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On q-Opial Type Inequality for Quantum Integral

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Abstract. In this paper, we establish some q-Opial type inequalities and generalization of q-Opial type inequalities.

1. Introduction

Inequalities which involve integrals of functions and their derivatives, whose study has a history of about one century, are of great importance in mathematics, with far-reaching applications in the theory of differential equations, approximations and probability, among others. This class of inequalities includes the Wirtinger, Lyapunov, Landau-Kolmogorov, and Hardy types to which an abundance of literature, including several monographs, have been devoted. Of these inequalities, the earliest one which appeared in print is believed to be a Wirtinger type inequality by L. Sheeffer in 1885 (actually before the result by Wirtinger), which found its motivation in the calculus of variations. Improvements, generalizations, extensions, discretizations, and new applications of these inequalities are constantly being found, making their study an extremely prolific field. These inequalities and their manifold manifestations occupy a central position in mathematical analysis and its applications, [1]-[3], [9], [11], [14], [15].

In the year 1960, Opial [1] established the following interesting integral inequalities :

Theorem 1.1. *Let $x(t) \in C^{(1)}[0,h]$ be such that $x(t) > 0$ in $(0,h)$. Then, the following inequalities holds:*
i) If $x(0) = x(h) = 0$, then

$$\int_0^h |x(t)x'(t)|\,dt \leq \frac{h}{4}\int_0^h |x'(t)|^2\,dt. \tag{1}$$

ii) If $x(0) = 0$, then

$$\int_0^h |x(t)x'(t)|\,dt \leq \frac{h}{2}\int_0^h |x'(t)|^2\,dt. \tag{2}$$

In (1), the constant $h/4$ is the best possible.

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Opial's inequality and its generalizations, extensions and discretizations, play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations. Over the last twenty years a large number of papers have been appeared in the literature which deals with the simple proofs, various generalizations and discrete analogues of Opial inequality and its generalizations, see [4]-[6], [8], [12]-[20].

In this paper we obtain Opial type inequalities for quantum integral. If $q \to 1^-$ is taken, all the results we have obtained provide valid results for classical analysis.

2. Preliminaries of q-Calculus

Throughout this paper, let $0 < q < 1$ be a constant. The following definitions for q- derivative and q-integral of a function f on $[0, h]$. Here and further we use the following notations(see [10]):

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + ... + q^{n-1}.$$

Definition 2.1. *For a continuous function $f : [0, h] \to \mathbb{R}$ then q- derivative of f at $x \in [0, h]$ is characterized by the expression*

$$D_q f(x) \ = \frac{f(x) - f(qx)}{(1-q)x}, \ x \neq 0.$$

Definition 2.2. *Let $f : [0, h] \to \mathbb{R}$ be a continuous function. Then the q-definite integral on $[0, h]$ is delineated asfor $x \in [0, h]$.*

q-definite integral on $[0, x]$ defined by the expression

$$\int_0^x f(t) \ {}_0d_q t \ = \int_0^x f(t)\, d_q t = (1-q)\, x \sum_{n=0}^{\infty} q^n f(q^n x).$$

If $c \in (0, x)$, then the q- definite integral on $[c, x]$ is expressed as

$$\int_c^x f(t)\, d_q t = \int_0^x f(t)\, d_q t - \int_0^c f(t)\, d_q t.$$

3. Main Results

First we will prove the q-Opial inequality below and some results

Theorem 3.1 (q-Opial Inequality). *Let $x(t) \in C^{(1)}[0, h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0, h)$. Then, the following inequality holds:*

$$\int_0^h \left| \{x(t) + x(qt)\}\, D_q x(t) \right| d_q t \leq \frac{h}{1+q} \int_0^h \left| D_q x(t) \right|^2 d_q t. \tag{3}$$

Proof. Let choosing $y(t)$ and $z(t)$ functions as

$$y(t) \ = \int_0^t \left| D_q x(s) \right| d_q s \tag{4}$$

$$z(t) \ = \int_t^h \left| D_q x(s) \right| d_q s,$$

such that

$$\left|D_q x\,(t)\right| = D_q y\,(t) = -D_q z\,(t) \tag{5}$$

and for $t \in [0, h]$, it follows that

$$\begin{aligned} |x\,(t)| &= \left|\int_0^t D_q x\,(s)\,d_q s\right| \leq \int_0^t \left|D_q x\,(s)\right| d_q s = y(t) \\ |x\,(t)| &= \left|\int_t^h D_q x\,(s)\,d_q s\right| \leq \int_t^h \left|D_q x\,(s)\right| d_q s = z(t). \end{aligned} \tag{6}$$

$$\begin{aligned} \left|x\,(qt)\right| &= \left|\int_0^{qt} D_q x\,(s)\,d_q s\right| \leq \int_0^{qt} \left|D_q x\,(s)\right| d_q s = y(qt) \\ \left|x\,(qt)\right| &= \left|\int_{qt}^h D_q x\,(s)\,d_q s\right| \leq \int_{qt}^h \left|D_q x\,(s)\right| d_q s = z(qt). \end{aligned} \tag{7}$$

Now let calculating the following q-integral by using partial q-integration method

$$\int_0^{\frac{h}{1+q}} y(t) D_q y\,(t)\,d_q t = y^2\left(\frac{h}{1+q}\right) - \int_0^{\frac{h}{1+q}} y(qt) D_q y\,(t)\,d_q t$$

and then

$$\int_0^{\frac{h}{1+q}} \{y(t) + y(qt)\}\,D_q y\,(t)\,d_q t = y^2\left(\frac{h}{1+q}\right). \tag{8}$$

By using (5), (6), (7) and (8) we have the following inequality

$$\begin{aligned} \int_0^{\frac{h}{1+q}} \left|\{x(t) + x(qt)\}\,D_q x\,(t)\right| d_q t &\leq \int_0^{\frac{h}{1+q}} \left\{|x(t)| + \left|x(qt)\right|\right\} \left|D_q x\,(t)\right| d_q t \\ &\leq \int_0^{\frac{h}{1+q}} \{y(t) + y(qt)\}\,D_q y\,(t)\,d_q t \\ &= y^2\left(\frac{h}{1+q}\right). \end{aligned} \tag{9}$$

Similarly we can write that

$$\begin{aligned} \int_{\frac{h}{1+q}}^h \left|\{x(t) + x(qt)\}\,D_q x\,(t)\right| d_q t &\leq \int_{\frac{h}{1+q}}^h \left\{|x(t)| + \left|x(qt)\right|\right\} \left|D_q x\,(t)\right| d_q t \\ &\leq -\int_{\frac{h}{1+q}}^h \{z(t) + z(qt)\}\,D_q z\,(t)\,d_q t \end{aligned} \tag{10}$$

$$= z^2\left(\frac{h}{1+q}\right).$$

Adding (9) and (10), we find that

$$\int_0^h \left|\{x(t)+x(qt)\}D_qx(t)\right| d_qt \le y^2\left(\frac{h}{1+q}\right)+z^2\left(\frac{h}{1+q}\right).$$

Finally using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} y^2\left(\frac{h}{1+q}\right) &= \left[\int_0^{\frac{h}{1+q}} \left|D_qx(t)\right| d_qt\right]^2 \qquad (11)\\ &= \left[\left(\int_0^{\frac{h}{1+q}} 1^2 d_qt\right)^{1/2}\left(\int_0^{\frac{h}{1+q}} \left|D_qx(t)\right|^2 d_qt\right)^{1/2}\right]^2\\ &= \frac{h}{1+q}\int_0^{\frac{h}{1+q}} \left|D_qx(t)\right|^2 d_qt. \end{aligned}$$

Similarly we have

$$z^2\left(\frac{h}{1+q}\right)=\left[\int_{\frac{h}{1+q}}^{h} \left|D_qx(t)\right| d_qt\right]^2 = \frac{h}{1+q}\int_{\frac{h}{1+q}}^{h} \left|D_qx(t)\right|^2 d_qt. \qquad (12)$$

Therefore, from (11) and (12) we obtain that

$$\int_0^h \left|\{x(t)+x(qt)\}D_qx(t)\right| d_qt \le \frac{h}{1+q}\int_0^h \left|D_qx(t)\right|^2 d_qt$$

and the proof is completed. □

Remark 3.2. *In Theorem 3.1 if we take $q \to 1^-$, we recapture the (1) inequality.*

Theorem 3.3. *Let $x(t) \in C^{(1)}[0,h]$ be such that $x(0) = 0$ and $x(t) > 0$ in $(0,h)$. Then, the following inequality holds:*

$$\int_0^h \left|\{x(t)+x(qt)\}D_qx(t)\right| d_qt \le h\int_0^h \left|D_qx(t)\right|^2 d_qt. \qquad (13)$$

Proof. Let choosing $y(t)$ functions as (4) such that

$$\begin{aligned} |x(t)| &\le y(t) \qquad (14)\\ \left|D_qx(t)\right| &= D_qy(t) \end{aligned}$$

and then

$$\int_0^h y(t)D_qy(t)\,d_qt = y^2(h) - \int_0^h y(qt)D_qy(t)\,d_qt,$$

i.e

$$\int_0^h \{y(t) + y(qt)\} D_q y(t)\, d_q t = y^2(h). \tag{15}$$

Now by using Cauchy-Schwarz inequality for $y^2(h)$, we have

$$y^2(h) = \left[\int_0^h \left|D_q x(s)\right| d_q s\right]^2 \leq h \int_0^h \left|D_q x(s)\right|^2 d_q s.$$

Finaly by using (14), then we have

$$\begin{aligned} \int_0^h \left|\{x(t) + x(qt)\} D_q x(t)\right| d_q t &\leq \int_0^h \{y(t) + y(qt)\} D_q y(t)\, d_q t \\ &\leq h \int_0^h \left|D_q x(t)\right|^2 d_q t \end{aligned}$$

and the proof is completed. □

Remark 3.4. *In Theorem 3.3 if we take $q \to 1^-$, we recapture the (2) inequality.*

Theorem 3.5. *Let $p(t)$ be a nonnegative and continuous function on $[0, h]$ and $x(t) \in C^{(1)}[0, h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0, h)$. Then, the following inequality holds:*

$$\int_0^h p(t) \left|\{x(t) + x(qt)\} D_q x(t)\right| d_q t \leq \left(h \int_0^h p^2(t)\, d_q t\right)^{\frac{1}{2}} \int_0^h \left|D_q x(t)\right|^2 d_q t \tag{16}$$

Proof. In proof of Theorem 3.1, we obtained that

$$|x(t)| \leq y(t) \quad \text{and} \quad |x(t)| \leq z(t)$$

Thus we get

$$|x(t)| \leq \frac{y(t) + z(t)}{2} = \frac{1}{2} \int_0^h \left|D_q x(s)\right| d_q s. \tag{17}$$

$$\begin{aligned} \left|x(qt)\right| &\leq \frac{y(qt) + z(qt)}{2} \\ &= \frac{\int_0^{qt} D_q x(s)\, d_q s + \int_{qt}^h D_q x(s)\, d_q s}{2} = \frac{1}{2} \int_0^h \left|D_q x(s)\right| d_q s. \end{aligned} \tag{18}$$

By using the (17) and from Cauchy-Schwarz inequality for q-integral,

$$\int_0^h p(t) |x(t)|^2 d_q t \leq \frac{1}{4} \int_0^h p(t) \left[\int_0^h \left|D_q x(s)\right| d_q s\right]^2 d_q t \tag{19}$$

(20)

$$\begin{aligned}
&\leq \frac{1}{4}\left(\int_0^h p(t)\,d_qt\right)\left(\int_0^h d_qs\right)\left(\int_0^h \left|D_qx(s)\right|^2 d_qs\right)\\
&\leq \frac{h}{4}\left(\int_0^h p(t)\,d_qt\right)\left(\int_0^h \left|D_qx(t)\right|^2 d_qt\right)
\end{aligned}$$

and similarly by using (18) we have

$$\int_0^h p(t)\left|x(qt)\right|^2 d_qt \leq \frac{h}{4}\left(\int_0^h p(t)\,d_qt\right)\left(\int_0^h \left|D_qx(t)\right|^2 d_qt\right). \tag{21}$$

From Cauchy-Schwarz inequality and (19), we have

$$\begin{aligned}
\int_0^h p(t)\left|x(t)\,D_qx(t)\right| d_qt &\leq \left(\int_0^h p^2(t)\left|x(t)\right|^2 d_qt\right)^{\frac{1}{2}}\left(\int_0^h \left|D_qx(t)\right|^2 d_qt\right)^{\frac{1}{2}} \qquad (22)\\
&\leq \left(\frac{h}{4}\left(\int_0^h p^2(t)\,d_qt\right)\left(\int_0^h \left|D_qx(t)\right|^2 d_qt\right)\right)^{\frac{1}{2}}\left(\int_0^h \left|D_qx(t)\right|^2 d_qt\right)^{\frac{1}{2}}\\
&\leq \frac{1}{2}\left(h\int_0^h p^2(t)\,d_qt\right)^{\frac{1}{2}}\left(\int_0^h \left|D_qx(t)\right|^2 d_qt\right).
\end{aligned}$$

Similarly, by using (21) we can write

$$\int_0^h p(t)\left|x(qt)\,D_qx(t)\right| d_qt \leq \frac{1}{2}\left(h\int_0^h p^2(t)\,d_qt\right)^{\frac{1}{2}}\left(\int_0^h \left|D_qx(t)\right|^2 d_qt\right) \tag{23}$$

Finally by adding (22) and (23) we have

$$\begin{aligned}
\int_0^h p(t)\left|\{x(t)+x(qt)\}\,D_qx(t)\right| d_qt &\leq \int_0^h p(t)\left\{|x(t)|+\left|x(qt)\right|\right\}D_qx(t)\,d_qt\\
&\leq \left(h\int_0^h p^2(t)\,d_qt\right)^{\frac{1}{2}}\left(\int_0^h \left|D_qx(t)\right|^2 d_qt\right)
\end{aligned}$$

which is complete the proof. □

Remark 3.6. *In Theorem 3.5 if we take $q \to 1^-$, we recapture the following inequality*

$$\int_0^h p(t)\left|x(t)x'(t)\right| dt \leq \left(\frac{h}{4}\int_0^h p^2(t)\,dt\right)^{\frac{1}{2}}\left(\int_0^h \left|x'(t)\right|^2 dt\right)$$

which is proved by Trable in [19].

Theorem 3.7. *Let $x(t) \in C^{(1)}[0,h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then, the following inequality holds:*

$$\int_0^h |x(t)|^{m(p+r)} \, d_q t \leq [K(m)]^{(p+r)} \int_0^h \left|D_q x(s)\right|^{m(p+r)} \left| \sum_{i=0}^{p+r-1} \left(\frac{x(qs)}{x(s)} \right)^i \right|^{m(p+r)} d_q s \tag{24}$$

where

$$K(m) = \int_0^h \left[t^{1-m} + (h-t)^{1-m} \right]^{-1} d_q t.$$

Proof. Firstly we can write q-derivative of $x^n(t)$

$$D_q x^n(t) = \sum_{i=0}^{n-1} x^{n-1-i}(t) x^i(qt) D_q x(t) \tag{25}$$

using (25) we have

$$\int_0^t D_q x^{p+r}(s) d_q s = x^{p+r}(t) \tag{26}$$

on the other hand we can write

$$\int_0^t D_q x^{p+r}(s) d_q s = \int_0^t \sum_{i=0}^{p+r-1} x^{p+r-1-i}(s) x^i(qs) D_q x(s) \, d_q s. \tag{27}$$

From (26)-(27) we get

$$x^{p+r}(t) = \int_0^t \sum_{i=0}^{p+r-1} x^{p+r-1-i}(s) x^i(qs) D_q x(s) \, d_q s. \tag{28}$$

Similarly, we can write

$$x^{p+r}(t) = -\int_t^h \sum_{i=0}^{p+r-1} x^{p+r-1-i}(s) x^i(qs) D_q x(s) \, d_q s. \tag{29}$$

Using the Hölder's inequality for q-integral with indices m, $\frac{m}{m-1}$ in (28) and (29), we have

$$\begin{aligned} |x(t)|^{m(p+r)} &\leq \left(\int_0^t \left| \sum_{i=0}^{p+r-1} x^{p+r-1-i}(s) x^i(qs) D_q x(s) \right| d_q s \right)^m \\ &\leq \left(\int_0^t \left| \sum_{i=0}^{p+r-1} x^{p+r-1-i}(s) x^i(qs) \right|^m \left|D_q x(s)\right|^m d_q s \right) \left(\int_0^t d_q s \right)^{m-1} \\ &\leq t^{m-1} \left(\int_0^t \left| \sum_{i=0}^{p+r-1} x^{p+r-1-i}(s) x^i(qs) \right|^m \left|D_q x(s)\right|^m d_q s \right). \end{aligned} \tag{30}$$

Similarly, we get

$$|x(t)|^{m(p+r)} \leq (h-t)^{m-1}\left(\int_0^t \left|\sum_{i=0}^{p+r-1} x^{p+r-1-i}(s)x^i(qs)\right|^m \left|D_q x(s)\right|^m d_q s\right). \tag{31}$$

Multiplying the (30) and (31) respectively by t^{1-m} and $(h-t)^{1-m}$ and summing these inequalities, we have

$$\left[t^{1-m}+(h-t)^{1-m}\right]|x(t)|^{m(p+r)} \leq \left(\int_0^h \left|\sum_{i=0}^{p+r-1} x^{p+r-1-i}(s)x^i(qs)\right|^m \left|D_q x(s)\right|^m d_q s\right) \tag{32}$$

and for $t \in [0,h]$ we get

$$\begin{aligned} |x(t)|^{m(p+r)} &\leq \left[t^{1-m}+(h-t)^{1-m}\right]^{-1}\left(\int_0^h \left|\sum_{i=0}^{p+r-1} x^{p+r-1-i}(s)x^i(qs)\right|^m \left|D_q x(s)\right|^m d_q s\right) \\ &= \left[t^{1-m}+(h-t)^{1-m}\right]^{-1}\left(\int_0^h |x(s)|^{m(p+r-1)}\left|\sum_{i=0}^{p+r-1}\left(\frac{x(qs)}{x(s)}\right)^i\right|^m \left|D_q x(s)\right|^m d_q s\right) \\ &= \left[t^{1-m}+(h-t)^{1-m}\right]^{-1}\left(\int_0^h |x(s)|^{mp/r}\left|D_q x(s)\right|^m |x(s)|^{m(p+r-1)-mp/r}\left|\sum_{i=0}^{p+r-1}\left(\frac{x(qs)}{x(s)}\right)^i\right|^m d_q s\right). \end{aligned} \tag{33}$$

Integrating (33) on $[0,h]$ and using the Hölder's inequality for q-integral with indices r, $\frac{r}{r-1}$ we have

$$\begin{aligned} \int_0^h |x(t)|^{m(p+r)} d_q t &\leq \int_0^h \left[t^{1-m}+(h-t)^{1-m}\right]^{-1} d_q t \\ &\quad \times\left(\int_0^h |x(s)|^{mp/r}\left|D_q x(s)\right|^m |x(s)|^{m(p+r-1)-mp/r}\left|\sum_{i=0}^{p+r-1}\left(\frac{x(qs)}{x(s)}\right)^i\right|^m d_q s\right) \\ &\leq K(m)\left(\int_0^h |x(s)|^{mp}\left|D_q x(s)\right|^{mr}\left|\sum_{i=0}^{p+r-1}\left(\frac{x(qs)}{x(s)}\right)^i\right|^{mr} d_q s\right)^{\frac{1}{r}}\left(\int_0^h |x(s)|^{m(p+r)} d_q s\right)^{\frac{r-1}{r}} \end{aligned} \tag{34}$$

which by dividing the both sides of (34) with $\left(\int_0^h |x(s)|^{m(p+r)} d_q s\right)^{\frac{r-1}{r}}$ and taking the rth power on both sides of resulting inequaliy. Finally by using the Hölder's inequality for q-integral with indices $\frac{p+r}{p}$, $\frac{p+r}{r}$ then, we get

$$\begin{aligned} \int_0^h |x(t)|^{m(p+r)} d_q t &\leq [K(m)]^r\left(\int_0^h |x(s)|^{mp}\left|D_q x(s)\right|^{mr}\left|\sum_{i=0}^{p+r-1}\left(\frac{x(qs)}{x(s)}\right)^i\right|^{mr} d_q s\right) \\ &\leq [K(m)]^r\left(\int_0^h |x(s)|^{m(p+r)} d_q s\right)^{\frac{p}{p+r}}\left(\int_0^h \left|D_q x(s)\right|^{m(p+r)}\left|\sum_{i=0}^{p+r-1}\left(\frac{x(qs)}{x(s)}\right)^i\right|^{m(p+r)} d_q s\right)^{\frac{r}{p+r}} \end{aligned} \tag{35}$$

which by dividing the both sides of (35) with $\left(\int\limits_0^h |x(s)|^{m(p+r)} d_q s\right)^{\frac{p}{p+r}}$ and taking the $\frac{p+r}{r}$th power on both sides of (35) we get

$$\int\limits_0^h |x(t)|^{m(p+r)} d_q t \leq [K(m)]^{(p+r)} \int\limits_0^h \left|D_q x(s)\right|^{m(p+r)} \left|\sum_{i=0}^{p+r-1} \left(\frac{x(qs)}{x(s)}\right)^i\right|^{m(p+r)} d_q s$$

and the proof is completed. □

Remark 3.8. *In Theorem 3.7 if we take $q \to 1^-$, we recapture the following result*

$$\int\limits_0^h |x(t)|^{m(p+r)} dt \leq \left[(p+r)^m K(m)\right]^{(p+r)} \int\limits_0^h |x'(s)|^{m(p+r)} ds$$

which is proved by Pachpatte in [12].

Theorem 3.9. *Let $x(t)$ be absulately continuous on $[0,h]$, and $x(0) = 0$. Further let $\alpha \geq 0$. Then, the following inequality holds:*

$$\int\limits_0^h \left|\sum_{i=0}^{\alpha-1} x^{\alpha-1-i}(t) x^i(qt) D_q x(t)\right| d_q t \leq h^\alpha \int\limits_0^h \left|D_q x(s)\right|^{\alpha+1} d_q s.$$

Proof. By q-derivative of $x^n(t)$

$$D_q y^{\alpha+1}(t) = \sum_{i=0}^{\alpha} y^{\alpha-i}(t) y^i(qt) D_q y(t). \tag{36}$$

and choosing $y(t)$ as

$$y(t) = \int\limits_0^t \left|D_q x(s)\right| d_q s \tag{37}$$

such that

$$|x(t)| \leq y(t).$$

From (36) and we get

$$\begin{aligned} \int\limits_0^h \left|\sum_{i=0}^{\alpha} x^{\alpha-i}(t) x^i(qt) D_q x(t)\right| d_q t &\leq \int\limits_0^h \sum_{i=0}^{\alpha} y^{\alpha-i}(t) y^i(qt) D_q y(t)\, d_q t \\ &= \int\limits_0^h D_q y^{\alpha+1}(t)\, d_q t \\ &= y^{\alpha+1}(h). \end{aligned} \tag{38}$$

By using the Hölder's inequality and (38) with (37) for q-integral with indices $\alpha+1$, $\frac{\alpha+1}{\alpha}$, we get

$$\begin{aligned} y^{\alpha+1}(h) &= \left[\int_0^h \left|D_q x(s)\right| d_q s\right]^{\alpha+1} \\ &\leq \left[\left(\int_0^h d_q s\right)^{\frac{\alpha}{\alpha+1}} \left(\int_0^h \left|D_q x(s)\right|^{\alpha+1} d_q s\right)^{\frac{1}{\alpha+1}}\right]^{\alpha+1} \\ &= h^{\alpha} \int_0^h \left|D_q x(s)\right|^{\alpha+1} d_q s \end{aligned}$$

and

$$\int_0^h \left|\sum_{i=0}^{\alpha} x^{\alpha-i}(t) x^i(qt) D_q x(t)\right| d_q t \leq h^{\alpha} \int_0^h \left|D_q x(s)\right|^{\alpha+1} d_q s$$

which is completes the proof. □

Remark 3.10. *In Theorem 3.9 if we take $q \to 1^-$, we recapture the following result*

$$\int_0^h \left|x^{\alpha}(t) x'(t)\right| dt \leq \frac{h^{\alpha}}{\alpha+1} \int_0^h \left|x'(s)\right|^{\alpha+1} ds$$

which is proved by Hua in [8].

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