $X_0 \in \widetilde{\mathcal{P}}(D)$ such that $|X_0| \geq n - 2$. By Claim 4.19, $D'[X_0]$ and $D[X_0]$ are isomorphic. Furthermore, for every $X \in \widetilde{\mathcal{P}}(D) \setminus \{X_0\}$, $D'[X]$ and $D[X]$ are isomorphic because $|X| \leq 2$. Second, assume that $|Y| \leq n - 3$ for every $Y \in \widetilde{\mathcal{P}}(D)$. Consider an element X of $\widetilde{\mathcal{P}}(D)$. Since D and D' are (≤ 4) -isomorphic, we may assume that $|X| \geq 5$, and that $D[X]$ is not an acyclic tournament, and hence $X \in \mathcal{P}(D)$. Let B be a 2-element subset of X . Since $X \in S(D, D[X]; B)$ and $|X| \leq n - 3$, Corollary 3.21 implies that $n(D', D[X]; B) \neq 0$. Thus there exists a vertex subset K containing B such $D'[K] \cong D[X]$. To the contrary, suppose that $K \neq X$. Thus $\{K \cap X, K \setminus X\}$ is a modular partition of $D'[K]$ such that the quotient $D'[K]/\{K \cap X, K \setminus X\}$ is a 2-vertex acyclic tournament. Since $D'[K] \cong D[X]$, it follows that the subdigraph $D[X]$ has a modular partition Q such that $|Q| = 2$ and $D[X]/Q$ is an acyclic tournament. Therefore, the first assertion of Lemma 3.4 implies that the frame of $D[X]$ is an acyclic tournament; which contradicts the second assertion of Lemma 3.4. Therefore, $K = X$, and hence $D'[X] \cong D[X]$. \square

Now we continue the proof of Lemma 4.4.

Let X be an element of $\widetilde{\mathcal{P}}(D)$. Since D and D' are (≤ 4) -isomorphic, we may assume that $|X| \geq 5$, and that $D[X]$ is not an acyclic tournament. By the preceding claims and the induction hypothesis, we may assume that $6 \leq |X| \leq 8$. Moreover, $D'[X]$ and $D[X]$ are isomorphic and $\{-3, -2\}$ -isomorphic by Claims 4.20 and 4.18. Therefore, by Corollary 1.2 it is sufficient to prove that $D'[X]$ and $D[X]$ are (-1) -isomorphic. Consider an element x of X , and a 2-element subset B of $(V \setminus X)$. Let $Q = Q_1 \cup \{X \setminus \{x\}\}$, where Q_1 is the set of the nonempty elements of the set $\{Y \setminus B : Y \in (\widetilde{\mathcal{P}}(D) \setminus \{X\})\}$. It is clear that Q is a common modular partition of the two isomorphic subdigraphs $D - (B \cup \{x\})$ and $D' - (B \cup \{x\})$ such that the quotients $(D - (B \cup \{x\}))/Q$ and $(D' - (B \cup \{x\}))/Q$ are two equal acyclic tournaments. Thus Claim 4.17 implies that $f_{B \cup \{x\}}(X \setminus \{x\}) = X \setminus \{x\}$, and hence $D'[X]$ and $D[X]$ are (-1) -isomorphic. \square

4.6. Proof of Lemma 4.5

By interchanging (D, D') and $(\overline{D}, \overline{D}')$, we may assume that the frame of D is an empty digraph. Lemma 4.5 is then immediately deduced from the following five claims. Recall that $\mathcal{P}(D') = \mathcal{P}(D)$ and $D'/\mathcal{P}(D) = D/\mathcal{P}(D)$ by Lemma 4.1.

Claim 4.21. *If there is an element X_0 of $\mathcal{P}(D)$ such that $|X_0| \geq n - 2$, then $D[X]$ and $D'[X]$ are hereditarily isomorphic for each element X of $\mathcal{P}(D)$.*

Proof. For each element Y of $\mathcal{P}(D) \setminus \{X_0\}$, the subdigraphs $D[Y]$ and $D'[Y]$ are hereditarily isomorphic because $|Y| \leq 2$. Thus it remains only to prove that $D[X_0]$ and $D'[X_0]$ are hereditarily isomorphic. Consider a 3-element subset B of X_0 , and let $Q = \{X_0 \setminus B, V \setminus X_0\}$. Clearly, Q is a common modular partition of the two isomorphic subdigraphs $D - B$ and $D' - B$ such that $(D - B)/Q = (D' - B)/Q$. Moreover, the subdigraphs $D[V \setminus X_0]$ and $D'[V \setminus X_0]$ are isomorphic because $|V \setminus X_0| \leq 2$. Thus Corollary 3.24 implies that $D'[X_0 \setminus B] \cong D[X_0 \setminus B]$, and hence $D'[X_0]$ and $D[X_0]$ are (-3) -isomorphic. Thus by the induction hypothesis, we may assume that $|X_0| \leq 8$. Since $|X_0| \geq n - 2$ and $n \geq 10$, it follows that $|X_0| = 8$ and $|V \setminus X_0| = 2$, and hence $D'[X_0]$ and $D[X_0]$ are 5-isomorphic. Moreover D and D' are (≤ 4) -isomorphic. Therefore, by Corollary 1.2 it is sufficient to prove that $D'[X_0]$ and $D[X_0]$ are (-2) -isomorphic. To do so, consider a 2-element subset C of X_0 , and denote by a and b the two elements of $V \setminus X_0$. Clearly, $\{X_0 \setminus C, \{b\}\}$ is a common modular partition of the two isomorphic subdigraphs $D - (C \cup \{a\})$ and $D' - (C \cup \{a\})$ such that $(D - (C \cup \{a\})) / \{X_0 \setminus C, \{b\}\} = (D' - (C \cup \{a\})) / \{X_0 \setminus C, \{b\}\}$. Thus Corollary 3.24 implies that $D'[X_0 \setminus C] \cong D[X_0 \setminus C]$, and hence $D[X_0]$ and $D'[X_0]$ are (-2) -isomorphic. \square

Claim 4.22. *The subdigraphs $D[X]$ and $D'[X]$ are isomorphic for each element X of $\mathcal{P}(D)$.*

Proof. By Claim 4.21, we may assume that $|Y| \leq n - 3$ for every $Y \in \mathcal{P}(D)$. Consider an element X of $\mathcal{P}(D)$. Since D and D' are (≤ 4) -isomorphic, we may assume that $|X| \geq 5$. Let B be a 2-element subset of X . Since $X \in S(D, D[X]; B)$ and $|X| \leq n - 3$, Corollary 3.21 implies that $n(D', D[X]; B) \neq 0$. Thus there exists a vertex subset K including B such $D'[K] \cong D[X]$. If $K \neq X$, then $\{K \cap X, K \setminus X\}$ is a modular partition of $D'[K]$ such that the quotient $D'[K] / \{K \cap X, K \setminus X\}$ is an empty digraph. Since $D'[K] \cong D[X]$, it follows that the subdigraph $D[X]$ has a modular partition Q such that $|Q| = 2$ and $D[X]/Q$ is an empty digraph. Therefore, the first assertion of Lemma 3.4 implies that the frame of $D[X]$ is an empty digraph; which contradicts the third assertion of Lemma 3.4. Therefore, $K = X$, and hence $D'[X] \cong D[X]$. \square

Claim 4.23. *Given an element X of $\mathcal{P}(D)$, the following assertions hold.*

1. *The subdigraphs $D[X]$ and $D'[X]$ are (-3) -isomorphic.*
2. *The subdigraphs $D[V \setminus X]$ and $D'[V \setminus X]$ are (-3) -isomorphic.*

Proof. Assume that $|X| \geq 4$ (resp. $|V \setminus X| \geq 4$), and consider a 3-element subset B of X (resp. of $V \setminus X$). Denote by Q the set $\{X - B, V \setminus X\}$ (resp. $\{X, V \setminus (X \cup B)\}$). Clearly, Q is a common modular partition of the two isomorphic subdigraphs $D - B$ and $D' - B$ such that $(D - B)/Q = (D' - B)/Q$. By Claim 4.22, $D'[X] \cong D[X]$, and $D'[Y] \cong D[Y]$ for each element Y of $\mathcal{P}(D) \setminus \{X\}$. Since the common frame of D and D' is an empty digraph, it follows that $D'[V \setminus X] \cong D[V \setminus X]$. Therefore, Corollary 3.24 implies that $D'[X - B] \cong D[X - B]$ (resp. $D'[V \setminus (X \cup B)] \cong D[V \setminus (X \cup B)]$). Thus the assertions 1 and 2 hold. \square

Claim 4.24. *If $|\mathcal{P}(D)| \geq 3$, then the subdigraphs $D[X]$ and $D'[X]$ are hereditarily isomorphic for each element X of $\mathcal{P}(D)$.*

Proof. Assume that $|\mathcal{P}(D)| \geq 3$, and consider an element X of $\mathcal{P}(D)$. Since D and D' are (≤ 4) -isomorphic, by Claim 4.22, Claim 4.23 and the induction hypothesis, we may assume $6 \leq |X| \leq 8$.

First, assume that there is $Y \in (\mathcal{P}(D) \setminus \{X\})$ such that $|V \setminus Y| \geq 9$. By Claim 4.23, the subdigraphs $D'[V \setminus Y]$ and $D[V \setminus Y]$ are (-3) -isomorphic. Thus the induction hypothesis implies that $D'[V \setminus Y]$ and $D[V \setminus Y]$ are hereditarily isomorphic. Consequently, the subdigraphs $D'[X]$ and $D[X]$ are hereditarily isomorphic.

Second, assume that $|V \setminus Y| \leq 8$ for each element Y of $\mathcal{P}(D) \setminus \{X\}$. Thus $|Y| \geq 2$ for each $Y \in (\mathcal{P}(D) \setminus \{X\})$. Since $|\mathcal{P}(D)| \geq 3$, $|V| \geq 10$, and $6 \leq |X| \leq 8$, it follows that $|V| = 10$, $|X| = 6$, and $\mathcal{P}(D) = \{X, Y_1, Y_2\}$ where $|Y_1| = |Y_2| = 2$. Since D and D' are (≤ 4) -isomorphic and $|X| = 6$, by Claim 4.22 and Corollary 1.2, it suffices to prove that $D'[X]$ and $D[X]$ are (-1) -isomorphic. To do so, consider an element x of X , and let $Q = \{X \setminus \{x\}, Y_2\}$. Clearly, Q is a common modular partition of the two isomorphic subdigraphs $D - (Y_1 \cup \{x\})$ and $D' - (Y_1 \cup \{x\})$ such that $(D - (Y_1 \cup \{x\})) / Q = (D' - (Y_1 \cup \{x\})) / Q$. Moreover, the subdigraphs $D[Y_2]$ and $D'[Y_2]$ are isomorphic because $|Y_2| = 2$. Thus Corollary 3.24 implies that $D'[X \setminus \{x\}] \cong D[X \setminus \{x\}]$, and hence $D'[X]$ and $D[X]$ are (-1) -isomorphic. \square

Claim 4.25. *If $|\mathcal{P}(D)| = 2$, then the subdigraphs $D[X]$ and $D'[X]$ are hereditarily isomorphic for each element X of $\mathcal{P}(D)$.*

Proof. Assume that $\mathcal{P}(D)$ is a pair $\{X_1, X_2\}$, where $|X_1| \leq |X_2|$.

First, we will prove that the subdigraphs $D'[X_1]$ and $D[X_1]$ are 5-isomorphic. Since D and D' are (≤ 4) -isomorphic, by Claim 4.22, Claim 4.23 and the induction hypothesis, we may assume that $|X_1| \in \{6, 7\}$. Consider a p -element subset B of X_1 , where $p = |X_1| - 5$. If $D[X_1 \setminus B]$ has a modular partition Q such that $|Q| = 2$ and $D[X_1 \setminus B]/Q$ is an empty digraph, then the subdigraphs $D'[X_1 \setminus B]$ and $D[X_1 \setminus B]$ are isomorphic because they are (≤ 4) -isomorphic. Thus, we may assume that $D[X_1 \setminus B]$ has no such a modular partition. Consider an element y_1 of $X_1 \setminus B$. Clearly, the subdigraphs $D - B$ and $D' - B$ are $(-(3 - p))$ -isomorphic with $p \in \{1, 2\}$, and $|X_1 \setminus B| \leq n - 3$ because the subset $(V \setminus X_1)$ includes X_2 . Thus Corollary 3.21 implies that $n(D - B, D[X_1 \setminus B]; \{y_1\}) = n(D' - B, D[X_1 \setminus B]; \{y_1\})$. Thus there is a subset K of $V \setminus B$ containing y_1 such that $D'[K] \cong D[X_1 \setminus B]$. If $K \neq X_1 \setminus B$, then $\{K \cap X_1, K \cap X_2\}$ is a modular partition of $D'[K]$ such that the quotient $D'[K]/\{K \cap X_1, K \cap X_2\}$ is an empty digraph. Since $D'[K] \cong D[X_1 \setminus B]$, it follows that the subdigraph $D[X_1 \setminus B]$ has a modular partition Q such that $|Q| = 2$ and $D[X_1 \setminus B]/Q$ is an empty digraph; which contradicts our assumption. Therefore, $K = X_1 \setminus B$, and hence $D'[X_1 \setminus B] \cong D[X_1 \setminus B]$. Therefore, $D'[X_1]$ and $D[X_1]$ are 5-isomorphic. Thus $D'[X_1]$ and $D[X_1]$ are (≤ 5) -isomorphic.

Second, we will prove that the subdigraphs $D'[X_1]$ and $D[X_1]$ are hereditarily isomorphic. By Claim 4.22, Claim 4.23 and the induction hypothesis, we may assume that $|X_1| \in \{6, 7, 8\}$. By Corollary 1.2, it remains only to prove that the subdigraphs $D'[X_1]$ and $D[X_1]$ are 6-isomorphic. By Claim 4.22, we may assume that $|X_1| \in \{7, 8\}$. Consider a p -element subset B of X_1 , where $p = |X_1| - 6$. If $D[X_1 \setminus B]$ has a modular partition Q such that $|Q| = 2$ and $D[X_1 \setminus B]/Q$ is an empty digraph, then the subdigraphs $D'[X_1 \setminus B]$ and $D[X_1 \setminus B]$ are isomorphic because they are (≤ 5) -isomorphic. Thus, we may assume that $D[X_1 \setminus B]$ has no such a modular partition. Consider an element y_1 of $X_1 \setminus B$. Since the subdigraphs $D - B$ and $D' - B$ are $(-(3 - p))$ -isomorphic and $p \in \{1, 2\}$, Corollary 3.21 implies that $n(D - B, D[X_1 \setminus B]; \{y_1\}) = n(D' - B, D[X_1 \setminus B]; \{y_1\})$. Thus there is a subset K of $V \setminus B$ containing y_1 such that $D'[K] \cong D[X_1 \setminus B]$. As previously, we obtain that $K = X_1 \setminus B$, and hence $D'[X_1 \setminus B] \cong D[X_1 \setminus B]$. Therefore, $D'[X_1]$ and $D[X_1]$ are 6-isomorphic.

Third, we will prove that the subdigraphs $D'[X_2]$ and $D[X_2]$ are hereditarily isomorphic. Since D and D' are (≤ 4) -isomorphic, by Claim 4.22, Claim 4.23 and the induction hypothesis, we may assume that $|X_2| \in \{6, 7, 8\}$, and hence $|X_1| \geq 2$ because $|V| \geq 10$. Therefore, by Claim 4.23 and Corollary 1.2, it suffices to prove that the subdigraphs $D'[X_2]$ and $D[X_2]$ are $\{-1, -2\}$ -isomorphic. To do so, consider a p -element subset B_2 of X_2 , with $p \in \{1, 2\}$, and a $(3 - p)$ -element subset B_1 of X_1 . If $B_1 = X_1$, then $D'[X_2 \setminus B_2] \cong D[X_2 \setminus B_2]$ because D and D' are (-3) -isomorphic. Otherwise, let $Q = \{X_1 \setminus B_1, X_2 \setminus B_2\}$. Clearly, Q is a common modular partition of the two isomorphic subdigraphs $D - (B_1 \cup B_2)$ and $D' - (B_1 \cup B_2)$ such that $(D - (B_1 \cup B_2))/Q = (D' - (B_1 \cup B_2))/Q$. Moreover, $D'[X_1 \setminus B_1]$ and $D[X_1 \setminus B_1]$ are isomorphic because $D'[X_1]$ and $D[X_1]$ are hereditarily isomorphic. Therefore, Corollary 3.24 implies that $D'[X_2 - B_2] \cong D[X_2 - B_2]$. Thus the subdigraphs $D'[X_2]$ and $D[X_2]$ are $\{-1, -2\}$ -isomorphic. \square

Lemma 4.5 follows from Lemmas 4.24 and 4.25. \square

The end of the proof of Theorem 1.8 follows from Lemmas 4.3, 4.4, and 4.5. \square

5. Proofs of Proposition 1.9 and Corollary 1.10

5.1. Proof of Proposition 1.9

First, let us consider some particular prechains.

1. For each integer $h \geq 1$, the *rotational tournament* T_{2h+1} is the prechain tournament defined on $\{0, \dots, 2h\}$ as follows. For all $i, j \in \{0, \dots, 2h\}$, $(i, j) \in E(T_{2h+1})$, if there exists $k \in \{1, \dots, h\}$ such that $j = i + k$ modulo $(2h + 1)$.
2. The tournament P_6 is a 6-vertex prechain tournament of which the frame is a 3-cycle such that $\mathcal{P}(P_6) = \{X_1, X_2, X_3\}$ where, $T[X_i]$ is an i -vertex acyclic tournament, for each $i \in \{1, 2, 3\}$.

3. For each integer h with $h \geq 2$, a wheel W_{2h} is a prechain of which the neutral pairs have the same type, defined on $\{0, \dots, 2h - 1\}$ as follows, where the integers are considered modulo $2h$. For each $i \in \{0, \dots, 2h - 1\}$, $\{i, i + h\}$ is a neutral pair, and for each $k \in \{1, \dots, h - 1\}$, $\{i, i + k\}$ is an oriented pair with $(i, i + k) \in E(W_{2h})$.

Notice that it is easy to verify that any prechain tournament is (≤ 5) -self converse, and that the tournament P_6 is not self converse.

The following lemma is easy to check and can be deduced from [7, 8].

Lemma 5.1. *Given an integer h with $h \geq 4$, the tournament T_{2h+1} and every wheel W_{2h} are $\{4, -2, -1\}$ -self converse, and are not hereditarily self converse.*

Now Proposition 1.9 is an immediate consequence of Lemma 5.1. \square

5.2. Proof of Corollary 1.10

Consider an integer k with $k \geq 4$, and two $(-k)$ -isomorphic digraphs D and D' on the same vertex set V with $|V| \geq k + 6$. By Corollary 1.2 it is sufficient to prove that D and D' are (≤ 6) -isomorphic. Consider a vertex subset X with at most 6 elements. To obtain the result, we will prove that the subdigraphs $D'[X]$ and $D[X]$ are isomorphic. Consider a subset B of $V \setminus X$ with $|B| = k - 3$. By Corollary 3.19, the digraphs D and D' are (≤ 4) -isomorphic. Moreover, the subdigraphs $D' - B$ and $D - B$ are (-3) -isomorphic because the digraphs D and D' are $(-k)$ -isomorphic. Thus the subdigraphs $D' - B$ and $D - B$ are $\{4, -3\}$ -isomorphic, and have at least 9 vertices. Hence Theorem 1.8 implies that $D' - B$ and $D - B$ are hereditarily isomorphic. Consequently, the subdigraphs $D'[X]$ and $D[X]$ are isomorphic. \square

5.3. Remark

In this section, we present two examples to show that the lower bounds of the orders of the digraphs given in Theorem 1.8 and Corollary 1.10 are the best possible.

First, consider an 8-vertex prechain tournament H of which the frame is a 3-cycle such that $\mathcal{P}(H) = \{X_1, X_2, X_3\}$ where, $H[X_1]$ and $H[X_2]$ are two 3-vertex acyclic tournaments, and $H[X_3]$ is a 2-vertex tournament. Since any prechain tournament is (≤ 5) -self converse, the prechain H is $\{4, -3\}$ -self converse. On the other hand, H is not hereditarily self converse because the subtournament $H - \{u, v\}$ is P_6 , where $u \in X_1$ and $v \in X_3$. Thus the value 9 is the best possible as a lower bound of n in Theorem 1.8.

Second, consider an integer k with $k \geq 4$, and two $(k + 5)$ -vertex tournaments K and K' , obtained from a k -vertex acyclic tournament by dilating the same vertex by the tournaments P_6 and P_6^* respectively. Since P_6 is not self converse, Lemma 3.22 implies that the tournaments K and K' are not isomorphic, and hence they are not hereditarily isomorphic. On the other hand, since P_6 is (≤ 5) -self converse, K and K' are $(-k)$ -isomorphic. Thus the value $k + 6$ is the best possible as a lower bound of n in Corollary 1.10.

5.4. Open problems

First, recall that the determination of the exact value of the integer k in Problem 1.7 remains an open problem. Now this problem becomes more precise:

Problem 5.2. *For the smallest positive integer k such that if D and D' are two $(-k)$ -isomorphic n -vertex digraphs, n large enough, then D and D' are hereditarily isomorphic, decide whether $k = 3$ or $k = 4$?*

Second, we pose the following problem.

Problem 5.3. *For $k \in \{1, 2\}$, characterize the digraphs D such that every digraph $\{4, -k\}$ -isomorphic to D is hereditarily isomorphic to D .*

ACKNOWLEDGEMENT

The Author is pleased to thank the anonymous referees for their careful reading and helpful suggestions.

The author would like to extend his sincere appreciation to the Deanship of Scientific Research at King Saud University for its funding of this research through the Research Group Project No RGP-056.

References

- [1] R. Ben Hamadou, Y. Boudabbous, N. El Amri, G. Lopez, *Description of the Tournaments Which are Reconstructible from Their k -cycle Partial Digraphs for $k \in \{3, 4\}$* , Graphs and Combinatorics 32 (5) (2016) 1881–1901.
- [2] Z. W. Birnbaum, J. D. Esary, *Modules of coherent binary systems*, J. Soc. Indust. Appl. Math. 13 (1965) 444–462.
- [3] J.A. Bondy, *A graph reconstructor's manual*, in: O. Keendwell (ed.), *Surveys in Combinatorics*, in: London. Math. Soc. Lecture Note Ser., Cambridge Univ. Press (1991) pp. 221–252.
- [4] A. Bondy, R.L. Hemminger, *Graph reconstruction, a survey*, J. Graph Theory 1 (1977) 227–268.
- [5] M. Bouaziz, Y. Boudabbous, *La demi-isomorphie et les tournois fortement connexes finis*, C. R. Math. Acad. Sci. Paris 335 (2) (2002) 105–110.
- [6] H. Bouchaala, *Études sur la répartition des diamants dans un tournoi et la $\{-k\}$ -autodualité dans les graphes orientés*, Ph.D. Thesis, Faculty of Sciences of Sfax, Tunisia (2005).
- [7] H. Bouchaala, Y. Boudabbous, *La $\{-k\}$ -autodualité des sommes lexicographiques finies de tournois suivant un 3-cycle ou un tournoi critique*, Ars Combinatoria 81 (2006) 33–64.
- [8] H. Bouchaala, Y. Boudabbous, G. Lopez, *$\{-1\}$ -self dual finite prechains and applications*, C. R. Math. Acad. Sci. Paris 351 (23–24) (2013) 859–864.
- [9] Y. Boudabbous, *Sur la détermination d'une relation binaire à partir d'informations locales*, Math. Logic Quart. 44 (1998) 265–276.
- [10] Y. Boudabbous, *La 5-Reconstructibilité et l'Indécomposabilité des Relations Binaires*, Eur. J. Combin. 23 (5) (2002) 507–522.
- [11] Y. Boudabbous, *Isomorphie héréditaire et $\{-4\}$ -hypomorphie pour les tournois*, C. R. Math. Acad. Sci. Paris 347 (15–16) (2009) 841–844.
- [12] Y. Boudabbous, J. Dammak, *Sur la $\{-k\}$ -demi-reconstructibilité des tournois finis. (French) [On the $\{-k\}$ -half-reconstructibility of finite tournaments]*, C. R. Acad. Sci. Paris Sér. I Math. 326 (9) (1998) 1037–1040.
- [13] Y. Boudabbous, C. Delhommé, *Prechains and self duality*, Discrete Math. 312 (10) (2012) 1743–1765.
- [14] Y. Boudabbous, C. Delhommé, *$(\leq k)$ -Reconstructible binary relations*, European J. Combin. 37 (2014) 43–67.
- [15] Y. Boudabbous, G. Lopez, *The minimal non- $(\leq k)$ -reconstructible relations*, Discrete Math. 291 (2005) 19–40.
- [16] J. Dammak, *La dualité dans la demi-reconstruction des relations binaires finies. (French) [Duality in the half-reconstruction of finite binary relations]*, C. R. Acad. Sci. Paris Sér. I Math. 327 (10) (1998) 861–864.
- [17] J. Dammak, *Isomorphy and dilatation in digraphs*, Advances in Pure and Applied Math. 5 (3) (2014) 161–171.
- [18] J. Dammak, G. Lopez, M. Pouzet, H. Si Kaddour, *Hypomorphy of graphs up to complementation*, J. Combin. Theory Ser. B 99 (1) (2009) 84–96.
- [19] J. Dammak, G. Lopez, M. Pouzet, H. Si Kaddour, *Boolean sum of graphs and reconstruction up to complementation*, Adv. Pure Appl. Math. 4 (3) (2013) 315–349.
- [20] A. Ehrenfeucht, T. Harju, G. Rozenberg, *The Theory of 2-Structures. A Framework for Decomposition and Transformation of Graphs*, World Scientific Publishing Co., Inc., River Edge, NJ, 1999.
- [21] A. Ehrenfeucht, G. Rozenberg, *Primitivity is hereditary for 2-structures*, Theoret. Comput. Sci. 70 (3) (1990) 343–358.
- [22] R. Fraïssé, *Abriement entre relations et spécialement entre chaînes*, Symposi. Math., Instituto Nazionale di Alta Matematica 5 (1970) pp. 203–251.
- [23] R. Fraïssé, *L'intervalle en théorie des relations; ses généralisations; filtre intervallaire et clôture d'une relation*, Orders: description and roles (L'Arbresle, 1982), 313–341, North-Holland Math. Stud., 99, Ann. Discrete Math., 23, North-Holland, Amsterdam, 1984.
- [24] T. Gallai, *Transitiv orientierbare Graphen*, Acta Math. Acad. Sci. Hungar. 18 (1967) 25–66.
- [25] J. G. Hagendorf, G. Lopez, *La demi-reconstructibilité des relations binaires d'au moins 13 éléments*, C. R. Acad. Sci. Paris Sér. I Math. 317 (1) (1993) 7–12.
- [26] P. Ille, *La La reconstructibilité des relations binaires*, C. R. Acad. Sci. Paris Sér. I Math. 306 (15) (1988) 635–638.
- [27] G. Lopez, *Deux résultats concernant la détermination d'une relation par les types d'isomorphie de ses restrictions*, C. R. Acad. Sci. Paris Série A 274 (1972) 1525–1528.
- [28] G. Lopez, *Sur la détermination d'une relation par les types d'isomorphie de ses restrictions*, C. R. Acad. Sci. Paris Série A 275 (1972) 951–953.
- [29] G. Lopez, *L'indéformabilité des relations et multirelations binaires*, Z. Math. Logik Grundlag. Math. 24 (4) (1978) 303–317.
- [30] G. Lopez, C. Rauzy, *Reconstruction of binary relations from their restrictions of cardinality 2, 3, 4 and $(n-1)$, I*, Z. Math. Logik Grundlag. Math. 38 (1) (1992) 27–37.
- [31] G. Lopez, C. Rauzy, *Reconstruction of binary relations from their restrictions of cardinality 2, 3, 4 and $(n-1)$, II*, Z. Math. Logik Grundlag. Math. 38 (2) (1992) 157–168.
- [32] F. Maffray, M. Preissmann, *A translation of Tibor Gallai's paper: Transitiv orientierbare Graphen*, in: J.L. Ramirez-Alfonsin and B.A. Reed (eds.), *Perfect Graphs*, Wiley, New York, pp. 25–66, 2001.
- [33] M. Pouzet, *Application d'une propriété combinatoire des parties d'un ensemble aux groupes et aux relations*, Math. Z. 150 (1976) 117–134.
- [34] M. Pouzet, *Relations non reconstructibles par leurs restrictions*, J. Combin. Theory Ser. B 26 (1) (1979) 22–34.
- [35] M. Pouzet, H. Si Kaddour, *Isomorphy up to complementation*, J. Comb. 7 (2–3) (2016) 285–305.
- [36] M. Pouzet, H. Si Kaddour, N. Trotignon, *Claw-freeness, 3-homogeneous subsets of a graph and a reconstruction problem*, Contrib. Discrete Math. 6 (1) (2011) 86–97.
- [37] K. B. Reid, C. Thomassen, *Strongly Self-Complementary and Hereditarily Isomorphic Tournaments*, Monatshefte Math. 81 (1976) 291–304.
- [38] H. Spinoza, D. B. West, *Reconstruction from the deck of k -vertex induced subgraphs*, J. Graph Theory 90(4) (2019) 497–522.
- [39] J. Spinrad, *P_4 -trees and substitution decomposition*, Discrete Appl. Math. 39 (1992) 263–291.
- [40] P. K. Stockmeyer, *The falsity of the reconstruction conjecture for tournaments*, J. Graph Theory 1 (1977) 19–25.
- [41] S. M. Ulam, *A collection of Mathematical Problems*, Intersciences Publishers, New York, 1960.
- [42] D. West, *Introduction to Graph Theory*, Second edition, Prentice Hall, 2001.