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On a Cosine Operator Function Framework of Approximation Processes in Banach Space

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Abstract. We introduce the cosine-type approximation processes in abstract Banach space setting. The historical roots of these processes go back to W. W. Rogosinski in 1926. The given new definitions use a cosine operator functions concept. We proved that in presented setting the cosine-type operators possess the order of approximation, which coincide with results known in trigonometric approximation. Moreover, a general method for factorization of certain linear combinations of cosine operator functions is presented. The given method allows to find the order of approximation using the higher order modulus of continuity. Also applications for the different type of approximations are given.

1. Introduction

The aim of this paper is to introduce an abstract framework of certain approximation processes and to estimate the order of approximation using a cosine operator functions concept. Historical roots of these processes go back to W.W. Rogosinski [14], who proved that the arithmetical mean of shifted Fourier partial sums converges uniformly to a given 2π-periodic continuous function $f \in C_{2\pi}$. In notations: for $f \in C_{2\pi}$ the Fourier partial sums

$$S_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$$

define the Rogosinski means by

$$R_n(f,x) := \frac{1}{2}\left(S_n\left(f, x + \frac{\pi}{2(n+1)}\right) + S_n\left(f, x - \frac{\pi}{2(n+1)}\right)\right). \tag{1}$$

Let X be an arbitrary (real or complex) Banach space, and $[X]$ the Banach algebra of all bounded linear operators U of X into itself.

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Let $\{P_k\}_{k=0}^{\infty} \subset [X]$ be a given sequence of mutually orthogonal projections, i.e. $P_jP_k = \delta_{jk}P_k$, (δ_{jk} being the Kronecker symbol). Moreover, let us assume that the sequence of projections is total, i.e. $P_kf = 0$ for all $k = 0, 1, 2, ...$ implies $f = 0$, and fundamental, i.e. the linear span of $\bigcup_{k=0}^{\infty} P_k(X)$ is dense in X. Then with each $f \in X$ one may associate its unique Fourier series expansion

$$f \sim \sum_{k=0}^{\infty} P_k f$$

with the Fourier partial sums operator or Fourier projection operator

$$S_n f = \sum_{k=0}^{n} P_k f.$$

As we know from trigonometric Fourier approximation the strong convergence of the Fourier partial sums is not guaranteed for all $f \in X$. The improvement of that situation will be given by some matrix transformation like

$$U_n f = \sum_{k=0}^{n} \theta_k(n) P_k f.$$

The first matrix transformation with $\theta_k(n) = 1 - \frac{k}{n+1}$ for the trigonometric Fourier series was introduced by L. Fejér in 1904 [5]. Later on, W.W. Rogosinski [14] introduced the arithmetical mean of shifted Fourier partial sums (1), which appeared to be the matrix transformation with $\Theta_k(n) = \cos \frac{k\pi}{2(n+1)}$.

In this paper we introduce in abstract setting the Rogosinski- and Blackman-type operators and find the order of approximation via a modulus of continuity (smoothness), which is defined by a general operator cosine functions. A little less abstract setting we used in [9]. The Rogosinski- and Blackman-type operators are interesting, because they are applicable in approximation by Fourier expansions of different orthogonal systems [18], in summation of Fourier transforms [3] and in approximation by generalized Shannon sampling operators [10].

Definition 1.1. *(compare [13])A cosine operator function $C_h \in [X]$ $(h \geq 0)$ is defined by the properties:*

(i) $C_0 = I$*(identity operator),*

(ii) $C_{h_1} \cdot C_{h_2} = \frac{1}{2}(C_{h_1+h_2} + C_{|h_1-h_2|})$,

(iii) $\|C_h f\| \leq T\|f\|$*, the constant* $T > 0$ *is not depending on* $h > 0$.

Remark 1.2. *Let* $\tau_h \in [X]$, $h \in \mathbb{R}$*, be a translation operator, defined by the properties*

(i) $\tau_0 = I$,

(ii) $\tau_{h_1} \cdot \tau_{h_2} = \tau_{h_1+h_2}$,

(iii) $\|\tau_h f\| \leq T\|f\|, 0 < T-$ *not depending on* $h \in \mathbb{R}$.

Then $C_h := \frac{1}{2}(\tau_h + \tau_{-h})$, $h \geq 0$*, is a cosine operator function.*

The following example demonstrates why sometimes we should use the cosine operator function.

System of symmetric trigonometric functions with respect to π. Let $X = C_{2\pi}^{-}$ denote the space of symmetric functions with respect to π (shortly π-symmetric) and in addition 4π-periodic, i.e. we suppose that $f(\pi - x) = f(\pi + x)$ (or equivalently, $f(2\pi - x) = f(x)$) and $f(4\pi + x) = f(x)$ for all $x \in \mathbb{R}$. The space $C_{2\pi}^{-}$ and the corresponding $\pi-$symmetric orthogonal system $\{\cos kx, \sin(k + 1/2)x\}$ $(k = 0, 1, ...)$ on $[-\pi, \pi]$ were studied in [11]. For example, for $k = 0, 1, 2, ...$ the functions $y = \cos kx$, $y = \sin(k + \frac{1}{2})x$, are in space $C_{2\pi}^{-}$, but $y = \sin((k + 1/2)x)$ are not in $C_{2\pi}$. An interesting phenomenon of π-symmetry is that for any continuous function f on $[-\pi, \pi]$ its π-symmetric and 4π-periodic extension is always continuous on $\mathbb{R}$. This is not the case of 2π-periodic extension of any continuous function f on $[-\pi, \pi]$ - to be continuous in addition the equality $f(-\pi) = f(\pi)$ should be valid.

If a function $f \in C_{2\pi}$, it is obvious that for the ordinary translation operator $\tau_h(f,x) = f(x+h)$, $h \in \mathbb{R}$, we have $\tau_h f \in C_{2\pi}$ as well. But here we may note that the ordinary translation operator $\tau_h(f,x) = f(x+h)$, $h \in \mathbb{R}$, is not good for the π-symmetric functions, since, for example, $\tau_h\left(\sin\left(\frac{1}{2}\circ\right),x\right) = \sin\frac{1}{2}(x+h) \notin C^-_{2\pi}$ for every $h \neq 2k\pi, k \in \mathbb{Z}$. But for the operator cosine function

$$C_h(f,x) = \frac{1}{2}\left(f(x+h) + f(x-h)\right),\ h \geq 0, \tag{2}$$

if $f \in C^-_{2\pi}$, then for every $h \geq 0$ the cosine operator function $C_h f \in C^-_{2\pi}$.

Fourier-Chebyshev series. For $f \in C_{[-1,1]}$ let us consider the Fourier-Chebyshev partial sums operator

$$S^T_n(f,x) = \hat{f}_T(0) + 2\sum_{k=1}^{n} \hat{f}_T(k)T_k(x),$$

where

$$\hat{f}_T(k) := \frac{1}{\pi}\int_{-1}^{1} f(u)T_k(u)\frac{du}{\sqrt{1-u^2}}$$

is the k-th Fourier-Chebyshev coefficient, and $T_k(u) = \cos(k\arccos u)$ is the k-th Chebyshev polynomial of the first kind. For this case a suitable cosine operator function (see [2], [4]) is

$$C^T_h(f,x) := \frac{1}{2}\left\{f(x\cos h + \sqrt{1-x^2}\sin h) + f(x\cos h - \sqrt{1-x^2}\sin h)\right\},\ 0 \leq h \leq \pi.$$

2. Modulus of continuity, best approximations, general cosine-type approximation operators

In the present section we will define the general cosine-type approximation operators and the apparatus that is needed for estimating the order of approximation. The leading idea for definitions below appeared from the trigonometric approximation (see [14], [3], [12], [15]) and references cited there.

An abstract modulus of continuity, defined by the cosine operator function, will play an important role in our paper.

Definition 2.1. *The modulus of continuity of order $k \in \mathbb{N}$ is defined for $\delta \geq 0$ via the cosine operator function by*

$$\omega_k(f,\delta) := \sup_{0\leq h\leq\delta} \|(C_h - I)^k f\|. \tag{3}$$

The next properties are adaptions of the well-known properties of the ordinary modulus of continuity (see, e.g. [3], [15], [17]).

Proposition 2.2. *The modulus of continuity $\omega_k(f,\delta)$ ($\omega(f,\delta) := \omega_1(f,\delta)$) in Definition 2.1 has the following properties:*

(i) $\omega_k(f,m\delta) \leq m^k(1+(m-1)T)^k\omega_k(f,\delta)$, $m \in \mathbb{N}$;
(ii) $\omega_k(f,\lambda\delta) \leq ([\lambda]+1)^k(1+[\lambda]T)^k\omega_k(f,\delta)$, $\lambda > 0$, ($[\lambda] \leq \lambda$ *is the entire part of* $\lambda \in \mathbb{R}$);
(iii) $\omega_k(f,\delta) \leq (1+T)^{k-l}\omega_l(f,\delta)$, $k \geq l$ *and* $k,l \in \mathbb{N}$.

Remark 2.3. *Let $\tau_h : X \to X,\ h \in \mathbb{R}$, be a translation operator and let us define another modulus of continuity of order $k \in \mathbb{N}$ by*

$$\widetilde{\omega}_k(f,\delta) := \sup_{0\leq h\leq\delta} \left\|(\tau_{h/2} - \tau_{-h/2})^k f\right\|.$$

Then, by Remark 1, $C_h := \frac{1}{2}(\tau_h + \tau_{-h})$, $h \geq 0$, defines the modulus of continuity ω_k by (3). Since $C_h - I = \frac{1}{2}(\tau_{h/2} - \tau_{-h/2})^2$, we have

$$\omega_k(f, \delta) = \frac{1}{2^k}\widetilde{\omega}_{2k}(f, \delta). \tag{4}$$

Another quantity we need is the best approximation. Let $A_\sigma \subset X$ be a dense family of linear subspaces with $A_{\sigma_1} \subset A_{\sigma_2}, 0 < \sigma_1 < \sigma_2$, meaning that for every $f \in X$ there exists a family $\{g_\sigma\}_{\sigma>0} \subset \bigcup_{\sigma>0} A_\sigma$ such that $\lim_{\sigma\to\infty} \|f - g_\sigma\| = 0$. Let $A_\sigma \subset X$ consist of the fixed points of a linear operator $S_\sigma : A_\sigma \to A_\sigma$, i.e. for any $g \in A_\sigma$ we have $S_\sigma g = g$.

Definition 2.4. *The best approximation of $f \in X$ by elements of A_σ is defined by*

$$E_\sigma(f) := \inf_{g\in A_\sigma} \|f - g\|.$$

Remark 2.5. *We often may suppose that there exists an element $g_* \in A_\sigma$ of the best approximation, i.e. $E_\sigma(f) = \|f - g_*\|$.*

First, let us define our approximation processes as operators only on the subspace A_σ. In the following definitions instead of $S_\sigma g, (g \in A_\sigma)$, we could write just g, because by our assumption $S_\sigma g = g$. But we prefer the given definitions, since in some cases we are able to define the operators S_σ on the whole space X, still with a set of fixed points A_σ, and in this case also the approximation operators will be defined on the whole space X. To clarify the situation let us give two characteristic examples.

1) Fourier projections are defined on the whole space X having the fixed point set as corresponding generalized polynomials.

2) Let $X = C(\mathbb{R})$ be the space of uniformly continuous and bounded functions on $\mathbb{R}$ (for what follows, see, e.g. [7]) with a family of the dense subsets $B^\infty_\sigma \subset C(\mathbb{R})$ consisting of the bounded functions on $\mathbb{R}$, which are entire functions $f(z)$ $(z \in \mathbb{C})$ of exponential type σ, i.e. $|f(z)| \leq e^{\sigma|y|}\|f\|_C \quad (z = x + iy \in \mathbb{C})$. In this case the linear operator $S_\sigma : B^\infty_\sigma \to B^\infty_\sigma$ is the classical Whittaker-Kotel'nikov-Shannon operator, for $g \in B^\infty_\sigma, \sigma < \pi w$, defined by

$$(S^{\text{sinc}}_w g)(t) := \sum_{k=-\infty}^{\infty} g\left(\frac{k}{w}\right) \text{sinc}(wt - k),$$

where the kernel function $\text{sinc}(t) := \frac{\sin \pi t}{\pi t}$. The fact that for $S^{\text{sinc}}_w : B^\infty_\sigma \to B^\infty_\sigma$ the set of fixed points is $B^\infty_\sigma, \sigma < \pi w$, is the statement of the famous Whittaker-Kotel'nikov-Shannon theorem: if $g \in B^\infty_\sigma, \sigma < \pi w$, then

$$(S^{\text{sinc}}_w g)(t) = g(t).$$

We define approximation operators in this paper as follows.

Definition 2.6. *The cosine-type operators $\widetilde{U}_{\sigma,h,\mathbf{a}} : A_\sigma \to X$ are defined by*

$$\widetilde{U}_{\sigma,h,\mathbf{a}} g := \sum_{k=0}^{m} a_k C_{kh}(S_\sigma g), \; h \geq 0, \tag{5}$$

where $\mathbf{a} = (a_0, ..., a_m) \in \mathbb{R}^{m+1}$, $m \geq 1$, and

$$\sum_{k=0}^{m} a_k = 1. \tag{6}$$

The two typical cases of the operators (5) are the Rogosinski-type operators and the Blackman-type operators.

Definition 2.7. *The Rogosinski-type operators $\widetilde{R}_{\sigma,h,\mathbf{a}} : A_\sigma \to X$ are defined by*

$$\widetilde{R}_{\sigma,h,\mathbf{a}}g := \sum_{k=0}^{m} a_k C_{kh}\left(S_\sigma g\right), \ h \geq 0,$$

where (6) is supposed to be valid and, moreover,

$$\sum_{k=0}^{m} a_k \cos\frac{k\pi}{2} = 0. \tag{7}$$

Remark 2.8. *The case $\boldsymbol{a} = (0,1) \in \mathbb{R}^2$ leads to the original Rogosinski operator $R_{n,h} : C_{2\pi} \to C_{2\pi}$ which in trigonometric approximation was introduced by W. W. Rogosinski [14] and afterwards elaborated by S. B. Stečkin in [16], see also [3], [15], [17]. In our notations the classical Rogosinski means (1) are in the form $R_n(f,x) = C_{\frac{\pi}{2(n+1)}}(S_n f, x)$.*

Definition 2.9. *The Blackman-type operators $\widetilde{B}_{\sigma,h,\mathbf{a}} : A_\sigma \to X$ are defined by*

$$\widetilde{B}_{\sigma,h,\mathbf{a}}g := \sum_{k=0}^{m} a_k C_{kh}\left(S_\sigma g\right), \ h \geq 0,$$

where (6) is supposed to be valid and, moreover,

$$\sum_{k=0}^{m} (-1)^k a_k = 0. \tag{8}$$

Remark 2.10. *In Definition 2.9 the Blackman operator in the case $\boldsymbol{a} = (1/2, 1/2) \in \mathbb{R}^2$ is called the Hann operator, denoted here by $\widetilde{H}_{\sigma,h}$, and that is in Communications Engineering the original Hann operator [1]. If the projector operator $S_\sigma : A_\sigma \to A_\sigma$ is translation invariant, i.e. $C_h S_\sigma = S_\sigma C_h$, then it is easy to prove that $\widetilde{R}^2_{\sigma,h} = \widetilde{H}_{\sigma,2h}$ and $\widetilde{B}_{\sigma,h,3/8} = \widetilde{H}^2_{\sigma,h}$, where $\widetilde{B}_{\sigma,h,3/8} \equiv \widetilde{B}_{\sigma,h,\boldsymbol{a}}$ with $\boldsymbol{a} = (3/8, 1/2, 1/8) \in \mathbb{R}^3$.*

The following Bounded Linear Transformation Theorem allows us to define our approximation operators on the whole space X.

Theorem 2.11. *([8], Sect. 8.2, 8.3) Let $A \subset X$ be a dense subset of a Banach space X and $\widetilde{B} : A \to X$ be a bounded linear operator with the operator norm $\|\widetilde{B}\|$. Then $\widetilde{B}$ has the unique bounded linear extension $B : X \to X$ with $\|B\| = \|\widetilde{B}\|$. For $f \in X$ the operator $B \in [X]$ is defined by $Bf = \lim_{\sigma\to\infty} \widetilde{B}g_\sigma$, where $\{g_\sigma\}_{\sigma>0} \subset A$ is an arbitrary family with $f = \lim_{\sigma\to\infty} g_\sigma$.*

Thus, if the approximation operators $\widetilde{U}_{\sigma,h,\mathbf{a}} : A_\sigma \to X$ are defined, their extensions will be denoted by $U_{\sigma,h,\mathbf{a}} : X \to X$, correspondingly.

3. Order of approximation by general cosine-type operators

In this section we discuss the order of approximation of the general cosine-type operators by the modulus of continuity. Let us define an accompaning operator to the approximation operator (5).

Definition 3.1. *Let for* $\mathbf{a} = (a_0, ..., a_m) \in \mathbb{R}^{m+1}$, $m \in \mathbb{N}$ *with*

$$\sum_{k=0}^{m} a_k = 1 \tag{9}$$

the operators $\Theta_{h,\mathbf{a}} \in [X]$ *be defined with*

$$\Theta_{h,\mathbf{a}} f := \sum_{k=0}^{m} a_k C_{kh} f, \quad f \in X. \tag{10}$$

Lemma 3.2. *For every* $f \in X$ *for the operator* $U_{\sigma,h,\mathbf{a}} : X \to X$ *it holds that*

$$\left\| U_{\sigma,h,\mathbf{a}} f - f \right\| \le \left(\left\| U_{\sigma,h,\mathbf{a}} \right\|_{[X]} + \|\Theta_{h,\mathbf{a}}\|_{[X]} \right) E_\sigma(f) + \left\| \Theta_{h,\mathbf{a}} f - f \right\|. \tag{11}$$

Proof. Let $g_* \in A_\sigma$ be an element of the best approximation of $f \in X$. Since $S_\sigma g_* = g_*$, by Definitions 2.6 and 3.1

$$\Theta_{h,\mathbf{a}} g_* = \widetilde{U}_{\sigma,h,\mathbf{a}} g_*,$$

and we get

$$\|\widetilde{U}_{\sigma,h,\mathbf{a}} g_* - f\| \le \|\Theta_{h,\mathbf{a}} g_* - \Theta_{h,\mathbf{a}} f\| + \|\Theta_{h,\mathbf{a}} f - f\|. \tag{12}$$

For the first term in the right-hand side of (12) we obtain

$$\|\Theta_{h,\mathbf{a}} g_* - \Theta_{h,\mathbf{a}} f\| \le \|\Theta_{h,\mathbf{a}}\|_{[X]} E_\sigma(f). \tag{13}$$

The operator $\widetilde{U}_{\sigma,h,\mathbf{a}} : A_\sigma \to X$ and its extension $U_{\sigma,h,\mathbf{a}} : X \to X$ coincide on the subspace A_σ, therefore

$$\|U_{\sigma,h,\mathbf{a}} f - f\| \le \|U_{\sigma,h,\mathbf{a}} f - \widetilde{U}_{\sigma,h,\mathbf{a}} g_*\| + \|\widetilde{U}_{\sigma,h,\mathbf{a}} g_* - f\| \le \|U_{\sigma,h,\mathbf{a}}\|_{[X]} E_\sigma(f) + \|\widetilde{U}_{\sigma,h,\mathbf{a}} g_* - f\|.$$

Combining all inequalities together we obtain the assertion. □

The rest of this Section deals with the problem how to estimate the term $\left\|\Theta_{h,\mathbf{a}} f - f\right\|$ in (11). The original trigonometric Rogosinski means (1) are defined, in our notations, as (see, e.g., [3], formula (1.3.9), Th. 2.4.8)

$$R_n f = C_{\frac{\pi}{2(n+1)}} \left(S_n f \right),$$

which is the special case of (5) with $m = 1$, $a_0 = 0$, $a_1 = 1$. Therefore, in the present case $\left(h = \frac{\pi}{2(n+1)}\right)$, by Definition 2.1

$$\left\|\Theta_{h,a} f - f\right\| = \left\|C_h f - f\right\| \le \omega(f, \delta) \;\; (0 \le h \le \delta).$$

In the following we are interested in whether it is possible to infer the higher order modulus of continuity. The next result is an important tool for what follows.

Lemma 3.3. *Let* $\mathbf{u} = (u_0, ..., u_m) \in \mathbb{R}^{m+1}$, $m \in \mathbb{N}$ *satisfy*

$$\sum_{k=0}^{m} u_k = 0. \tag{14}$$

Then

$$U_m := \sum_{k=0}^{m} u_k C_{kh} = 2 \sum_{k=0}^{m-1}{}' C_{kh} \left(C_h - I \right) \sum_{l=k+1}^{m} (l-k)\, u_l \tag{15}$$

($\sum'$ means here and in the following that the first term is halved).

Proof. For $m = 1$ the equation (15) follows immediately. Let $m \geq 2$ and define

$$D_0 := \frac{1}{2}I, \;\; D_{lh} := \frac{1}{2}I + C_h + ... + C_{lh}, \; l = 1, 2, ...$$

Then by induction on $j \geq 2$ we obtain

$$\sum_{l=0}^{j-1} D_{lh} = \frac{j}{2}I + \sum_{l=1}^{j-1} (j-l)\, C_{lh}. \tag{16}$$

Multiplying (16) by $2(C_h - I)$ we get

$$\begin{aligned} 2\,(C_h - I) \sum_{l=0}^{j-1} D_{lh} &= j\,(C_h - I) + 2 \sum_{l=1}^{j-1} (j-l)\,(C_h - I)\, C_{lh} \\ &= j\,(C_h - I) + \sum_{l=1}^{j-1} (j-l) \Big(C_{(l+1)h} + C_{(l-1)h} - 2C_{lh} \Big). \end{aligned}$$

For the sum on the right-hand side we have

$$\begin{aligned} & j\,(C_h - I) + \sum_{l=1}^{j-1} (j-l) \Big(C_{(l+1)h} - C_{lh} - (C_{lh} - C_{(l-1)h}) \Big) \\ = \;& \sum_{l=1}^{j} (j-l+1) \Big(C_{lh} - C_{(l-1)h} \Big) - \sum_{l=1}^{j-1} (j-l) \Big(C_{lh} - C_{(l-1)h} \Big) \\ = \;& C_{jh} - C_{(j-1)h} + \sum_{l=1}^{j-1} \Big(C_{lh} - C_{(l-1)h} \Big) = C_{jh} - I. \end{aligned}$$

Therefore,

$$2\,(C_h - I) \sum_{l=0}^{j-1} D_{lh} = C_{jh} - I,$$

which together with (16) gives the equation

$$C_{jh} - I = (C_h - I) \left(jI + 2 \sum_{l=1}^{j-1} (j-l)\, C_{lh} \right), \; j \geq 2. \tag{17}$$

By the assumption (14) and equation (17) we may write

$$U_m \;\; := \;\; \sum_{j=0}^{m} u_j C_{jh} = \sum_{j=1}^{m} u_j \Big(C_{jh} - I \Big) = (C_h - I) \left(\sum_{j=1}^{m} j u_j I + 2 \sum_{j=2}^{m} u_j \sum_{l=1}^{j-1} (j-l)\, C_{lh} \right).$$

The equation (15) follows by changing the order of summation in the sum

$$\sum_{j=2}^{m} \sum_{l=1}^{j-1} ... = \sum_{l=1}^{m-1} \sum_{j=l+1}^{m} ...$$

□

Proposition 3.4. *For the operators $\Theta_{h,\mathbf{a}} : X \to X$ in (10) the following equation holds*

$$\Theta_{h,\mathbf{a}} - I = 2 \sum_{k=0}^{m-1}{}' C_{kh} \left(C_h - I\right) \sum_{l=k+1}^{m} (l-k)\, a_l. \tag{18}$$

Proof. Take $u_0 = a_0 - 1$, $u_k = a_k$ $(k = 1, ..., m)$ in Lemma 3.3. □

For that follows we need certain combinatorial identities.

Lemma 3.5. *For $l \geq p,\ l, p \in \mathbb{N}$ the following equations hold*

$$\sum_{k=0}^{l-p} \binom{l+p-k-1}{2p-1} = \binom{l+p}{2p}, \tag{19}$$

$$\sum_{k=0}^{l-p}{}' \binom{l+p-k-1}{2p-1} = \frac{l}{2p}\binom{l+p-1}{2p-1} \tag{20}$$

Proof. The equation (19) is a modification of the known equation (see [6], p. 174, table 174)

$$\sum_{k=0}^{n} \binom{r+k}{k} = \binom{r+n+1}{n},\ r \in \mathbb{R},\ n = 0, 1, 2, ...$$

Indeed,

$$\begin{aligned} S_{lp} \quad &:= \quad \sum_{k=0}^{l-p} \binom{l+p-k-1}{2p-1} = \sum_{k=0}^{l-p} \binom{l+p-k-1}{l-p-k} = [l-p-k = k'] \\ &= \quad \sum_{k'=0}^{l-p} \binom{2p-1+k'}{k'} = \binom{l+p}{l-p} = \binom{l+p}{2p}. \end{aligned}$$

The equation (20) follows from (19):

$$\begin{aligned} \sum_{k=0}^{l-p}{}' \binom{l+p-k-1}{2p-1} \quad &= \quad S_{lp} - \frac{1}{2}\binom{l+p-1}{2p-1} = \binom{l+p}{2p} - \frac{1}{2}\binom{l+p-1}{2p-1} \\ &= \quad \binom{l+p-1}{2p-1}\left(\frac{l+p}{2p} - \frac{1}{2}\right) = \frac{l}{2p}\binom{l+p-1}{2p-1}. \end{aligned}$$

□

Now we are able to estimate the order of the norm $\left\|\Theta_{h,\mathbf{a}} f - f\right\|$.

Proposition 3.6. *For every $f \in X$ for the operator (10) it holds that*

$$\left\|\Theta_{h,\mathbf{a}} f - f\right\| \leq \max(T, 1)\omega(f, \delta) \sum_{l=1}^{m} l^2 \left|a_l\right|,$$

where $h \leq \delta$ and $T > 0$ is the uniform bound of the cosine operator function C_h, i.e. $\|C_h\|_{[X]} \leq T$ for every $h > 0$.

Proof. Using (18) and the definition of the modulus of continuity yields

$$\begin{aligned}\left\|\Theta_{h,\mathbf{a}}f-f\right\| &\leq \omega(f,h)\left(\sum_{l=1}^{m} l\,|a_l| + 2T\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}(l-k)|a_l|\right)\\ &= \omega(f,h)\left(\sum_{l=1}^{m} l|a_l| + 2T\sum_{l=2}^{m}|a_l|\sum_{k=1}^{l-1}(l-k)\right).\end{aligned}$$

Suppose $T \geq 1$, then by (19) with $p = 1$ we obtain

$$\left\|\Theta_{h,a}f-f\right\| \leq T\omega(f,h)\left(\sum_{l=1}^{m} l\,|a_l| + \sum_{l=1}^{m}(l-1)l\,|a_l|\right)$$

and we are done. The case $T \leq 1$ follows similarly. □

We give our first main result, which corresponds to the classical trigonometric Rogosinski means.

Theorem 3.7. *Assume (9) is valid. Then for every $f \in X$ for the operators $U_{\sigma,h,\mathbf{a}} : X \to X$ we have*

$$\left\|U_{\sigma,h,\mathbf{a}}f-f\right\| \leq \left(\left\|U_{\sigma,h,\mathbf{a}}\right\|_{[X]} + \left\|\Theta_{h,\mathbf{a}}\right\|_{[X]}\right)E_\sigma(f) + \max(T,1)\omega(f,\delta)\sum_{l=1}^{m} l^2\,|a_l|$$

for every $0 < h \leq \delta$.

Proof. Use Lemma 3.2 and Proposition 3.6. □

Remark 3.8. *In trigonometric approximation, in space $X_{2\pi}$ (which is $C_{2\pi}$ – the space of 2π-periodic continuous functions on $\mathbb{R}$ or $L^p_{2\pi}$ – the space of 2π-periodic functions, Lebesgue integrable to the p-th power over $(-\pi,\pi)$) the corresponding operator cosine function (2) is uniformly bounded by $T = 1$. As mentioned before, the trigonometric Rogosinski means are defined by $\boldsymbol{a} = (0,1)$ and $h = \frac{\pi}{2(n+1)}$. Therefore, Theorem 3.7 yields (compare [3], Theorem 2.4.8)*

$$\left\|R_nf-f\right\| \leq \left(\|R_n\|_{[X_{2\pi}]} + 1\right)E_nf + \omega\left(f,\frac{\pi}{2(n+1)}\right).$$

Recall that here $\omega(f,\delta) = \frac{1}{2}\widetilde{\omega}_2(f,\delta)$, where $\widetilde{\omega}_2$ is the ordinary modulus of continuity of order 2, that is $\widetilde{\omega}_2$ is defined by the central differences.

Our next intention is to prove a similar result as Theorem 3.7, where instead of $\omega(f,h)$ the modulus of continuity of higher order is used. We need the following combinatorial lemma.

Lemma 3.9. *The following equation holds*

$$\sum_{k=1}^{l-p} k\binom{l-k}{p} = \binom{l+1}{p+2},\ l \geq p+1,\ \ p = 0,1,2,\ldots;\ \ l \in \mathbb{N}.$$

Proof. By [6], p.169, equation (5.26), we may write

$$\sum_{k=0}^{l}\binom{l-k}{m}\binom{q+k}{n} = \binom{l+q+1}{m+n+1}\ (m,l = 0,1,2,\ldots;\ n \geq q \geq 0).$$

Taking $n = 1$ and $q = 0$ we obtain

$$\sum_{k=1}^{l} k\binom{l-k}{m} = \binom{l+1}{m+2}.$$

Since by definition $\binom{l-k}{m} = 0$ for $m > l-k$, we get

$$\sum_{k=1}^{l-m} k\binom{l-k}{m} = \binom{l+1}{m+2}.$$

□

Our main tool, working with the higher order modulus of continuity, is Lemma 3.3. Applying Lemma 3.3 we obtain the following result.

Proposition 3.10. *Assume for* $\mathbf{a} = (a_0, ..., a_m) \in \mathbb{R}^{m+1}$, $m \geq 2$, *that (9) is valid and, moreover, suppose that*

$$\sum_{l=1}^{m} l^2 a_l = 0. \tag{21}$$

Then

$$\Theta_{h,\mathbf{a}} - I = 4\sum_{k=0}^{m-2}{}' C_{kh}(C_h - I)^2 \sum_{l=k+2}^{m} \binom{l-k+1}{3} a_l. \tag{22}$$

Proof. Denote the sum in (18) by

$$V_{m-1} := \sum_{k=0}^{m-1}{}' C_{kh} \sum_{l=k+1}^{m} (l-k)a_l \equiv \sum_{k=0}^{m-1} u_k C_{kh}, \tag{23}$$

i.e.

$$u_0 = \frac{1}{2}\sum_{l=1}^{m} l a_l, \quad u_k = \sum_{l=k+1}^{m} (l-k)a_l,\ k = 1, ..., m-1. \tag{24}$$

If we choose the coefficients a_l in (23) in such a way that for the coefficients u_k in (24) the assumption (14) of Lemma 3.3 for $m \longmapsto m-1$ is fulfilled, then by Lemma 3.3 we obtain

$$V_{m-1} = 2(C_h - I)\sum_{k=0}^{m-2}{}' C_{kh} \sum_{l=k+1}^{m-1} (l-k)u_l.$$

Therefore, by Proposition 3.4,

$$\Theta_{h,\mathbf{a}} - I = 4(C_h - I)^2 \sum_{k=0}^{m-2}{}' C_{kh} \sum_{l=k+1}^{m-1} (l-k)u_l. \tag{25}$$

Let us calculate by (24) the sum

$$\sum_{k=0}^{m-1} u_k = \frac{1}{2}\sum_{l=1}^{m} l a_l + \sum_{k=1}^{m-1}\sum_{l=k}^{m-1} (l+1-k)a_{l+1},$$

where, interchanging the order of summation and using Lemma 3.5, equation (19) with $p = 1$, we get

$$\sum_{k=1}^{m-1}\sum_{l=k}^{m-1} (l+1-k)a_{l+1} = \sum_{l=1}^{m-1} a_{l+1} \sum_{k=0}^{l-1} (l-k) = \frac{1}{2}\sum_{l=1}^{m-1} l(l+1)a_{l+1}.$$

So, by the assumption (21) we obtain

$$\sum_{k=0}^{m-1} u_k = \frac{1}{2}\sum_{l=0}^{m-1}(l+1)a_{l+1} + \frac{1}{2}\sum_{l=0}^{m-1} l(l+1)a_{l+1} = \frac{1}{2}\sum_{l=1}^{m} l^2 a_l = 0$$

and therefore the assumption of Lemma 3.3 is fulfilled. Finally, using (24), we have to calculate in (25) the sum ($k = 0, ..., m-2$)

$$\begin{aligned} \sum_{l=k+1}^{m-1}(l-k)u_l &= \sum_{l=k+1}^{m-1}(l-k)\sum_{j=l}^{m-1}(j+1-l)a_{j+1} \\ &= \sum_{j=k+1}^{m-1} a_{j+1}\sum_{l=k+1}^{j}(l-k)(j+1-l) = \sum_{j=k+2}^{m} a_j \sum_{l=1}^{j-k-1} l\binom{j-k-l}{1} \\ &= \sum_{j=k+2}^{m}\binom{j-k+1}{3}a_j. \end{aligned}$$

Here the last equality is valid due to Lemma 3.9, and so (22) is proved by (25). □

Now we shall generalize the result of Proposition 3.10.

Proposition 3.11. *Let us fix the integer q with $m \geq q \geq 2$. Suppose the coefficients $\mathbf{a} = (a_0, ..., a_m)$ satisfy the condition (9) and for every $p = 1, 2, 3, ..., q-1$ we have*

$$\sum_{l=p}^{m} l\binom{l+p-1}{2p-1}a_l = 0. \tag{26}$$

Then

$$\Theta_{h,\mathbf{a}} - I = 2^q \sum_{k=0}^{m-q}{}' C_{kh}(C_h - I)^q \sum_{l=k+q}^{m}\binom{l-k+q-1}{2q-1}a_l. \tag{27}$$

Proof. To prove by induction on q first we see that the case $q = 2$ is exactly the result of Proposition 3.10. Suppose (26) is valid for every $p = 1, 2, 3, ..., q$ and suppose (27) holds. Denote the sum in (27) by

$$T_{m-q} := \sum_{k=0}^{m-q}{}' C_{kh}\sum_{l=k+q}^{m}\binom{l-k+q-1}{2q-1}a_l \equiv \sum_{k=0}^{m-q} u_k C_{kh} \tag{28}$$

with

$$u_0 = \frac{1}{2}\sum_{l=q}^{m}\binom{l+q-1}{2q-1}a_l,\quad u_k = \sum_{l=k+q}^{m}\binom{l-k+q-1}{2q-1}a_l,\quad (k = 1, ..., m-q). \tag{29}$$

Let us choose the coefficients a_l in (28) in such a way that for the coefficients u_k in (29) the assumption (14) of Lemma 3.3 for $m \longmapsto m-q$ is fulfilled. Then by Lemma 3.3

$$T_{m-q} = 2(C_h - I)\sum_{k=0}^{m-q-1}{}' C_{kh}\sum_{l=k+1}^{m-q}(l-k)u_l.$$

Hence, by (28) and (27) (as the assumption of the induction) we obtain

$$\Theta_{h,\mathbf{a}} - I = 2^{q+1}(C_h - I)^{q+1}\sum_{k=0}^{m-q-1}{}' C_{kh}\sum_{l=k+1}^{m-q}(l-k)u_l. \tag{30}$$

Let us calculate by (29) the sum

$$\begin{aligned} U_{m-q} &:= \sum_{k=0}^{m-q} u_k = \frac{1}{2}\sum_{l=q}^{m}\binom{l+q-1}{2q-1}a_l + \sum_{k=1}^{m-q}\sum_{l=k}^{m-q}\binom{l-k+2q-1}{2q-1}a_{l+q} \\ &= \frac{1}{2}\sum_{l=0}^{m-q}\binom{l+2q-1}{2q-1}a_{l+q} + \sum_{l=1}^{m-q}a_{l+q}\sum_{k=1}^{l}\binom{l-k+2q-1}{2q-1}. \end{aligned}$$

Since Lemma 3.5, equation (19) with $p = q$, $l \longmapsto q + l - 1$, yields

$$\sum_{k=1}^{l}\binom{l-k+2q-1}{2q-1} = \sum_{k=0}^{l-1}\binom{l-k+2q-2}{2q-1} = \binom{l+2q-1}{2q},$$

we obtain

$$\begin{aligned} U_{m-q} &= \frac{1}{2}\sum_{l=0}^{m-q}\binom{l+2q-1}{2q-1}a_{l+q} + \sum_{l=1}^{m-q}a_{l+q}\binom{l+2q-1}{2q} \\ &= \frac{1}{2}a_q + \sum_{l=1}^{m-q}a_{l+q}\left(\frac{1}{2}\binom{l+2q-1}{2q-1} + \binom{l+2q-1}{2q}\right) \\ &= \frac{1}{2q}\sum_{l=0}^{m-q}(l+q)\binom{l+2q-1}{2q-1}a_{l+q} = \frac{1}{2q}\sum_{l=q}^{m} l\binom{l+q-1}{2q-1}a_l. \end{aligned}$$

We see that the assumption of Lemma 3.3 is fulfilled by condition (26) for $p = q + 1$, i.e.

$$\sum_{k=0}^{m-q} u_k = \frac{1}{2q}\sum_{l=q}^{m} l\binom{l+q-1}{2q-1}a_l = 0,$$

hence, (30) is valid. To finish the proof we calculate by (29) in (30) the sum ($k = 0, ..., m - q - 1$)

$$\sum_{l=k+1}^{m-q}(l-k)u_l = \sum_{l=k+1}^{m-q}(l-k)\sum_{j=l}^{m-q}\binom{j-l+2q-1}{2q-1}a_{j+q}$$

$$= \sum_{j=k+1}^{m-q}a_{j+q}\sum_{l=k+1}^{j}(l-k)\binom{j-l+2q-1}{2q-1} = \sum_{j=k+q+1}^{m}a_j\sum_{l=k+1}^{j-q}(l-k)\binom{j-l+q-1}{2q-1}.$$

Here for the inner sum by Lemma 3.9 with $l = j - k + q - 1,\ p = 2q - 1$ we obtain

$$\sum_{l=k+1}^{j-q}(l-k)\binom{j-l+q-1}{2q-1} = \sum_{l=1}^{j-k-q} l\binom{j-l-k+q-1}{2q-1} = \binom{j-k+q}{2q+1}.$$

Finally,

$$\sum_{l=k+1}^{m-q}(l-k)u_l = \sum_{j=k+q+1}^{m}a_j\binom{j-k+q}{2q+1},$$

and (30) takes the form

$$\Theta_{h,\mathbf{a}} - I = 2^{q+1}(C_h - I)^{q+1}\sum_{k=0}^{m-q-1}{}' C_{kh}\sum_{j=k+q+1}^{m}\binom{j-k+q}{2q+1}a_j,$$

which is (27) for $q \longmapsto q + 1$. □

Proposition 3.12. *Under the assumptions of Proposition 3.11 we get*

$$\left\|\Theta_{h,\mathbf{a}}f - f\right\| \le \frac{2^{q-1}C_{T,q}}{q}\omega_q(f,h)\sum_{l=q}^{m} l|a_l|\binom{l+q-1}{2q-1},$$

where $C_{T,q} = \max(T,1)$ *for* $2 \le q \le m-1$ *and* $C_{T,q} = 1$ *for* $q = m$.

Proof. The case $q = m$ follows immediately from Proposition 3.11. Let $2 \le q \le m-1$. Then by Proposition 3.11

$$\left\|\Theta_{h,\mathbf{a}}f - f\right\| \le 2^{q-1}\omega_q(f,h)\sum_{l=q}^{m}\binom{l+q-1}{2q-1}|a_l| + 2^qT\omega_q(f,h)\sum_{k=1}^{m-q}\sum_{l=k+q}^{m}\binom{l-k+q-1}{2q-1}|a_l|.$$

Suppose $T \ge 1$. Then

$$\left\|\Theta_{h,\boldsymbol{a}}f - f\right\| \le 2^qT\omega_q(f,h)\left(\frac{1}{2}\sum_{l=q}^{m}\binom{l+q-1}{2q-1}|a_l| + \sum_{k=1}^{m-q}\sum_{l=k+q}^{m}\binom{l-k+q-1}{2q-1}|a_l|\right).$$

Let us calculate the sum in brackets:

$$\begin{aligned}
V_{m-q} &:= \frac{1}{2}\sum_{l=q}^{m}\binom{l+q-1}{2q-1}|a_l| + \sum_{k=1}^{m-q}\sum_{l=k}^{m-q}\binom{l-k+2q-1}{2q-1}|a_{l+q}| \\
&= \frac{1}{2}\sum_{l=0}^{m-q}\binom{l+2q-1}{2q-1}|a_{l+q}| + \sum_{l=1}^{m-q}|a_{l+q}|\sum_{k=1}^{l}\binom{l-k+2q-1}{2q-1} \\
&= \frac{1}{2}|a_q| + \sum_{l=1}^{m-q}|a_{l+q}|\left(\frac{1}{2}\binom{l+2q-1}{2q-1} + \sum_{k=1}^{l}\binom{l-k+2q-1}{2q-1}\right) \\
&= \frac{1}{2}|a_q| + \sum_{l=1}^{m-q}|a_{l+q}|\sum_{k=0}^{l}{}'\binom{l-k+2q-1}{2q-1}.
\end{aligned}$$

By Lemma 3.5, equation (20) ($1 \le l \le m-q$), we obtain

$$\sum_{k=0}^{l}{}'\binom{l-k+2q-1}{2q-1} = \sum_{k=0}^{l+q-q}{}'\binom{l-k+2q-1}{2q-1} = \frac{l+q}{2q}\binom{l+2q-1}{2q-1},$$

hence

$$V_{m-q} = \frac{1}{2}|a_q| + \frac{1}{2q}\sum_{l=q+1}^{m} l|a_l|\binom{l+q-1}{2q-1} = \frac{1}{2q}\sum_{l=q}^{m} l|a_l|\binom{l+q-1}{2q-1},$$

which proves the assertion in the case $T \ge 1$. The case $T \le 1$ follows similarly. □

The next main result improves the order of approximation in Theorem 3.7.

Theorem 3.13. *Let the cosine-type operators* $U_{\sigma,h,\boldsymbol{a}} : X \to X$ *be bounded linear extensions of the operators* $\widetilde{U}_{\sigma,h,\boldsymbol{a}} : A_\sigma \to X$,

$$\widetilde{U}_{\sigma,h,\boldsymbol{a}}g := \sum_{k=0}^{m} a_kC_{kh}(S_\sigma g),\ h > 0,\ m \in \mathbb{N},$$

where the cosine operator functions $C_h : X \to X$ are equibounded, i.e. $\|C_h\|_{[X]} \leq T$. Let us fix the integer q with $m \geq q \geq 2$ and suppose that the coefficients $\boldsymbol{a} = (a_0, ..., a_m)$ satisfy the equalities

$$\sum_{k=0}^{m} a_k = 1$$

and

$$\sum_{k=p}^{m} k \binom{k+p-1}{2p-1} a_k = 0$$

for every $p = 1, 2, .., q-1$. Then for every $f \in X$ we have

$$\left\| U_{\sigma,h,\mathbf{a}} f - f \right\| \leq \left(\left\| U_{\sigma,h,\mathbf{a}} \right\|_{[X]} + \left\| \Theta_{h,\mathbf{a}} \right\|_{[X]} \right) E_\sigma(f) + \frac{2^{q-1} C_{T,q}}{q} \omega_q(f,h) \sum_{l=q}^{m} l|a_l| \binom{l+q-1}{2q-1}, \tag{31}$$

where $C_{T,q} = \max(T, 1)$ for $2 \leq q \leq m-1$ and $C_{T,q} = 1$ for $q = m$.

Proof. Use Lemma 3.2 and Proposition 3.12. □

4. Applications

In this Section we apply Theorem 3.13 to special cases of operators including Blackman- and Rogosinski-type operators. Note that in these cases the estimate (31) is much simplified.

Example 1 a) (Blackman) Take in Theorem 3.13 $m = 3$ and $q = 3$. Then the conditions (9), (8) and (26) give us

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & 2^2 & 3^2 \\ 0 & 0 & 1 & 6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system corresponds to the system in [10] (see Th.3 and Example 1), which was proved by completely different method.

After solving the linear system we get the unique solution $\mathbf{a} = \frac{1}{32}(22, 15, -6, 1)$. From (31) we obtain

$$\left\| U_{\sigma,h,\mathbf{a}} f - f \right\| \leq \left(\left\| U_{\sigma,h,\mathbf{a}} \right\|_{[X]} + \left\| \Theta_{h,\mathbf{a}} \right\|_{[X]} \right) E_\sigma(f) + \frac{4}{3} \omega_3(f,h) \sum_{l=3}^{3} l\,|a_l| \binom{l+2}{5},$$

for which by (10) we have

$$\left\| \Theta_{h,\mathbf{a}} \right\|_{[X]} \leq |a_0| + T \sum_{k=1}^{m} |a_k|.$$

Using the coefficients $\mathbf{a} = \frac{1}{32}(22, 15, -6, 1)$ it follows

$$\left\| U_{\sigma,h,\mathbf{a}} f - f \right\| \leq \left(\left\| U_{\sigma,h,\mathbf{a}} \right\|_{[X]} + \frac{11}{16}(1+T) \right) E_\sigma(f) + \frac{1}{8} \omega_3(f,h).$$

b) (Rogosinski) Take in Theorem 3.13 $m = 3$ and $q = 3$. Then the condition (9), (7) and (26) give us

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 2^2 & 3^2 \\ 0 & 0 & 1 & 6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

After solving the linear system we get the unique solution $\mathbf{a} = \frac{1}{4}(-6, 15, -6, 1)$. From (31) using the coefficients and (10) follows

$$\left\|U_{\sigma,h,\mathbf{a}}f - f\right\| \le \left(\left\|U_{\sigma,h,\mathbf{a}}\right\|_{[X]} + \frac{1}{2}(3 + 11T)\right) E_\sigma(f) + \omega_3(f, h).$$

Example 2 a) (Blackman) Take in Theorem 3.13 $m = 3$ and $q = 2$. The conditions (9), (8) and (26) give together (compare [10], Example 2) the system

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & 2^2 & 3^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The solution of the linear system is $\mathbf{a} = \frac{1}{32}(22 - 2C, 15 + C, -6 + 2C, 1 - C)$, where C is a parameter. Using the solution and equation (10) for (31) we get

$$\left\|U_{\sigma,h,\mathbf{a}}f - f\right\| \le \left(\left\|U_{\sigma,h,\mathbf{a}}\right\|_{[X]} + \frac{1}{32}(|22 - C| + T(|15 + C| + |-6 + 2C| + |1 - C|))\right) E_\sigma(f)$$

$$+\frac{1}{16}\max(T, 1)\omega_2(f, h)(|-6 + 2C| + 6|1 - C|).$$

If we put $C = 3$, which minimizes the constants, then for $T \ge \frac{1}{4}$ it follows

$$\left\|U_{\sigma,h,\mathbf{a}}f - f\right\| \le \left(\left\|U_{\sigma,h,\mathbf{a}}\right\|_{[X]} + \frac{1}{32}(19 + 20T)\right) E_\sigma(f) + \frac{3}{4}\max(T, 1)\omega_2(f, h).$$

b) (Rogosinski) Take in Theorem 3.13 $m = 3$ and $q = 2$. Then the condition (9), (7) and (26) together give us the system

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & 2^2 & 3^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The solution of the system is $\mathbf{a} = \frac{1}{2}(-1 - 8C, 4 + 14C, -1 - 8C, 2C)$, where C is a parameter. Using the solution and (10) for (31) we get

$$\left\|U_{\sigma,h,\mathbf{a}}f - f\right\| \le \left(\left\|U_{\sigma,h,\mathbf{a}}\right\|_{[X]} + \frac{1}{2}(|-1 - 8C| + T(|4 + 14C| + |-1 - 8C| + |2C|))\right) E_\sigma(f)$$

$$+\max(T, 1)\omega_2(f, h)(|-1 - 8C| + 12|C|).$$

If we put $C = -\frac{1}{8}$, then for $0 < T \le 2$ it follows

$$\left\|U_{\sigma,h,\mathbf{a}}f - f\right\| \le \left(\left\|U_{\sigma,h,\mathbf{a}}\right\|_{[X]} + \frac{5}{4}T\right) E_\sigma(f) + \frac{3}{2}\max(T, 1)\omega_2(f, h).$$

CONCLUSION We introduced the general cosine-type approximation operators, in particular case the Blackman- and Rogosinski-type operators, using the cosine operator function. This abstract setting is useful, because now we were able to consider different approximation problems from the unique point of view.

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