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A Generalization on Character Connes-Amenability

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Abstract. In the current paper, we introduce the concepts of left φ-approximate Connes-amenability and left character approximate Connes-amenability of a dual Banach algebra $\mathcal{A}$ that φ is a ω^*-continuous homomorphism from $\mathcal{A}$ to $\mathbb{C}$. We also characterize left φ-approximate Connes-amenability of $\mathcal{A}$ in terms of certain derivations and study some hereditary properties for such Banach algebras. Some examples show that these new notions are different from approximate Connes-amenability and left character Connes-amenability for dual Banach algebras.

1. Introduction

In [12], Johnson, Kadison and Ringrose introduced a notion of amenability for von Neumann algebras which modified Johnson's original definition for Banach algebras in the sense that it takes the dual space structure of a von Neumann algebra into account. This notion of amenability was called later Connes-amenability by A. Ya. Helemskii [11]. Runde extended the notion of Connes-amenability to the larger class of dual Banach algebras in [17] and studied certain concrete Banach algebras in the subsequent papers [18–21].

The concept of approximate amenability of Banach algebras was introduced by Ghahramani and Loy in [9] for the first time. They characterized the structure of approximately amenable Banach algebras through several different ways. After that, this notion was generalized for dual Banach algebras, namely, approximate Connes-amenability [8] and ideal Connes-amenability [15].

Let $\mathcal{A}$ be a Banach algebra and φ a character on $\mathcal{A}$, that is, a non-zero multiplicative linear functional on $\mathcal{A}$. Then $\mathcal{A}$ is said to be φ-amenable if there exists a $m \in \mathcal{A}^{**}$ such that $m(a^* \cdot a) = \varphi(a)m(a^*)$ and $m(\varphi) = 1$ for all $a \in \mathcal{A}$ and $a^* \in \mathcal{A}^*$. This notion was introduced by kaniuth, Lau and Pym in [13]. In the same time, Monfared [23] presented the notions of left and right character amenability for Banach algebras (see also [24]); for the module version of character amenability and its generalizations refer to [2] and [3]. Furthermore, we say that $\mathcal{A}$ is approximately-φ amenable [1] (in this paper is called left φ-approximately amenable), if there is a net $(m_\alpha) \subseteq \mathcal{A}^{**}$ such that $m_\alpha(\varphi) \to 1$ and $a \cdot m_\alpha - \varphi(a)m_\alpha \to 0$ for all $a \in \mathcal{A}$.

It is well-known there are two product, as the first and second Arens products on $\mathcal{A}^{**}$, the second dual of a Banach algebra $\mathcal{A}$; see [4] for more details. Then $\mathcal{A}$ is called *Arens regular* if these two products coincide on $\mathcal{A}^{**}$. A Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is dual if there is a closed submodule $\mathcal{X}_*$ of $\mathcal{X}^*$ such that $\mathcal{X} = (\mathcal{X}_*)^*$. A Banach

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algebra $\mathcal{A}$ is dual if there is a closed submodule $\mathcal{A}_*$ of $\mathcal{A}^*$ such that $\mathcal{A} = (\mathcal{A}_*)^*$. For a dual Banach algebra $\mathcal{A}$, a dual Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is called normal if the module actions of $\mathcal{A}$ on $\mathcal{X}$ are ω^*-continuous. Examples of dual Banach algebras (besides von Neumann algebras) include the measure algebra $M(G) = C_0(G)^*$ and the Fourier-Stieltjes algebra $B(G) = C^*(G)^*$ of a locally compact group G. Also, the algebra $\mathfrak{B}(E) = (E \otimes_\gamma E^*)^*$ of all bounded operators on a reflexive Banach space E where $\otimes_\gamma$ stands for the projective tensor product and the second dual B^{**} of an Arens regular Banach algebra B.

Let $\mathcal{A}$ be a dual Banach algebra and $\sigma_{\omega^*}(\mathcal{A})$ the set of all non-zero ω^*-continuous characters on $\mathcal{A}$ (note that $\sigma_{\omega^*}(\mathcal{A}) \subseteq \mathcal{A}_*$). We say that $\mathcal{A}$ is left φ-Connes-amenable if there exists $m \in \mathcal{A}$ such that $m(\varphi) = 1$ and $am = \varphi(a)m$ for all $a \in \mathcal{A}$. Moreover, $\mathcal{A}$ is called left 0-Connes-amenable if for every normal dual Banach $\mathcal{A}$-bimodule $\mathcal{X}^*$ where the left module action of $\mathcal{X}$ is zero, every ω^*-continuous derivation $D : \mathcal{A} \to \mathcal{X}^*$ is inner. In addition, $\mathcal{A}$ is called left character Connes-amenable if it left φ-Connes-amenable for each $\varphi \in \sigma_{\omega^*}(\mathcal{A}) \cup \{0\}$.

Motivated to [8, 13, 23–25], in this paper we introduce and study the concepts of left φ-approximate Connes-amenability and left character approximate Connes-amenability for dual Banach algebras where $\varphi \in \sigma_{\omega^*}(\mathcal{A})$. We firstly investigate the basic properties of left φ-approximately Connes-amenable dual Banach algebra. Then we characterize left φ-approximate Connes-amenability and left character approximate Connnes-amenability in some different ways (Theorems 2.5, 2.14 and 2.16). For an Arens regular Banach algebra $\mathcal{A}$, we show that $\mathcal{A}$ is left φ-approximately amenable if $\mathcal{A}^{**}$ is left φ^{**}-approximately Connes-amenable (Theorem 2.12). Finally, we present some examples that show the concepts of left φ-approximate Connes-amenability and left character approximate Connes-amenability are weaker than [approximate] Connes-amenability, [approximate] amenability and left character Connes-amenability.

2. Character approximate Connes-amenability

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. Then $\mathcal{X}^*$, the conjugate of $\mathcal{X}$, has an $\mathcal{A}$-bimodule structure defined by the usual way as follows:

$$\langle a \cdot x^*, x\rangle = \langle x^*, x \cdot a\rangle, \quad \langle x^* \cdot a, x\rangle = \langle x^*, a \cdot x\rangle \quad (a \in \mathcal{A}, x \in \mathcal{X}, x^* \in \mathcal{X}^*).$$

A derivation $D : \mathcal{A} \longrightarrow \mathcal{X}$ is a continuous linear map such that $D(ab) = a \cdot Db + Da \cdot b$ for $a, b \in \mathcal{A}$. Given $x \in \mathcal{X}$, the inner derivation $\delta_x : \mathcal{A} \longrightarrow \mathcal{X}$ is defined by $\delta_x(a) = a \cdot x - x \cdot a$. A derivation $D : \mathcal{A} \longrightarrow \mathcal{X}$ is called approximately inner if there exists a net $(x_\alpha) \subseteq \mathcal{X}$ such that $Da = \lim_\alpha a \cdot x_\alpha - x_\alpha \cdot a$ for each $a \in \mathcal{A}$.

Definition 2.1. *Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \sigma_{\omega^*}(\mathcal{A})$. We say that $\mathcal{A}$ is left φ-approximately Connes-amenable if there exists a net $(m_\alpha) \subseteq \mathcal{A}$ such that $m_\alpha(\varphi) \to 1$ and $am_\alpha - \varphi(a)m_\alpha \to 0$ for each $a \in \mathcal{A}$. Moreover, $\mathcal{A}$ is left 0-approximately Connes-amenable, if for every normal dual Banach $\mathcal{A}$-bimodule $\mathcal{X}^*$ which the left action of $\mathcal{X}$ is zero, every ω^*-continuous derivation from $\mathcal{A}$ into $\mathcal{X}^*$ is approximately inner. We also say that $\mathcal{A}$ is left character approximately Connes-amenable if it is left φ-approximately Connes-amenable for each $\varphi \in \sigma_{\omega^*}(\mathcal{A}) \cup \{0\}$.*

Definition 2.2. *Let $\mathcal{A}$ be a dual Banach algebra, $\varphi \in \sigma_{\omega^*}(\mathcal{A}) \cup \{0\}$ and $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule with the left module action $a \cdot x = \varphi(a)x$ ($a \in \mathcal{A}, x \in \mathcal{X}$). The dual Banach $\mathcal{A}$-bimodule $\mathcal{X}^*$ is called left φ-normal if for each $x^* \in \mathcal{X}^*$ the map $\mathcal{A} \to \mathcal{X}^*$; $a \mapsto a \cdot x^*$ is ω^*-continuous. It is easily seen that every left φ-normal dual Banach $\mathcal{A}$-bimodule is normal dual Banach $\mathcal{A}$-bimodule.*

Proposition 2.3. *Let $\mathcal{A}$ be a dual Banach algebra. Then $\mathcal{A}$ is left 0-approximately Connes-amenable if and only if it has a right approximate identity.*

Proof. Assume that $\mathcal{X}$ is the Banach $\mathcal{A}$-bimodule, whose underlying linear space is $\mathcal{A}_*$ with the module actions $a.x = 0$ and $x \cdot a = xa$ for each $x \in \mathcal{X}$ and $a \in \mathcal{A}$. Then $\mathcal{X}^* = \mathcal{A}$ is a left 0-normal dual Banach $\mathcal{A}$-bimodule and the identity map from $\mathcal{A}$ into $\mathcal{X}^*$ is a ω^*-continuous derivation, so it is approximately inner, i.e. there exists a net $(a_\alpha) \subseteq \mathcal{X}^* = \mathcal{A}$ such that $a = \lim_\alpha aa_\alpha$ for each $a \in \mathcal{A}$.

Conversely, let (a_α) be a right approximate identity for $\mathcal{A}$ and D be a ω^*-continuous derivation from $\mathcal{A}$ into $\mathcal{X}^*$ where $\mathcal{X}^*$ be a left 0-normal dual Banach $\mathcal{A}$ bimodule. Since $D(a)\cdot a_\alpha = D(a_\alpha)\cdot a = 0$ for $a \in \mathcal{A}$ and each α, we have

$$\begin{aligned} D(a) &= \lim_\alpha D(aa_\alpha) = \lim_\alpha (Da\cdot a_\alpha + a\cdot D(a_\alpha)) = \lim_\alpha a\cdot D(a_\alpha) \\ &= \lim_\alpha (a\cdot D(a_\alpha) - D(a_\alpha)\cdot a). \end{aligned}$$

Hence D is approximately inner. It means that $\mathcal{A}$ is left 0-approximately Connes-amenable. □

Proposition 2.4. *Let $\mathcal{A}$ be a Banach algebra, $\mathcal{B}$ be a dual Banach algebra and let $\theta : \mathcal{A} \to \mathcal{B}$ be a continuous epimorphism and $\varphi \in \sigma_{\omega^*}(\mathcal{B})$.*

(i) If $\mathcal{A}$ is left $\varphi\circ\theta$-approximately amenable, then $\mathcal{B}$ is left φ-approximately Connes-amenable;
(ii) If θ is ω^*-continuous and $\mathcal{A}$ is a dual Banach algebra which is left $\varphi\circ\theta$-approximately Connes-amenable, then $\mathcal{B}$ is left φ-approximately Connes-amenable;
(iii) If $\mathcal{A}$ is left character approximately [*Connes*]*-amenable, then $\mathcal{B}$ is left character approximately Connes-amenable.*

Proof. (i) Since $\mathcal{A}$ is left $\varphi\circ\theta$-approximately amenable, there exists a net $(m_\alpha) \subseteq \mathcal{A}^{**}$ such that $am_\alpha - \varphi\circ\theta(a)m_\alpha \to 0$, $m_\alpha(\varphi\circ\theta) \to 1$. On the other hand, as [25, Proposition 2.6] for mapping $\psi = (\theta^*|_{\mathcal{B}_*})^* : \mathcal{A}^{**} \to \mathcal{B}$, we have $\psi(aa^{**}) = \theta(a)\psi(a^{**})$ where $a \in \mathcal{A}$ and $a^{**} \in \mathcal{A}^{**}$. Set $n_\alpha = \psi(m_\alpha) \in \mathcal{B}$, so $\theta(a)n_\alpha - \varphi\circ\theta(a)n_\alpha = \theta(a)\psi(m_\alpha) - \varphi\circ\theta(a)\psi(m_\alpha) = \psi(am_\alpha - \varphi\circ\theta(a)m_\alpha) \to 0$ and $\langle n_\alpha, \varphi\rangle = \langle \psi(m_\alpha), \varphi\rangle = \langle m_\alpha, \varphi\circ\theta\rangle \to 1$. Furthermore, θ is epimorphism and so we get $\mathcal{B}$ is left φ-approximately Connes-amenable. For the part (ii) it suffices to define $n_\alpha = \theta(m_\alpha)$. The Part (iii) is a direct consequence of statements (i) or (ii). □

Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \sigma_{\omega^*}(\mathcal{A})$. We can consider ker$\varphi$, the kernel of φ as a Banach $\mathcal{A}$-bimodule which its right module action is $ma = \varphi(a)m$, $a \in \mathcal{A}$, $m \in$kerφ and the left action is the natural one. Thus kerφ is a left φ-normal dual Banach $\mathcal{A}$-bimodule. In the next theorem we characterize left φ-approximate Connes-amenability of dual Banach algebra $\mathcal{A}$ through the cohomlogy groups $H^1_{\omega^*}(\mathcal{A},\mathcal{X}^*) := Z^1_{\omega^*}(\mathcal{A},\mathcal{X}^*)/B^1(\mathcal{A},\mathcal{X}^*)$ for certain normal dual Banach $\mathcal{A}$-bimodules $\mathcal{X}^*$.

Theorem 2.5. *Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \sigma_{\omega^*}(\mathcal{A})$. Then the following are equivalent:*

(i) $\mathcal{A}$ is left φ-approximately Connes-amenable;
(ii) For every left φ-normal dual Banach $\mathcal{A}$-bimodule $\mathcal{X}^*$ every ω^*-continuous derivation $D : \mathcal{A} \to \mathcal{X}^*$ is approximately inner;
(iii) Every ω^*-continuous derivation $D : \mathcal{A} \to$kerφ is approximately inner;
(iv) There is a net $(m_\alpha) \subseteq \mathcal{A}$ such that $m_\alpha(\varphi) = 1$ for each α and $am_\alpha - \varphi(a)m_\alpha \to 0$ for each $a \in \mathcal{A}$.

Proof. (i)⇒(ii) By the assumption there exists a net $(m_\alpha) \subseteq \mathcal{A}$ which $am_\alpha - \varphi(a)m_\alpha \to 0$ and $m_\alpha(\varphi) \to 1$. Now, suppose that $\mathcal{X}^*$ is a left φ-normal dual Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ is a ω^*-continuous derivation. Define $x^*_\alpha = D(m_\alpha) \in \mathcal{X}^*$. Then for each $a \in \mathcal{A}$,

$$\begin{aligned} 0 = \lim_\alpha D(am_\alpha - \varphi(a)m_\alpha) &= \lim_\alpha (D(am_\alpha) - \varphi(a)D(m_\alpha)) \\ &= \lim_\alpha (D(a)\cdot m_\alpha + a\cdot D(m_\alpha) - \varphi(a)D(m_\alpha)) \\ &= \lim_\alpha (m_\alpha(\varphi)D(a) + a\cdot x^*_\alpha - \varphi(a)x^*_\alpha). \end{aligned}$$

Therefore we obtain $Da = \lim_\alpha a\cdot(-x^*_\alpha) - \varphi(a)(-x^*_\alpha) = \lim_\alpha a\cdot(-x^*_\alpha) - (-x^*_\alpha)\cdot a$.

(ii)⇒(iii) It is obvious.

(iii)⇒(iv) We make $\mathcal{A}^*$ into a Banach $\mathcal{A}$-bimodule by taking the right module action as usual and the left module action to be defined by $a\cdot a^* = \varphi(a)a^*(a \in \mathcal{A}, a^* \in \mathcal{A}^*)$. Since $a\cdot\varphi = \varphi\cdot a = \varphi(a)\varphi$ and $\varphi \in \mathcal{A}_*$,

$\mathbb{C}\varphi$ is a closed $\mathcal{A}-$submodule $\mathcal{A}_*$. So $(\mathcal{A}_*/\mathbb{C}\varphi)^* \simeq \{\mathbb{C}\varphi\}^\perp = \{a \in \mathcal{A} : \varphi(a) = 0\} =ker\varphi$ is a left φ-normal dual Banach $\mathcal{A}-$bimodule with the module actions as follows:

$$a \cdot [a_*] = \varphi(a)[a_*], \quad [a_*] \cdot a = [a_* \cdot a] \quad (a \in \mathcal{A}, [a_*] \in \mathcal{A}_*/\mathbb{C}\varphi).$$

Now, choose $m_0 \in A$ such that $\varphi(m_0) = 1$ and define a derivation $D : \mathcal{A} \to \mathcal{A}$ by $Da = am_0 - \varphi(a)m_0$. Obviously, $D(\mathcal{A}) \subseteq$kerφ, and by the assumption there exists a net $(n_\alpha) \subseteq$kerφ such that $Da = \lim_\alpha an_\alpha - \varphi(a)n_\alpha$. Define $m_\alpha = m_0 - n_\alpha \in \mathcal{A}$. It follows that

$$m_\alpha(\varphi) = m_0(\varphi) = 1 \quad and \quad \lim_\alpha (am_\alpha - \varphi(a)m_\alpha) = 0.$$

(iv)⇒(i) It is routine. □

The upcoming corollary is a direct consequence of Theorem 2.5 and [1, Proposition 2.2]. We include it without proof.

Corollary 2.6. *Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \sigma_{\omega^*}(\mathcal{A})$. Then $\mathcal{A}$ is left φ-approximately amenable if and only if it is left φ-approximately Connes-amenable.*

Here, we bring the next lemmas which are useful tools to reach some aims.

Lemma 2.7. *Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \sigma_{\omega^*}(\mathcal{A})$. If I_φ =*kerφ *has a right approximate identity, then $\mathcal{A}$ is left φ-approximately Connes-amenable.*

Proof. Suppose that (e_α) is a right approximate identity for I_φ. Consider $e_0 \in \mathcal{A}$ such that $\varphi(e_0) = 1$ and set $m_\alpha = e_0 - e_0e_\alpha \in \mathcal{A}$. Since $e_0a - \varphi(a)e_0 \in I_\varphi$, for each $a \in \mathcal{A}$, we get

$$\begin{aligned} am_\alpha - \varphi(a)m_\alpha &= ae_0 - ae_0e_\alpha - \varphi(a)e_0 + \varphi(a)e_0e_\alpha \\ &= ae_0 - \varphi(a)e_0 - (ae_0 - \varphi(a)e_0)e_\alpha \to 0. \end{aligned}$$

In addition, $m_\alpha(\varphi) = 1$ for all α. By Theorem 2.5, $\mathcal{A}$ is left φ-approximately Connes-amenable. □

Lemma 2.8. *Let $\mathcal{A}$ be a dual Banach algebra with a bounded right approximate identity which be left φ-approximately Connes-amenable. Then I_φ =* kerφ *has a right approximate identity.*

Proof. Assume that (a_α) is a bounded right approximate identity for $\mathcal{A}$. Since it is dual Banach algebra, $a_\alpha \xrightarrow{\omega^*} e \in \mathcal{A}$, $ae = a$ for every $a \in \mathcal{A}$ and $\varphi(e) = 1$. On the other hand, $\mathcal{A}$ is left φ-approximately Connes-amenable. Theorem 2.5 implies that there exists net $(m_\alpha) \subseteq \mathcal{A}$ such that $am_\alpha - \varphi(a)m_\alpha \to 0$ and $\langle m_\alpha, \varphi\rangle = 1$ for each α. Put $e_\alpha = e - m_\alpha$. We have $\varphi(e_\alpha) = 0$ and so $e_\alpha \in I_\varphi$. Hence for each $a \in I$, $ae_\alpha = ae - am_\alpha = ae - (am_\alpha - \varphi(a)m_\alpha) \to a$, as required. □

The following corollary is immediate result of lemmas 2.7 and 2.8.

Corollary 2.9. *Let $\mathcal{A}$ be a dual Banach algebra with a bounded right approximate identity and $\varphi \in \sigma_{\omega^*}(\mathcal{A})$. Then $\mathcal{A}$ is left φ-approximately Connes-amenable if and only if $\mathcal{I}_\varphi$ =* kerφ *has a right approximate identity.*

Proposition 2.10. *Let $\mathcal{A}$ be a dual Banach algebra, $\mathcal{I}$ be a ω^*-closed ideal with a left bounded approximate identity and $\varphi \in \sigma_{\omega^*}(\mathcal{A})$ with $\varphi|_\mathcal{I} \neq 0$. If $\mathcal{A}$ is left φ-approximately Connes-amenable, then $\mathcal{I}$ is left $\varphi|_\mathcal{I}$-approximately Connes-amenable.*

Proof. Suppose that (e_α) is a bounded left approximate identity for $\mathcal{I}$ and $\varphi|_\mathcal{I} \neq 0$, there exists an $e \in \mathcal{I}$ such that $e_\alpha \xrightarrow{\omega^*} e$, $ea = a$ for each $a \in \mathcal{I}$ and $\varphi(e) = 1$. On the other hand, $\mathcal{A}$ is left φ-approximately Connes-amenable so there is a net $(m_\alpha) \subseteq \mathcal{A}$ such that $am_\alpha - \varphi(a)m_\alpha \to 0$ and $\langle m_\alpha, \varphi\rangle \to 1$. Set $n_\alpha = m_\alpha e \in I$, then for each $a \in \mathcal{I}$,

$$an_\alpha - \varphi(a)n_\alpha = am_\alpha e - \varphi(a)m_\alpha e = (am_\alpha - \varphi(a)m_\alpha)e \to 0,$$

and $\langle \varphi|_\mathcal{I}, n_\alpha\rangle = \langle \varphi, m_\alpha e\rangle = \langle \varphi, m_\alpha\rangle \to 1$. Therefore $\mathcal{I}$ is left $\varphi|_\mathcal{I}$-approximately Connes-amenable. □

Suppose that $(\mathcal{A}, \mathcal{A}_*)$ is a dual Banach algebra. Then its unitization, $\mathcal{A}^\# = \mathcal{A} \oplus_1 \mathbb{C}$ is a dual Banach algebra with predual $\mathcal{A}_* \oplus_\infty \mathbb{C}$, where $\oplus_1$ and $\oplus_\infty$ denote the ℓ^1 and ℓ^∞-direct sums respectively. More generally if $\mathcal{A}$ and $\mathcal{B}$ are dual Banach algebras, then $\mathcal{A} \oplus_1 \mathcal{B}$ is a dual Banach algebra with predual $\mathcal{A}_* \oplus_\infty \mathcal{B}_*$. In addition, for $\varphi \in \sigma_{\omega^*}(\mathcal{A})$ the mapping $\widetilde{\varphi} : \mathcal{A}^\# \longrightarrow \mathbb{C}$ is defined by $\widetilde{\varphi}(a + \lambda) = \varphi(a) + \lambda$ is the unique extension of φ and $\widetilde{\varphi} \in \sigma_{\omega^*}(\mathcal{A}^\#)$.

Proposition 2.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras and $\varphi \in \sigma_{\omega^*}(\mathcal{A})$. Then*

(i) $\mathcal{A}$ is left φ-approximately Connes-amenable if and only if $\mathcal{A}^\#$ is left $\widetilde{\varphi}$-approximately Connes-amenable;

(ii) $\mathcal{A}$ is left character approximately Connes-amenable if and only if $\mathcal{A}^\#$ is left character approximately Connes-amenable;

(iii) $\mathcal{A} \oplus_1 \mathcal{B}$ is left character approximately Connes-amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are left character approximately Connes-amenable.

Proof. (i) Assume that $\mathcal{A}$ is left φ-approximately Connes-amenable. Thus there exists a net $(m_\alpha) \subseteq \mathcal{A}$ such that $m_\alpha(\varphi) \to 1$ and $am_\alpha - \varphi(a)m_\alpha \to 0$. Consider m_α as an element of $\mathcal{A}^\#$. Hence

$$(a + \lambda)m_\alpha - \widetilde{\varphi}(a + \lambda)m_\alpha = am_\alpha - \varphi(a)m_\alpha \to 0 \quad (a \in \mathcal{A}, \quad \lambda \in \mathbb{C}),$$

and also $m_\alpha(\widetilde{\varphi}) = m_\alpha(\varphi) \to 1$. Thus $\mathcal{A}^\#$ is left $\widetilde{\varphi}$-approximately Connes-amenable. For the converse, suppose that $\mathcal{A}^\#$ is left $\widetilde{\varphi}$-approximately Connes-amenable and $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ is a ω^*-continuous derivation which $\mathcal{X}^*$ is a left φ-normal dual Banach $\mathcal{A}$-bimodule. Define $x \cdot e = x$ for $x \in \mathcal{X}$. So, $\mathcal{X}$ can be considered as a left $\widetilde{\varphi}$-normal dual Banach $\mathcal{A}^\#$-bimodule with module actions $(a + \lambda) \cdot x = (\varphi(a) + \lambda)x$ and $x \cdot (a + \lambda) = xa + \lambda x$. Therefore the mapping of $\widetilde{D} : \mathcal{A}^\# \longrightarrow \mathcal{X}^*$, $\widetilde{D}(a + \lambda) = D(a)$ is a ω^*-continuous derivation. By Theorem 2.5, there exists a net $(x^*_\alpha) \subseteq \mathcal{X}^*$ such that

$$D(a) = \widetilde{D}(a + \lambda) = \lim_\alpha (a + \lambda)x^*_\alpha - (\varphi(a) + \lambda)x^* = \lim_\alpha ax^*_\alpha - \varphi(a)x^*_\alpha,$$

once more by Theorem 2.5, it follows that $\mathcal{A}$ is left φ-approximately Connes-amenable. The Part (ii) is a immediately consequence of part (i).

(iii) Assume $\mathcal{A} \oplus_1 \mathcal{B}$ is left character approximately Connes-amenable. The natural projections on $\mathcal{A}$ and $\mathcal{B}$ are ω^*-continuous and so $\mathcal{A}$ and $\mathcal{B}$ are left character approximately Connes-amenable by the part (iii) of Proposition 2.4. Conversely, suppose that $\mathcal{A}$ and $\mathcal{B}$ are left character approximately Connes-amenable. Without loss of generality in view of part(ii), we can consider $\mathcal{A}$ and $\mathcal{B}$ with identities $e_\mathcal{A}$ and $e_\mathcal{B}$, respectively and $\varphi \in \sigma_{\omega^*}(\mathcal{A} \oplus_1 \mathcal{B})$. There exists two cases, if $\varphi|_\mathcal{A} \neq 0$, then $\varphi(e_\mathcal{A}) = 1$. Since $\varphi(e_\mathcal{A} \oplus e_\mathcal{B}) = 1$, so $\varphi(e_\mathcal{B}) = 0$, $\varphi|_\mathcal{B} = 0$. On the other hand, $\mathcal{A}$ is left $\varphi|_\mathcal{A}$-approximately Connes amenable so there is a net $(m_\alpha) \subseteq \mathcal{A}$ such that $am_\alpha - \varphi(a)m_\alpha \to 0$ for each $a \in \mathcal{A}$ and $\langle m_\alpha, \varphi \rangle \to 1$. Consider the net $(m_\alpha \oplus 0) \subseteq \mathcal{A} \oplus_1 \mathcal{B}$, then for each $a \oplus b \in \mathcal{A} \oplus_1 \mathcal{B}$, we have $(a \oplus b)(m_\alpha \oplus 0) - \varphi(a \oplus b)(m_\alpha \oplus 0) = am_\alpha - \varphi(a)m_\alpha \to 0$ and $\langle m_\alpha \oplus 0, \varphi \rangle = \langle m_\alpha, \varphi \rangle \to 1$, it follows that $\mathcal{A} \oplus_1 \mathcal{B}$ is left φ-approximately Connes-amenable. For the other case, that is $\varphi|_\mathcal{B} \neq 0$. Similarly, one can show that $\mathcal{A} \oplus_1 \mathcal{B}$ is left φ-approximately Connes-amenable. Moreover, $\mathcal{A} \oplus_1 \mathcal{B}$ has an identity $e_\mathcal{A} \oplus e_\mathcal{B}$ and hence it is left 0-approximately Connes-amenable by Proposition 2.3. Therefore, $\mathcal{A} \oplus_1 \mathcal{B}$ is left character approximately Connes-amenable. □

Here, we have two observations for Arens regular Banach algebra $\mathcal{A}$ with extra condition that $\mathcal{A}$ is an ideal in $\mathcal{A}^{**}$.

• If $\mathcal{A}^{**}$ is Connes-amenable, then so is $\mathcal{A}$ [17, Theorem 4.4].

• If $\mathcal{A}^{**}$ is approximately Connes-amenable then so is $\mathcal{A}$ [8, Theorem 7.1(ii)].

We prove the above assertions for left character approximate Connes-amenability case only with Arens regularity condition. Recall that every $\varphi \in \sigma(\mathcal{A})$ has a unique extension to $\varphi^{**} \in \sigma_{\omega^*}(\mathcal{A}^{**})$.

Theorem 2.12. *Let $\mathcal{A}$ be an Arens regular Banach algebra and $\varphi \in \sigma(\mathcal{A})$. If $\mathcal{A}^{**}$ is left φ^{**}-approximately Connes-amenable, then $\mathcal{A}$ is left φ-approximately amenable.*

Proof. By the hypothesis there exists a net $(m_\alpha) \subseteq \mathcal{A}^{**}$ such that $m_\alpha(\varphi^{**}) \to 1$ and $a^{**}m_\alpha - \varphi^{**}(a^{**})m_\alpha \to 0$. For each α, consider a net $(a^\alpha_\beta) \subseteq \mathcal{A}$ which $a^\alpha_\beta \to m_\alpha$ in the ω^*-topology. Since φ^{**} is ω^*-continues, $\varphi^{**} \in \mathcal{A}^*$ and we get

$$\langle m_\alpha, \varphi\rangle = \lim_\beta \langle a^\alpha_\beta, \varphi\rangle = \lim_\beta \langle a^\alpha_\beta, \varphi^{**}\rangle = \langle m_\alpha, \varphi^{**}\rangle \to 1.$$

Moreover for each $a \in \mathcal{A}$, $am_\alpha - \varphi(a)m_\alpha \to 0$, which finishes the proof. □

Let $L^2(\mathcal{A}, \mathbb{C})$ be the space of all bounded bilinear functionals on $\mathcal{A}$ and $L^2_{\omega^*}(\mathcal{A}, \mathbb{C})$ be the space of separately ω^* continuous elements of $L^2(\mathcal{A}, \mathbb{C})$. Following [18], we turn $L^2(\mathcal{A}, \mathbb{C})$ into a Banach $\mathcal{A}$-bimodule through the identification $L^2(\mathcal{A}, \mathbb{C}) \simeq (\mathcal{A}\widehat{\otimes}\mathcal{A})^*$. The module actions of $\mathcal{A}$ on $L^2(\mathcal{A}, \mathbb{C})$ are as follow:

$$(a.F)(b, c) = F(b, ca), \quad (F.a)(b, c) = F(ab, c) \quad (a, b, c \in \mathcal{A}, \quad F \in L^2(\mathcal{A}, \mathbb{C})).$$

Clearly, $L^2_{\omega^*}(\mathcal{A}, \mathbb{C})$ is a Banach $\mathcal{A}$-submodule of $L^2(\mathcal{A}, \mathbb{C})$. We also have a natural $\mathcal{A}$-bimodule map

$$\theta : \mathcal{A} \otimes \mathcal{A} \longrightarrow L^2_{\omega^*}(\mathcal{A}, \mathbb{C})^*, \quad \theta(a \otimes b)(F) = F(a, b).$$

Since $\mathcal{A}_* \otimes \mathcal{A}_* \subseteq L^2_{\omega^*}(\mathcal{A}, \mathbb{C})$ and $\mathcal{A}_* \otimes \mathcal{A}_*$ separates the points of $\mathcal{A} \otimes \mathcal{A}$, θ is one-to-one. We will identify $\mathcal{A} \otimes \mathcal{A}$ with its image as

$$\mathcal{A} \otimes \mathcal{A} \subseteq L^2_{\omega^*}(\mathcal{A}, \mathbb{C})^*.$$

For the map

$$\Delta_{\mathcal{A}} : \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}; \quad a \otimes b \mapsto ab \quad (a, b \in \mathcal{A}).$$

Since the multiplication in a dual Banach algebra is separately $\omega^* - \omega^*$-continuous, we have

$$\Delta^*_{\mathcal{A}}(\mathcal{A}_*) \subset L^2_{\omega^*}(\mathcal{A}, \mathbb{C}).$$

Therefore the restriction of $\Delta^{**}_{\mathcal{A}}$ to $L^2_{\omega^*}(\mathcal{A}, \mathbb{C})^*$ turns into a Banach $\mathcal{A}$-bimodule homomorphism

$$\Delta_{\omega^*} : L^2_{\omega^*}(\mathcal{A}, \mathbb{C})^* \longrightarrow \mathcal{A}.$$

Definition 2.13. *Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \sigma_{\omega^*}(\mathcal{A})$. A net $(M_\alpha) \subseteq L^2_{\omega^*}(\mathcal{A}, \mathbb{C})^*$ is said to be a left φ-approximately normal virtual diagonal for $\mathcal{A}$ if*

$$a.M_\alpha - \varphi(a)M_\alpha \longrightarrow 0 \quad (a \in \mathcal{A}) \quad \text{and} \quad \langle M_\alpha, \varphi \otimes \varphi\rangle = \langle \Delta_{\omega^*}(M_\alpha), \varphi\rangle \to 1.$$

For a dual Banach algebra $\mathcal{A}$, if $\mathcal{A}$ has a (approximate) normal virtual diagonal, then it is (approximately) Connes-amenable (see [8, Theorem 3.1] and [18, Theorem 4.4.15] for more details). The converse of the mentioned results is no longer valid for the Connes-amenability case [22]. However, we can characterize left φ-approximate Connes-amenability of dual Banach algebras in terms of left φ-approximate normal virtual diagonals as follows:

Theorem 2.14. *Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \sigma_{\omega^*}(\mathcal{A})$. Then the following are equivalent:*

(i) $\mathcal{A}$ is left φ-approximately Connes-amenable;
(ii) There exists a net $(M_\alpha) \subseteq L^2_{\omega^*}(\mathcal{A}, \mathbb{C})^*$ such that $a.M_\alpha - \varphi(a)M_\alpha \to 0$ $(a \in \mathcal{A})$ and $\langle M_\alpha, \varphi \otimes \varphi\rangle = \langle \Delta_{\omega^*}(M_\alpha), \varphi\rangle = 1$ for each α;
(iii) $\mathcal{A}$ has a left φ-approximately normal virtual diagonal.

Proof. (i)⇒(ii) By Theorem 2.5 there exists a net $(m_\alpha) \subseteq \mathcal{A}$ such that $am_\alpha - \varphi(a)m_\alpha \to 0$ and $m_\alpha(\varphi) = 1$ for each $a \in \mathcal{A}$ and for each α. Choosing a $m_0 \in \mathcal{A}$ which $\langle m_0, \varphi\rangle = 1$ and in view of $\mathcal{A} \otimes \mathcal{A} \subseteq L^2_{\omega^*}(\mathcal{A}, \mathbb{C})^*$, define $M_\alpha = m_\alpha \otimes m_0 \in L^2_{\omega^*}(\mathcal{A}, \mathbb{C})^*$. Thus for $a \in \mathcal{A}$ we get

$$aM_\alpha - \varphi(a)M_\alpha = am_\alpha \otimes m_0 - \varphi(a)m_\alpha \otimes m_0 = (am_\alpha - \varphi(a)m_0) \otimes m_0 \to 0,$$

and $\langle \Delta_{\omega^*}(M_\alpha), \varphi\rangle = \langle m_\alpha \otimes m_0, \varphi \otimes \varphi\rangle = \langle m_\alpha, \varphi\rangle\langle m_0, \varphi\rangle = 1$ for each α.

(ii)⇒(iii) It is clear.

(iii)⇒(i) Assume that (M_α) is a left φ-approximately normal virtual diagonal for $\mathcal{A}$, define $m_\alpha = \Delta_{\omega^*}(M_\alpha) \in \mathcal{A}$. Hence $am_\alpha - \varphi(a)m_\alpha = \Delta_{\omega^*}(a.M_\alpha - \varphi(a)M_\alpha) \to 0$ and $\langle m_\alpha, \varphi\rangle = \langle \Delta_{\omega^*}(M_\alpha), \varphi\rangle \to 1$. It implies that $\mathcal{A}$ is left φ-approximately Connes-amenable. □

In the next theorem we characterize left φ-Connes-amenability of dual Banach algebra $\mathcal{A}$ in terms of some bounded nets in $\mathcal{A}$ and $\mathcal{A}\widehat{\otimes}\mathcal{A}$.

Theorem 2.15. *Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \sigma_{\omega^*}(\mathcal{A})$. Then the following are equivalent:*

(i) $\mathcal{A}$ is left φ-Connes-amenable;

(ii) There is a bounded net $(m_\alpha) \subseteq \mathcal{A}$ such that

$$am_\alpha - \varphi(a)m_\alpha \to 0 \quad \textit{and} \quad \langle m_\alpha, \varphi\rangle \to 1;$$

(iii) There is a bounded net $(N_\alpha) \subseteq \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that

$$a.N_\alpha - \varphi(a)N_\alpha \to 0 \quad \textit{and} \quad \langle \Delta_{\mathcal{A}}(N_\alpha), \varphi\rangle \to 1.$$

Proof. (i)⇒(ii) It is obvious.

(ii)⇒(iii) Choose a $m_0 \in \mathcal{A}$ such that $\langle m_0, \varphi\rangle = 1$ and define $N_\alpha = m_\alpha \otimes m_0 \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ for each α. It is easy to check that the net (N_α) works as required.

(iii)⇒(i) Define $m_\alpha = \Delta_{\mathcal{A}}(N_\alpha)$ for each α. Therefore for each $a \in \mathcal{A}$,

$$\lim_\alpha (am_\alpha - \varphi(a)m_\alpha) = \lim_\alpha \Delta_{\mathcal{A}}(a.N_\alpha - \varphi(a)N_\alpha) = 0,$$

and $\langle m_\alpha, \varphi\rangle = \langle \Delta_{\mathcal{A}}(N_\alpha), \varphi\rangle \longrightarrow 1$. On the other hand the net (m_α) is bounded and $\mathcal{A}$ is a dual Banach algebra, thus there is a $m \in \mathcal{A}$ such that,

$$m_\alpha \xrightarrow{\omega^*} m \quad \text{and} \quad am_\alpha - \varphi(a)m_\alpha \xrightarrow{\omega^*} am - \varphi(a)m.$$

Consequently, $am = \varphi(a)m$ and $\langle m, \varphi\rangle = \lim_\alpha\langle m_\alpha, \varphi\rangle = 1$. It follows that $\mathcal{A}$ is left φ-Connes-amenable. □

In the next theorem we present other characterizations of left φ-approximate Connes-amenability of dual Banach algebras.

Theorem 2.16. *Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \sigma_{\omega^*}(\mathcal{A})$. Then the following are equivalent:*

(i) $\mathcal{A}$ is left φ-approximately Connes-amenable;

(ii) There is a net $(m_\alpha) \subseteq \mathcal{A}$ such that

$$am_\alpha - \varphi(a)m_\alpha \xrightarrow{\omega} 0 \quad \textit{and} \quad \langle m_\alpha, \varphi\rangle \to 1;$$

(iii) There is a net $(N_\alpha) \subseteq \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that

$$a.N_\alpha - \varphi(a)N_\alpha \to 0 \quad \textit{and} \quad \langle \Delta_{\mathcal{A}}(N_\alpha), \varphi\rangle \to 1.$$

Proof. (i)⇒(ii). It is clear.

(ii)⇒(iii) For each finite set $F = \{a_1, ..., a_n\} \subseteq \mathcal{A}$, we get

$$(a_1 m_\alpha - \varphi(a_1) m_\alpha, ..., a_n m_\alpha - \varphi(a_n) m_\alpha, \langle \varphi, m_\alpha \rangle) \xrightarrow{\omega} (0, \cdots, 0, 1)$$

in $\mathcal{A}^n \oplus \mathbb{C}$. With applying Mazur's theorem, we can obtain another net $(n_\alpha) \subseteq \mathcal{A}$, which is called again (m_α) such that $am_\alpha - \varphi(a)m_\alpha \to 0$ in norm topology and $\langle m_\alpha, \varphi \rangle \to 1$. Now, consider $m_0 \in \mathcal{A}$ such that $\langle m_0, \varphi \rangle = 1$ and define $N_\alpha = m_\alpha \otimes m_0 \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ for each α. It is easily to verified that $a.N_\alpha - \varphi(a)N_\alpha \to 0$ and $\langle \Delta_{\mathcal{A}}(N_\alpha), \varphi \rangle \to 1$.

(iii)⇒(i) Define $m_\alpha = \Delta_{\mathcal{A}}(N_\alpha)$ for each α. Thus for each $a \in \mathcal{A}$, $am_\alpha - \varphi(a)m_\alpha \to 0$ and $\langle m_\alpha, \varphi \rangle \to 1$, as required. □

Let $\mathcal{A}$ be a dual Banach algebra. Recall that a point derivation d at a character $\varphi \in \sigma(\mathcal{A})$ is a linear functional $d : \mathcal{A} \longrightarrow \mathbb{C}$ such that $d(ab) = d(a)\varphi(b) + \varphi(a)d(b)$ for each $a, b \in \mathcal{A}$.

Proposition 2.17. *Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \sigma_{\omega^*}(\mathcal{A})$. If $\mathcal{A}$ has a non-zero continuous point derivation at φ, then $\mathcal{A}$ is not left φ-approximately Connes-amenable.*

Proof. Suppose that d is a non-zero continuous point derivation at φ and $\mathcal{A}$ is left φ-approximately Connes-amenable. Thus there is a net $(m_\alpha) \subseteq \mathcal{A}$ such that $am_\alpha - \varphi(a)m_\alpha \to 0$ and $\langle m_\alpha, \varphi \rangle \to 1$ for every $a \in \mathcal{A}$. Consequently

$$\begin{aligned} 0 &= \lim_\alpha d(am_\alpha - \varphi(a)m_\alpha) = \lim_\alpha (d(a)\varphi(m_\alpha) + \varphi(a)d(m_\alpha) - \varphi(a)d(m_\alpha)) \\ &= \lim_\alpha \varphi(m_\alpha)d(a) = d(a). \end{aligned}$$

Thus d=0, which contradicts the assumption. □

Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras, $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ a ω^*-continuous homomorphism and $\varphi \in \sigma_{\omega^*}(\mathcal{A}) \cup \{0\}$. Then $\mathcal{B}$ can be considered as a Banach $\mathcal{A}$-bimodule by the following module actions

$$a.b = \theta(a)b, \quad b.a = \varphi(a)b \quad (a \in \mathcal{A}, \quad b \in \mathcal{B}).$$

We denote the above left φ-normal dual Banach $\mathcal{A}$-bimodule by $\mathcal{B}_{(\theta,\varphi)}$. The next Theorem generalize [7, Theorem 2.3], and give another characterizations for left φ-approximate Connes- amenability of dual Banach algebras.

Theorem 2.18. *Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras and $\varphi \in \sigma_{\omega^*}(\mathcal{A}) \cup \{0\}$. The following are equivalent:*

(i) $\mathcal{A}$ is left φ-approximately Connes-amenable;

(ii) For every ω^*-continuous homomorphism $\theta : \mathcal{A} \to \mathcal{B}$, then every ω^*-continuous derivation $D : \mathcal{A} \to \mathcal{B}_{(\theta,\varphi)}$ is approximately inner;

(iii) For every injective ω^*-continuous homomorphism $\theta : \mathcal{A} \to \mathcal{B}$, then every ω^* continuous derivation $D : \mathcal{A} \to \mathcal{B}_{(\theta,\varphi)}$ is approximately inner.

Proof. According to Theorem 2.5, the implications (i)⇒(ii) and (ii)⇒(iii) are obvious. It suffices to prove that (iii) implies that (i). Assume that $\mathcal{X}^*$ is a left φ-normal dual Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \to \mathcal{X}^*$ is a ω^*-continuous derivation. In view of [7, Lemma 2.2], $\mathcal{A} \oplus_\infty \mathcal{X}^*$ is a dual Banach algebra, and the map $\theta : \mathcal{A} \to \mathcal{A} \oplus_\infty \mathcal{X}^*$ by $\theta(a) = (a, 0)$ is an injective ω^*-continuous homomorphism. Thus $(\mathcal{A} \oplus_\infty \mathcal{X}^*)_{(\theta,\varphi)}$ is a left

φ-normal dual Banach $\mathcal{A}$-bimodule. We now define $D_1 : \mathcal{A} \to (\mathcal{A} \oplus_\infty \mathcal{X}^*)_{(\theta,\varphi)}$ through $D_1(a) = (0, D(a))$, then for each $a, b \in \mathcal{A}$ we get

$$\begin{aligned} D_1(ab) &= (0, Dab) = (0, D(a) \cdot b + a \cdot D(b)) \\ &= \varphi(b)(0, D(a)) + (a, 0) \cdot (0, D(b)) \\ &= \varphi(b)(0, D(a)) + \theta(a)(0, D(b)) \\ &= \varphi(b)D_1(a) + \theta(a)D_1(b), \end{aligned}$$

consequently, D_1 is a ω^*-continuous derivation. Hence by the assumption D_1 is approximately inner, so there is a net $((a_\alpha, x^*_\alpha)) \subseteq (\mathcal{A} \oplus_\infty \mathcal{X}^*)_{(\theta,\varphi)}$ such that

$$\begin{aligned} (0, D(a)) = D_1(a) &= \lim_\alpha (a \cdot (a_\alpha, x^*_\alpha) - (a_\alpha, x^*_\alpha) \cdot a) \\ &= \lim_\alpha (\theta(a) \cdot (a_\alpha, x^*_\alpha) - \varphi(a)(a_\alpha, x^*_\alpha)) \\ &= \lim_\alpha ((a, 0)(a_\alpha, x^*_\alpha) - \varphi(a)(a_\alpha, x^*_\alpha)) \\ &= \lim_\alpha (aa_\alpha - \varphi(a)a_\alpha, a \cdot x^*_\alpha - \varphi(a)x^*_\alpha), \end{aligned}$$

and therefore $D(a) = \lim_\alpha (a.x^*_\alpha - \varphi(a)x^*_\alpha)$. Theorem 2.5 implies that $\mathcal{A}$ is left φ-approximately Connes-amenable. □

Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras and $\theta \in \sigma_{\omega^*}(\mathcal{B})$. Then the θ-Lau product $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \times_\theta \mathcal{B}$, is defined as the set $\mathcal{A} \times \mathcal{B}$ equipped with the multiplication

$$(a, b)(a', b') = (aa' + \theta(b)a' + \theta(b')a, bb'),$$

and norm $\|(a, b)\| = \|a\| + \|b\|$. It is known that $\mathcal{A}$ is a closed two sided ideal of $\mathcal{A} \times_\theta \mathcal{B}$ and $\mathcal{A} \times_\theta \mathcal{B}/\mathcal{A} \cong \mathcal{B}$. Similar to [23] one may prove that $\sigma_{\omega^*}(\mathcal{A} \times_\theta \mathcal{B}) = \sigma_{\omega^*}(\mathcal{A}) \times \{\theta\} \cup \{0\} \times \sigma_{\omega^*}(\mathcal{B})$.

Lemma 2.19. *Let net (m_α, n_α) and (a,b) be in $\mathcal{A} \times_\theta \mathcal{B}$.*

(i) $\varphi \in \sigma_{\omega^*}(\mathcal{A})$, $(a, b)(m_\alpha, n_\alpha) - (\varphi(a) + \theta(b))(m_\alpha, n_\alpha) \to 0$ and $\langle m_\alpha, \varphi\rangle + \langle n_\alpha, \theta\rangle \to 1$ if and only if $am_\alpha - \varphi(a)m_\alpha \to 0$, $\langle m_\alpha, \varphi\rangle \to 1$ and $n_\alpha \to 0$;

(ii) Suppose that $\theta(n_\alpha) = 1$ for each α. Then $(a, b)(m_\alpha, n_\alpha) - \theta(b)(m_\alpha, n_\alpha) \to 0$ if and only if $bn_\alpha - \theta(b)n_\alpha \to 0$ and net $(-m_\alpha)$ is a right approximate identity for $\mathcal{A}$;

(iii) Suppose that $\psi \in \sigma_{\omega^*}(\mathcal{B})$, $\theta \neq \psi$, $\theta(n_\alpha) = 0$ and $\psi(n_\alpha) = 1$ for each α. Then $(a, b)(m_\alpha, n_\alpha) - \psi(b)(m_\alpha, n_\alpha) \to 0$ if and only if $m_\alpha \to 0$ and $an_\alpha - \psi(b)n_\alpha \to 0$.

Proof. (i) Assume that $(a, b)(m_\alpha, n_\alpha) - (\varphi(a) + \theta(b))(m_\alpha, n_\alpha) \to 0$. Then $am_\alpha + \theta(n_\alpha)a - \varphi(a)m_\alpha \to 0$ and $bn_\alpha - \varphi(a)n_\alpha - \theta(b)n_\alpha \to 0$. Choosing $b = 0$ and $\varphi(a) = 1$, we obtain $n_\alpha \to 0$. Hence $am_\alpha - \varphi(a)m_\alpha \to 0$ and $\langle m_\alpha, \varphi\rangle \to 1$. The converse is clear.

(ii) From $(a, b)(m_\alpha, n_\alpha) - \theta(b)(m_\alpha, n_\alpha) \to 0$, we get $am_\alpha + \theta(n_\alpha)a \to 0$ and $bn_\alpha - \theta(b)n_\alpha \to 0$. Since $\theta(n_\alpha) = 1$, $am_\alpha + a \to 0$ and hence the net $(-m_\alpha)$ is a right approximate identity for $\mathcal{A}$. The converse is obvious.

(iii) According to $(a, b)(m_\alpha, n_\alpha) - \psi(b)(m_\alpha, n_\alpha) \to 0$, we obtain $am_\alpha + \theta(b)m_\alpha + \theta(n_\alpha)a - \psi(b)m_\alpha \to 0$ and $an_\alpha - \psi(b)n_\alpha \to 0$. Consider $a = 0$, then $(\theta(b) - \psi(b))m_\alpha \to 0$. Since $\theta \neq \psi$, we have $m_\alpha \to 0$. □

Proposition 2.20. *Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras. Then $\mathcal{A} \times_\theta \mathcal{B}$ is left character approximately Connes-amenable if and only $\mathcal{A}$ and $\mathcal{B}$ are so.*

Proof. Assume that $\mathcal{A} \times_\theta \mathcal{B}$ is left character approximately Connes-amenable. Then by the part (iii) Proposition 2.4, so is $\mathcal{B}$. Now, consider $(0, \theta) \in \sigma_{\omega^*}(\mathcal{A} \times_\theta \mathcal{B})$. By the assumption and Lemma 2.19 (ii), $\mathcal{A}$ has a right approximate identity. It follows from Proposition 2.3 that $\mathcal{A}$ is left 0-approximately Connes-amenable. On the other hand, for each $\varphi \in \sigma_{\omega^*}(\mathcal{A})$ and $(\varphi, \theta) \in \sigma_{\omega^*}(\mathcal{A} \times_\theta \mathcal{B})$, according to the assumption and Lemma 2.19 (i) we get $\mathcal{A}$ is φ-approximately Connes-amenable and so $\mathcal{A}$ is left character approximately Connes-amenable.

Conversely, by Proposition 2.11 (ii) we can consider that $\mathcal{B}$ is unitary with identity $e \in \mathcal{B}$. Since (0,e) is a right identity for $\mathcal{A} \times_\theta \mathcal{B}$, it is left 0-approximately Connes-amenable by Proposition 2.3. The rest of the proof can be obtained immediately from Lemma 2.19. □

3. Examples

It is clear that every dual Banach algebra which is either Connes-amenable or approximately Connes-amenable, then it is left character approximately Connes-amenable. In this section, by using some examples we show that the converse of these results do not hold in general.

Example 3.1. Consider the Banach algebra $\ell^1 = \ell^1(\mathbb{N})$ of all sequences $a = (a(n))$ of complex numbers with $\|a\| = \sum_{n=1}^{\infty} |a(n)| < \infty$ along with product $\circ$ defined by

$$a \circ b(n) = \begin{cases} a(1)b(1) & n=1 \\ a(1)b(n) + b(1)a(n) + a(n)b(n) & n > 1 \end{cases}$$

for $a, b \in \ell^1$. It is easy to check that ℓ^1 with this product is a dual Banach algebra and

$$\sigma_{\omega^*}(\ell^1) = \sigma(\ell^1) = \{\varphi_1\} \cup \{\varphi_1 + \varphi_n : n \geq 2\},$$

where $\varphi_n(a) = a(n)$ for each $a \in \ell^1$. It was shown in [16, Example 2.9(a)] that there is no bounded net $(m_\alpha) \subseteq \ell^1$ such that $am_\alpha - \varphi_1(a)m_\alpha \to 0$ and $\langle m_\alpha, \varphi_1 \rangle \to 1$. Theorem 2.15 implies that ℓ^1 is not left φ_1-Connes amenable and hence it is not Connes-amenable. However, ℓ^1 is $(\varphi_1 + \varphi_n)$-Connes amenable by [16, Example 2.9(a)]. On the other hand, the sequence $(m_k) \subseteq \ell^1$ defined by

$$m_k(n) = \begin{cases} 1 & n = 1 \\ -1 & 1 < n \leq k \\ 0 & n > k \end{cases}$$

satisfies in Definition 2.1 for every $\varphi \in \sigma_{\omega^*}(\ell^1)$ and therefore ℓ^1 is φ-approximately Connes-amenable for each $\varphi \in \sigma_{\omega^*}(\ell^1)$ [16, Example 2.9(a)]. Now, consider ℓ^1 as a dual Banach algebra with pointwise multiplication. It is easily verified that

$$\sigma_{\omega^*}(\ell^1) = \sigma(\ell^1) = \{\varphi_n : n \in \mathbb{N}, \langle \varphi_n, a \rangle = a(n), \quad (a \in \ell^1)\}.$$

For each $n \in \mathbb{N}$, set $m = \varphi_n \in \ell^1$. Then

$$am - \varphi_n(a)m = a\varphi_n - \varphi_n(a)\varphi_n = 0 \quad \textit{and} \quad \langle \varphi_n, \varphi_n \rangle = 1,$$

for every $a \in \ell^1$. Thus ℓ^1 is left φ_n-Connes amenable. Furthermore, ℓ^1 has an approximate identity and so it is left 0-approximately Connes-amenable by Proposition 2.3. Consequently, ℓ^1 is left character approximately Connes-amenable, but it is neither approximately Connes-amenable [14, Theorem 3.2] nor approximately amenable [6, Theorem 4.1].

Example 3.2. Consider the discrete convolution algebra ℓ^1 consist of all sequences of $a = (a(n))$ which $\|a\| = \sum_{n=1}^{\infty} |a(n)| < \infty$. Since $\mathbb{N}$ is weakly cancellative semigroup, ℓ^1 is a dual Banach algebra [5, Theorem 4.6]. Moreover, $\sigma(\ell^1) = \{\varphi_z : z \in \mathbb{Z}, |z| \leq 1\}$, where $\varphi_z(a) = \sum_{n=1}^{\infty} a(n)z^n$ for each $a \in \ell^1$. Thus for each $z \in \mathbb{C}$ and $|z| < 1$, $\varphi_z = (z, z^2, z^3, \cdots) \in c_0$. Consequently $\sigma_{\omega^*}(\ell^1) = \{\varphi_z : z \in \mathbb{C}, |z| < 1\}$. Therefore for each $|z| < 1$, the map $f \longmapsto f'(z)$ is a non-trivial bounded point derivation at φ_z, and hence ℓ^1 is not left φ_z-approximately Connes-amenable by Proposition 2.17.

Example 3.3. Let G be a locally compact group, then $\sigma_{\omega^*}(M(G)) = \emptyset$ by [25, Example 3.1]. So $M(G)$ is always left character approximately Connes-amenable. But for non-amenable G, $M(G)$ is neither Connes amenable [19] (approximately Connes amenable [8, Theorem 5.2]) nor character amenable [23, Corollary 2.5].

Example 3.4. Consider the dual Banach algebra $\mathcal{A} = \ell^1(\mathbb{N})$ with pointwise product which has an approximate identity. For each $n \in \mathbb{N}$, $M_n(\mathcal{A}) = \mathcal{A}\widehat{\otimes}M_n$ is a dual Banach algebra with approximate identity. Since $\sigma(M_n) = \emptyset$, we find $\sigma_{\omega^*}(M_n(\mathcal{A})) = \emptyset$ for each $n \geq 2$. Therefore $M_n(\mathcal{A})$ is always left character approximately Connes-amenable by Proposition 2.3. However $M_n(\mathcal{A})$ is not left character Connes-amenable [25, Lemma 2.9].

Example 3.5. Let $\mathbb{F}_2$ be a free group on two generators and $VN(\mathbb{F}_2)$ denote the von Neumann algebra generated by the left regular representation λ of $\mathbb{F}_2$ on $\ell^2(\mathbb{F}_2)$. Then $VN(\mathbb{F}_2)$ is left character approximately Connes-amenable [25, Example 3.1], but $VN(\mathbb{F}_2)$ is neither amenable [17, Corollary 6.7] (Connes-amenable [18, Theorem 4.4.13]) nor approximately Connes-amenable [8, Corollary 6.3]. Moreover, for a non-discrete, amenable locally compact group G, the dual Banach algebra $\mathcal{A} = M(G) \oplus_1 VN(\mathbb{F}_2)$ is left character approximately Connes-amenable by Proposition 2.11 part(iii). On the other hand, the projections $\pi_1 : \mathcal{A} \to M(G)$ and $\pi_2 : \mathcal{A} \to VN(\mathbb{F}_2)$ are norm and w^*-continuous homomorphisms, respectively. In addition, $M(G)$ is not left character amenable [23, Corollary 2.5] and $VN(\mathbb{F}_2)$ is not approximately Connes-amenable, and hence $\mathcal{A}$ is neither left character amenable [23, Theorem 2.6(i)] nor approximately Connes amenable [8, Proposition 2.3(ii)].

Note that the converse of the part (iii) from Proposition 2.4 is not true. Since for a non-amenable group G, $M(G)$ is always left character approximately Connes-amenable by Example 3.3 while it is not left character approximately amenable [1, Theorem 7.2].

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