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Wiener-Type Invariants and Hamiltonian Properties of Graphs

Qiannan Zhou[a,b], Ligong Wang[a,b], Yong Lu[c]

[a]*Department of Applied Mathematics, School of Science, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, P.R. China*
[b]*Xi'an-Budapest Joint Research Center for Combinatorics, Northwestern Polytechnical University, Xi'an, Shaanxi 710129, P.R. China*
[c]*School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, Jiangsu 221116, P.R. China.*

Abstract. The Wiener-type invariants of a simple connected graph $G = (V(G), E(G))$ can be expressed in terms of the quantities $W_f = \sum_{\{u,v\}\subseteq V(G)} f(d_G(u, v))$ for various choices of the function $f(x)$, where $d_G(u, v)$ is the distance between vertices u and v in G. In this paper, we give some sufficient conditions for a bipartite graph to be Hamiltonian or a connected general graph to be Hamilton-connected and traceable from every vertex in terms of the Wiener-type invariants of G or the complement of G.

1. Introduction

In this paper, we only consider finite undirected graphs without loops and multiple edges. We use Bondy and Murty [2] for terminology and notation not defined here. Let $G = (V(G), E(G))$ denote a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Denote by $d_i = d_{v_i} = d_G(v_i)$ the degree of v_i. Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of the graph G, where $d_1 \leq d_2 \leq \cdots \leq d_n$. In addition, $\overline{G}$ denotes the complement of G. Let $G := G[X, Y]$ be a bipartite graph with bipartition (X, Y). The bipartite graph $G^* := G^*[X, Y]$ is called the quasi-complement of G, which is constructed as follows: $V(G^*) = V(G)$ and $xy \in E(G^*)$ if and only if $xy \notin E(G)$ for $x \in X$, $y \in Y$. Let G and H be two disjoint graphs. The disjoint union of G and H, denoted by $G + H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The disjoint union of k graphs G is denoted by kG. The join of G and H, denoted by $G \vee H$, is the graph obtained from disjoint union of G and H by adding edges joining every vertex of G to every vertex of H.

A path (cycle) is called a Hamilton path (Hamilton cycle) if it contains every vertex of a graph. The graph is said to be traceable (Hamiltonian) if it has a Hamilton path (cycle). The graph G is called Hamilton-connected if every two vertices of G are connected by a Hamilton path. Surely all Hamilton-connected graphs are Hamiltonian. A graph is called traceable from a vertex x if it has a Hamilton x-path.

For $v_i, v_j \in V(G)$, let $d_G(v_i, v_j)$ denote the distance between v_i and v_j. The Wiener index $W(G)$ of a connected graph G is defined by

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u, v).$$

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Corresponding author: Ligong Wang
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Email addresses: qnzhoumath@163.com (Qiannan Zhou), lgwangmath@163.com (Ligong Wang), luyong@jsnu.edu.cn (Yong Lu)

The Wiener index was introduced in 1947 by Wiener [26], who used it for modeling the shape of organic molecules and for calculating several of their physiso-chemical properties. We can refer to [5, 6, 25, 26] to know more details on vertex distances and Wiener index.

The Harary index $H(G)$ of a graph G has been introduced independently by Ivanciuc et al. [11] and Plavšić et al. [23] in 1993 for the characterization of molecular graphs. It has been named in honor of Professor Frank Harary on the occasion of his 70th birthday. The definition of Harary index is as follows:

$$H(G) = \sum_{\{u,v\}\subseteq V(G)} \frac{1}{d_G(u,v)}.$$

We can refer to [3, 21, 27, 29, 31] to know more details on Harary index.

Some generalizations and modification of the Wiener index were put forward. Many of these Wiener-type invariants can be expressed in terms of the quantities

$$W_f = W_f(G) = \sum_{\{u,v\}\subseteq V(G)} f(d_G(u,v)),$$

for various choices of the function $f(x)$. We can see that when $f(x) = x$, W_x is the Wiener index; when $f(x) = \frac{1}{x}$, $W_{\frac{1}{x}}$ is the Harary index; when $f(x) = \frac{x^2+x}{2}$, $W_{\frac{x^2+x}{2}}$ is called the hyper-Wiener index [24], which is denoted by WW; when $f(x) = x^{\lambda}$, where $\lambda \neq 0$ is a real number, $W_{x^{\lambda}}$ is called the modified Wiener index [7], which is denoted by W_{λ}. We can refer to [4, 8, 13] to know more details on Wiener-type invariants.

The problem of deciding whether a given graph is Hamiltonian or not is one of the most difficult classical problems in graph theory. There are many sufficient conditions in terms of vertex degree, spectral radius or signless Laplace spectral radius for a graph to be Hamiltonian, traceable, Hamilton-connected or traceable from every vertex. In recent years, some sufficient conditions in terms of Wiener index and Harary index are given for a graph to be Hamiltonian and traceable. We can refer to [9, 15, 16, 19, 20, 28, 30] to see more details. In 2016, Kuang et al. [14] generalized some results to a more general version. Hua and Ning [10] remarked that some of known theorems can be unified in a short proof.

In this paper, we mainly give some sufficient conditions in terms of Wiener-type invariants for Hamilton-connectivity. In Section 2, firstly, we give a sufficient condition for a bipartite graph to be Hamiltonian in terms of its Wiener-type index, which is an improvement of the Theorem 14 in [14]. Furthermore, we present some sufficient conditions for a connected general graph to be Hamilton-connected and traceable from every vertex in terms of its Wiener-type index. In Section 3, we give a sufficient condition for a bipartite graph to be Hamiltonian in terms of the Wiener-type index of its quasi-complement. We also present some sufficient conditions for a general connected graph to be Hamiltonian, traceable, Hamilton-connected and traceable from every vertex in terms of the Wiener-type index of its complement.

2. Wiener-type index conditions on Hamiltonian bipartite graphs and Hamilton-connected graphs

In this section, we will give a sufficient condition for a bipartite graph to be Hamiltonian, which is an improvement of the Theorem 14 in [14], and some sufficient conditions for a connected general graph to be Hamilton-connected and traceable from every vertex in terms of the Wiener-type index.

Lemma 2.1. *([2]) Let $G := G[X, Y]$ be a bipartite graph with degree sequence $(d_1, d_2, \dots, d_{2n})$, where $|X| = |Y| = n$ and $d_1 \le d_2 \le \cdots \le d_{2n}$. If there is no integer $k \le \dfrac{n}{2}$ such that $d_k \le k$ and $d_n \le n - k$, then G is Hamiltonian.*

Let $K_{p,n-2} + 4e$ be a bipartite graph obtained from $K_{p,n-2}$ by adding two vertices which are adjacent to two common vertices with degree $n-2$ in $K_{p,n-2}$, where $p \ge n-1$.

Theorem 2.2. *Let $G := G[X, Y]$ be a bipartite graph with minimum degree $\delta \ge 2$, where $|X| = |Y| = n \ge 3$. If*

$$W_f(G) \le [f(1) + f(2)]n^2 - [2f(1) + f(2) - 2f(3)]n - 4[f(3) - f(1)],$$

for a monotonically increasing function $f(x)$ on $x \in [1, 2n-1]$, or

$$W_f(G) \geq [f(1) + f(2)]n^2 - [2f(1) + f(2) - 2f(3)]n + 4[f(1) - f(3)],$$

for a monotonically decreasing function $f(x)$ on $x \in [1, 2n-1]$, then G is Hamiltonian unless $G = K_{n,n-2} + 4e$.

Proof. Assume that G is not Hamiltonian and has degree sequence $(d_1, d_2, \ldots, d_{2n})$, where $d_1 \leq d_2 \leq \cdots \leq d_{2n}$. By Lemma 2.1, there is an integer k such that $2 \leq \delta \leq d_k \leq k \leq \frac{n}{2}$ and $d_n \leq n-k$. Note that G is connected. If $f(x)$ is a monotonically increasing function for $x \in [1, 2n-1]$, then

$$\begin{aligned}
W_f(G) &= \frac{1}{2}\sum_{i=1}^{2n}\sum_{j=1}^{2n} f(d_G(v_i, v_j)) \\
&\geq \frac{1}{2}\sum_{i=1}^{2n}[f(1)d_i + f(3)(n-d_i) + f(2)(n-1)] \\
&= \frac{1}{2}\sum_{i=1}^{2n}[f(3)n + f(2)(n-1) + (f(1) - f(3))d_i] \\
&= \frac{1}{2}f(3)n \cdot 2n + \frac{1}{2}f(2)(n-1)\cdot 2n - \frac{1}{2}[f(3) - f(1)]\sum_{i=1}^{2n} d_i \\
&= f(3)n^2 + f(2)n(n-1) - \frac{1}{2}[f(3) - f(1)](\sum_{i=1}^{k} d_i + \sum_{i=k+1}^{n} d_i + \sum_{i=n+1}^{2n} d_i) \\
&\geq f(3)n^2 + f(2)n(n-1) - \frac{1}{2}[f(3) - f(1)][k\cdot k + (n-k)\cdot(n-k) + n\cdot n] \\
&= f(3)n^2 + f(2)n(n-1) - [f(3) - f(1)][n^2 - 2n + 4 - (k-2)(n-k-2)] \\
&= [f(1) + f(2)]n^2 - [2f(1) + f(2) - 2f(3)]n - 4[f(3) - f(1)] + [f(3) - f(1)](k-2)(n-k-2).
\end{aligned}$$

Similarly, if $f(x)$ is a monotonically decreasing function for $x \in [1, 2n-1]$, then

$$W_f(G) \leq [f(1) + f(2)]n^2 - [2f(1) + f(2) - 2f(3)]n + 4[f(1) - f(3)] - [f(1) - f(3)](k-2)(n-k-2).$$

If $f(x)$ is a monotonically increasing function on $[1, 2n-1]$, by the condition of Theorem 2.2, we have $(k-2)(n-k-2) \leq 0$. Note that $k \geq 2$, $n-k \geq d_n \geq 2$, so $(k-2)(n-k-2) \geq 0$. Hence $(k-2)(n-k-2) = 0$. Then we have $W_f(G) = [f(1) + f(2)]n^2 - [2f(1) + f(2) - 2f(3)]n - 4[f(3) - f(1)]$. So all the inequalities in the above arguments should be equalities. Thus we have (a) the diameter of G is no more than three; (b) $d_1 = \cdots = d_k = k$, $d_{k+1} = \cdots = d_n = n-k$, $d_{n+1} = \cdots = d_{2n} = n$; and (c) $k = 2$ or $k = n-2$.

If $k = 2$, then $d_1 = d_2 = 2$, $d_3 = \cdots = d_n = n-2$ and $d_{n+1} = \cdots = d_{2n} = n$. This implies $G = K_{n,n-2} + 4e$. If $k = n-2$, then $2 \leq n-2 \leq \frac{n}{2}$ and hence $n = 4$. Then G is a bipartite graph with 12 edges and degree sequence $(2, 2, 2, 2, 4, 4, 4, 4)$. Thus $G = K_{4,2} + 4e$.

If $f(x)$ is a monotonically decreasing function on $[1, 2n-1]$, we can prove the result by the similar method. □

Remark 2.3. *If $f(x)$ is a monotonically increasing function, then the upper bound in Theorem 2.2 is an improvement of Theorem 14 in [14] since*

$$[f(1) + f(2)]n^2 - [2f(1) + f(2) - 2f(3)]n - 4[f(3) - f(1)] \geq$$

$$[f(1) + f(2)]n^2 - [f(1) + f(2) - f(3)]n + [f(1) - f(3)],$$

for $n \geq 3$. If $f(x)$ is a monotonically decreasing function, we can prove the lower bound in Theorem 2.2 is an improvement of Theorem 14 in [14] by the similar method.

Note that some previous works (see Theorem 4.7 in [20] and Theorem 4.7 in [19]) are two direct corollaries of Theorem 2.2 when $f(x) = x, \frac{1}{x}$, respectively. Moreover, Let $f(x) = \frac{x^2+x}{2}, x^\lambda$ in Theorem 2.2. We can get the following sufficient conditions in terms of the hyper-Wiener index, modified Wiener index, respectively, for a bipartite graph to be Hamiltonian.

Corollary 2.4. *Let $G := G[X, Y]$ be a bipartite graph with minimum degree $\delta \geq 2$, where $|X| = |Y| = n \geq 3$. If its hyper-Wiener index*

$$WW(G) \leq 4n^2 + 7n - 20,$$

then G is Hamiltonian unless $G = K_{n,n-2} + 4e$.

Corollary 2.5. *Let $G := G[X, Y]$ be a bipartite graph with minimum degree $\delta \geq 2$, where $|X| = |Y| = n \geq 3$. If its modified Wiener index*

$$W_\lambda(G) \leq (2^\lambda + 1)n^2 + 2(3^\lambda - 2^{\lambda-1} - 1)n - 4(3^\lambda - 1),$$

for $\lambda > 0$, or

$$W_\lambda(G) \geq (2^\lambda + 1)n^2 + 2(3^\lambda - 2^{\lambda-1} - 1)n + 4(1 - 3^\lambda),$$

for $\lambda < 0$, then G is Hamiltonian unless $G = K_{n,n-2} + 4e$.

In order to give sufficient conditions for a general connected graph to be Hamilton-connected or traceable from every vertex in terms of the Wiener-type index, we firstly give two sufficient conditions in terms of degree sequence for a graph to be Hamilton-connected or traceable from every vertex.

Lemma 2.6. *([1]) Let G be a graph of order $n \geq 3$ with degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \leq d_2 \leq \cdots \leq d_n$. If there is no integer $2 \leq k \leq \frac{n}{2}$ such that $d_{k-1} \leq k$ and $d_{n-k} \leq n - k$, then G is Hamilton-connected.*

Lemma 2.7. *([2]) Let G be a graph. Then G is traceable from every vertex if and only if $G \vee K_1$ is Hamilton-connected.*

Theorem 2.8. *Let G be a graph of order $n \geq 2$ with degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \leq d_2 \leq \cdots \leq d_n$. If there is no integer $2 \leq k \leq \frac{n+1}{2}$ such that $d_{k-1} \leq k - 1$ and $d_{n-k+1} \leq n - k$, then G is traceable from every vertex.*

Proof. Indeed, given any graph G we can construct a graph G^* by adding a new vertex u and new edges joining u to all the vertices of G. By Lemma 2.7, G^* is Hamilton-connected if and only if G is traceable from every vertex. Moreover, if the degree sequence of G satisfies the condition of Theorem 2.8, then the degree sequence of G^* satisfies the condition of Lemma 2.6. □

Theorem 2.9. *Let G be a connected simple graph of order $n \geq 12$ with minimum degree $\delta \geq 3$. If*

$$W_f(G) \leq \frac{f(1)}{2}n^2 + [2f(2) - \frac{5}{2}f(1)]n - 9[f(2) - f(1)],$$

for a monotonically increasing function $f(x)$ on $x \in [1, n-1]$, or

$$W_f(G) \geq \frac{f(1)}{2}n^2 + [2f(2) - \frac{5}{2}f(1)]n + 9[f(1) - f(2)],$$

for a monotonically decreasing function $f(x)$ on $x \in [1, n-1]$, then G is Hamilton-connected unless $G \in \{K_3 \vee (K_{n-5} + 2K_1), K_6 \vee 6K_1\}$.

Proof. Assume that G is not Hamilton-connected and has the degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \le d_2 \le \cdots \le d_n$ and $n \ge 12$. By Lemma 2.6, there is an integer k such that $3 \le \delta \le d_{k-1} \le k \le \frac{n}{2}$ and $d_{n-k} \le n-k$. Note that G is connected. If $f(x)$ is a monotonically increasing function for $x \in [1, n-1]$, then

$$\begin{aligned}
W_f(G) &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(d_G(v_i, v_j))\\
&\ge \frac{1}{2}\sum_{i=1}^{n}[f(1)d_i + f(2)(n-1-d_i)]\\
&= \frac{1}{2}\sum_{i=1}^{n}[(n-1)f(2) - (f(2)-f(1))d_i]\\
&= \frac{1}{2}n(n-1)f(2) - \frac{f(2)-f(1)}{2}\sum_{i=1}^{n} d_i\\
&= \frac{1}{2}n(n-1)f(2) - \frac{f(2)-f(1)}{2}(\sum_{i=1}^{k-1} d_i + \sum_{i=k}^{n-k} d_i + \sum_{i=n-k+1}^{n} d_i)\\
&\ge \frac{1}{2}n(n-1)f(2) - \frac{f(2)-f(1)}{2}[(k-1)k + (n-2k+1)(n-k) + k(n-1)]\\
&= \frac{1}{2}n(n-1)f(2) - \frac{f(2)-f(1)}{2}[n^2 - 5n + 18 - (k-3)(2n-3k-6)]\\
&= \frac{f(1)}{2}n^2 + [2f(2) - \frac{5}{2}f(1)]n - 9[f(2)-f(1)] + \frac{f(2)-f(1)}{2}(k-3)(2n-3k-6).
\end{aligned}$$

Similarly, if $f(x)$ is a monotonically decreasing function for $x \in [1, n-1]$, then

$$W_f(G) \le \frac{f(1)}{2}n^2 + [2f(2) - \frac{5}{2}f(1)]n + 9[f(1)-f(2)] - \frac{f(1)-f(2)}{2}(k-3)(2n-3k-6).$$

If $f(x)$ is a monotonically increasing function on $[1, n-1]$, by the condition of Theorem 2.9, we have $(k-3)(2n-3k-6) \le 0$. Then we discuss the following two cases.

Case 1. Assume that $(k-3)(2n-3k-6) = 0$. In this case, we get $W_f(G) = \frac{f(1)}{2}n^2 + [2f(2) - \frac{5}{2}f(1)]n - 9[f(2)-f(1)]$. So all the inequalities in the above arguments should be equalities. Thus, we have (a) the diameter of G is no more than two; (b) $d_1 = \ldots = d_{k-1} = k$, $d_k = \ldots = d_{n-k} = n-k$ and $d_{n-k+1} = \ldots = d_n = n-1$; and (c) $k = 3$ or $2n = 3k+6$.

If $k = 3$, then $d_1 = d_2 = 3$, $d_3 = \cdots = d_{n-3} = n-3$, $d_{n-2} = d_{n-1} = d_n = n-1$. It implies that $G = K_3 \vee (K_{n-5} + 2K_1)$. If $2n - 3k - 6 = 0$, since $n \ge 12$ and $k \le \frac{n}{2}$, then $n = 12, k = 6$. The corresponding permissible graphic sequence is $(6, 6, 6, 6, 6, 6, 11, 11, 11, 11, 11, 11)$, which implies $G = K_6 \vee 6K_1$.

Case 2. We assume $k \ge 4$ and $2n - 3k - 6 < 0$. In this case, we have $n \ge 2k \ge 8$. By the condition of Theorem 2.9, $n \ge 12$, then $2n - 3k - 6 \ge 2n - \frac{3}{2}n - 6 \ge 0$, a contradiction.

If $f(x)$ is a monotonically decreasing function on $[1, n-1]$, we can prove the result by the similar method. □

Note that some previous works (see Theorem 2.5 and Theorem 2.7 in [12]) are two direct corollaries of Theorem 2.9 when $f(x) = x, \frac{1}{x}$, respectively. Moreover, we also have the following two corollaries.

Corollary 2.10. *Let G be a connected simple graph of order $n \ge 12$ with minimum degree $\delta \ge 3$. If its hyper-Wiener index*

$$WW(G) \le \frac{1}{2}n^2 + \frac{7}{2}n - 18,$$

then G is Hamilton-connected unless $G \in \{K_3 \vee (K_{n-5} + 2K_1), K_6 \vee 6K_1\}$.

Corollary 2.11. *Let G be a connected simple graph of order $n \geq 12$ with minimum degree $\delta \geq 3$. If its modified Wiener index*

$$W_\lambda(G) \leq \frac{1}{2}n^2 + (2^{\lambda+1} - \frac{5}{2})n - 9(2^\lambda - 1),$$

for $\lambda > 0$, or

$$W_\lambda(G) \geq \frac{1}{2}n^2 + (2^{\lambda+1} - \frac{5}{2})n + 9(1 - 2^\lambda),$$

for $\lambda < 0$, then G is Hamilton-connected unless $G \in \{K_3 \vee (K_{n-5} + 2K_1), K_6 \vee 6K_1\}$.

Theorem 2.12. *Let G be a connected simple graph of order $n \geq 11$ with minimum degree $\delta \geq 2$. If*

$$W_f(G) \leq \frac{f(1)}{2}n^2 + [2f(2) - \frac{5}{2}f(1)]n - 7[f(2) - f(1)],$$

for a monotonically increasing function $f(x)$ on $x \in [1, n-1]$, or

$$W_f(G) \geq \frac{f(1)}{2}n^2 + [2f(2) - \frac{5}{2}f(1)]n + 7[f(1) - f(2)],$$

for a monotonically decreasing function $f(x)$ on $x \in [1, n-1]$, then G is traceable from every vertex unless $G \in \{K_2 \vee (K_{n-4} + 2K_1), K_5 \vee 6K_1\}$.

Proof. Suppose that G is not traceable from every vertex and has the degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \leq d_2 \leq \cdots \leq d_n$ and $n \geq 11$. By Theorem 2.8, there is an integer k such that $3 \leq d_{k-1} + 1 \leq k \leq \frac{n+1}{2}$ and $d_{n-k+1} \leq n - k$. Note that G is connected. If $f(x)$ is a monotonically increasing function for $x \in [1, n-1]$, as the proof of Theorem 2.9, then

$$\begin{aligned}
W_f(G) &\geq \frac{1}{2}n(n-1)f(2) - \frac{f(2) - f(1)}{2}\sum_{i=1}^{n} d_i \\
&= \frac{1}{2}n(n-1)f(2) - \frac{f(2) - f(1)}{2}(\sum_{i=1}^{k-1} d_i + \sum_{i=k}^{n-k+1} d_i + \sum_{i=n-k+2}^{n} d_i) \\
&\geq \frac{1}{2}n(n-1)f(2) - \frac{f(2) - f(1)}{2}[(k-1)(k-1) + (n-2k+2)(n-k) \\
&\quad + (k-1)(n-1)] \\
&= \frac{1}{2}n(n-1)f(2) - \frac{f(2) - f(1)}{2}[n^2 - 5n + 14 - (k-3)(2n-3k-4)] \\
&= \frac{f(1)}{2}n^2 + [2f(2) - \frac{5}{2}f(1)]n - 7[f(2) - f(1)] + \frac{f(2) - f(1)}{2}(k-3)(2n-3k-4).
\end{aligned}$$

Similarly, if $f(x)$ is a monotonically decreasing function for $x \in [1, n-1]$, then

$$W_f(G) \leq \frac{f(1)}{2}n^2 + [2f(2) - \frac{5}{2}f(1)]n + 7[f(1) - f(2)] - \frac{f(1) - f(2)}{2}(k-3)(2n-3k-4).$$

Since $W_f(G) \leq \frac{f(1)}{2}n^2 + [2f(2) - \frac{5}{2}f(1)]n - 7[f(2) - f(1)]$ for a monotonically increasing function $f(x)$ on $[1, n-1]$, $(k-3)(2n-3k-4) \leq 0$. Then we discuss the following two cases.

Case 1. Assume that $(k-3)(2n-3k-4) = 0$. In this case, we get $W_f(G) = \frac{f(1)}{2}n^2 + [2f(2) - \frac{5}{2}f(1)]n - 7[f(2) - f(1)]$. So, all inequalities in the above arguments should be equalities. Thus, we have (a) the diameter of G is no more than two; (b) $d_1 = \ldots = d_{k-1} = k-1$, $d_k = \ldots = d_{n-k+1} = n-k$ and $d_{n-k+2} = \ldots = d_n = n-1$; and (c) $k = 3$ or $2n = 3k + 4$.

If $k = 3$, then G is a graph with $d_1 = d_2 = 2$, $d_3 = \cdots = d_{n-2} = n-3$, $d_{n-1} = d_n = n-1$. It implies that $G = K_2 \vee (K_{n-4} + 2K_1)$. If $2n-3k-4 = 0$, since $n \geq 11$ and $k \leq \frac{n+1}{2}$, we can get $n = 11, k = 6$. The corresponding permissible graphic sequence is $(5,5,5,5,5,5,10,10,10,10,10)$, which implies $G = K_5 \vee 6K_1$.

Case 2. We assume $k \geq 4$ and $2n-3k-4 < 0$. In this case, if $n \geq 11$, then $2n-3k-4 \geq 2n - \frac{3}{2}(n+1) - 4 \geq 0$, a contradiction.

If $f(x)$ is a monotonically decreasing function on $[1, n-1]$, we can prove the result by the similar method. □

Note that some previous works (see Theorem 3.2 and Theorem 3.4 in [12]) are two direct corollaries of Theorem 2.12 when $f(x) = x, \frac{1}{x}$, respectively. Moreover, we also have the following two corollaries.

Corollary 2.13. *Let G be a connected simple graph of order $n \geq 11$ with minimum degree $\delta \geq 2$. If its hyper-Wiener index*

$$WW(G) \leq \frac{1}{2}n^2 + \frac{7}{2}n - 14,$$

then G is traceable from every vertex unless $G \in \{K_2 \vee (K_{n-4} + 2K_1), K_5 \vee 6K_1\}$.

Corollary 2.14. *Let G be a connected simple graph of order $n \geq 11$ with minimum degree $\delta \geq 2$. If its modified Wiener index*

$$W_\lambda(G) \leq \frac{1}{2}n^2 + (2^{\lambda+1} - \frac{5}{2})n - 7(2^\lambda - 1),$$

for $\lambda > 0$, or

$$W_\lambda(G) \geq \frac{1}{2}n^2 + (2^{\lambda+1} - \frac{5}{2})n + 7(1 - 2^\lambda),$$

for $\lambda < 0$, then G is traceable from every vertex unless $G \in \{K_2 \vee (K_{n-4} + 2K_1), K_5 \vee 6K_1\}$.

3. Wiener-type index conditions on $\overline{G}$ or G^* for G to be Hamiltonian and Hamilton-connected

In this section, firstly, we give two sufficient conditions on Wiener-type index of $\overline{G}$ for a graph G to be Hamiltonian and traceable. Then we will give a sufficient condition for a bipartite graph G to be Hamiltonian in terms of the Wiener-type index of G^*. Furthermore, we give two sufficient conditions on Wiener-type index of $\overline{G}$ for a graph G to be Hamilton-connected and traceable from every vertex.

Firstly, we give two sufficient conditions for a connected graph to be Hamiltonian and traceable in terms of the edge number.

Lemma 3.1. *([22]) Let G be a connected graph on $n \geq 4$ vertices and m edges with minimum degree $\delta \geq 1$. If*

$$m \geq \binom{n-2}{2} + 2,$$

then G is traceable unless $G \in \mathbb{NP}_1 = \{K_1 \vee (K_{n-3} + 2K_1), K_1 \vee (K_{1,3} + K_1), K_{2,4}, K_2 \vee 4K_1, K_2 \vee (K_2 + 3K_1), K_1 \vee K_{2,5}, K_3 \vee 5K_1, K_2 \vee (K_{1,4} + K_1), K_4 \vee 6K_1\}$.

Lemma 3.2. *([22]) Let G be a connected graph on $n \geq 5$ vertices and m edges with minimum degree $\delta \geq 2$. If*

$$m \geq \binom{n-2}{2} + 4,$$

then G is Hamiltonian unless $G \in \mathbb{NC}_1 = \{K_2 \vee (K_{n-4} + 2K_1), K_3 \vee 4K_1, K_2 \vee (K_{1,3} + K_1), K_1 \vee K_{2,4}, K_3 \vee (K_2 + 3K_1), K_4 \vee 5K_1, K_3 \vee (K_{1,4} + K_1), K_2 \vee K_{2,5}, K_5 \vee 6K_1\}$.

Theorem 3.3. *Let G be a connected graph of order $n \geq 4$ with minimum degree $\delta \geq 1$ and edge number m, and $\overline{G}$ be its complement graph. If*

$$W_f(\overline{G}) \geq \frac{f(n-1)n^2 - [5f(n-1) - 4f(1)]n + 10[f(n-1) - f(1)]}{2},$$

for a monotonically increasing function $f(x)$ on $x \in [1, n-1]$, or

$$W_f(\overline{G}) \leq \frac{f(n-1)n^2 - [5f(n-1) - 4f(1)]n - 10[f(1) - f(n-1)]}{2},$$

for a monotonically decreasing function $f(x)$ on $x \in [1, n-1]$, then G is traceable unless $G \in \mathbb{NP}_1$.

Proof. If $f(x)$ is a monotonically increasing function for $x \in [1, n-1]$, then

$$\begin{aligned}
W_f(\overline{G}) &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(d_{\overline{G}}(v_i, v_j)) \\
&\leq \frac{1}{2}\sum_{i=1}^{n}[f(1)d_{\overline{G}}(v_i) + f(n-1)(n-1-d_{\overline{G}}(v_i))] \\
&= \frac{1}{2}\sum_{i=1}^{n}[(n-1)f(n-1) + (f(1) - f(n-1))d_{\overline{G}}(v_i)] \\
&= \frac{f(n-1)}{2}n(n-1) + \frac{f(1) - f(n-1)}{2}\sum_{i=1}^{n} d_{\overline{G}}(v_i) \\
&= \frac{f(n-1)}{2}n(n-1) - \frac{f(n-1) - f(1)}{2}\sum_{i=1}^{n}(n-1-d_G(v_i)) \\
&= \frac{f(1)}{2}n(n-1) + \frac{f(n-1) - f(1)}{2}\sum_{i=1}^{n} d_i \\
&= \frac{f(1)}{2}n(n-1) + [f(n-1) - f(1)]m.
\end{aligned}$$

Since $W_f(\overline{G}) \geq \dfrac{f(n-1)n^2 - [5f(n-1) - 4f(1)]n + 10[f(n-1) - f(1)]}{2}$, we have $m \geq \binom{n-2}{2} + 2$. By Lemma 3.1, we get G is traceable unless $G \in \mathbb{NP}_1$.

If $f(x)$ is a monotonically decreasing function on $[1, n-1]$, we can prove the result by the similar method. □

Note that the Wiener index, hyper-Wiener index and modified Wiener index of an unconnected graph is meaningless, and the Harary index of an unconnected graph is the sum of the Harary index of all components [17]. Under the condition that G and $\overline{G}$ are both connected, the previous work (see Theorem 3.2 in [20]) is the direct corollary of Theorem 3.3 when $f(x) = x$. So there seems to have some flaws in their original theorem. Also, Theorem 3.2 in [19] is the direct corollary of Theorem 3.3 when $f(x) = \dfrac{1}{x}$. Moreover, we also have the following two corollaries.

Corollary 3.4. *Let G be a connected simple graph of order $n \geq 4$ with minimum degree $\delta \geq 1$, $\overline{G}$ be its complement graph and $\overline{G}$ is connected. If its hyper-Wiener index*

$$WW(\overline{G}) \geq \frac{n^4 - 6n^3 + 15n^2 - 2n - 20}{4},$$

then G is traceable unless $G \in \mathbb{NP}_1$.

Corollary 3.5. *Let G be a connected simple graph of order $n \geq 4$ with minimum degree $\delta \geq 1$, $\overline{G}$ be its complement graph and $\overline{G}$ is connected. If its modified Wiener index*

$$W_\lambda(\overline{G}) \geq \frac{(n^2-5n+10)(n-1)^\lambda + 4n - 10}{2},$$

for $\lambda > 0$, or

$$W_\lambda(\overline{G}) \leq \frac{(n^2-5n+10)(n-1)^\lambda + 4n - 10}{2},$$

for $\lambda < 0$, then G is traceable unless $G \in \mathbb{NP}_1$.

Theorem 3.6. *Let G be a connected graph of order $n \geq 4$ with minimum degree $\delta \geq 2$ and edge number m, and $\overline{G}$ be its complement graph. If*

$$W_f(\overline{G}) \geq \frac{f(n-1)n^2 - [5f(n-1) - 4f(1)]n + 14[f(n-1) - f(1)]}{2},$$

for a monotonically increasing function $f(x)$ on $x \in [1, n-1]$, or

$$W_f(\overline{G}) \leq \frac{f(n-1)n^2 - [5f(n-1) - 4f(1)]n - 14[f(1) - f(n-1)]}{2},$$

for a monotonically decreasing function $f(x)$ on $x \in [1, n-1]$, then G is Hamiltonian unless $G \in \mathbb{NC}_1$.

Proof. If $f(x)$ is a monotonically increasing function for $x \in [1, n-1]$. From the proof of Theorem 3.3, we have

$$W_f(\overline{G}) \leq \frac{f(1)}{2}n(n-1) + [f(n-1) - f(1)]m.$$

Since $W_f(\overline{G}) \geq \dfrac{f(n-1)n^2 - [5f(n-1) - 4f(1)]n + 14[f(n-1) - f(1)]}{2}$, we have $m \geq \binom{n-2}{2} + 4$. By Lemma 3.2, we can get G is Hamiltonian unless $G \in \mathbb{NC}_1$.

If $f(x)$ is a monotonically decreasing function on $[1, n-1]$, we can prove the result by the similar method. □

Under the condition that a graph G and its complement $\overline{G}$ are both connected, the previous work (see Theorem 3.5 in [20]) is the direct corollary of Theorem 3.6 when $f(x) = x$. So there seems to have some flaws in their original theorem. Also, Theorem 3.5 in [19] is the direct corollary of Theorem 3.6 when $f(x) = \frac{1}{x}$. Moreover, we also have the following two corollaries.

Corollary 3.7. *Let G be a connected simple graph of order $n \geq 5$ with minimum degree $\delta \geq 2$, $\overline{G}$ be its complement graph and $\overline{G}$ is connected. If its hyper-Wiener index*

$$WW(\overline{G}) \geq \frac{n^4 - 6n^3 + 19n^2 - 6n - 28}{4},$$

then G is Hamiltonian unless $G \in \mathbb{NC}_1$.

Corollary 3.8. *Let G be a connected simple graph of order $n \geq 5$ with minimum degree $\delta \geq 2$, $\overline{G}$ be its complement graph and $\overline{G}$ is connected. If its modified Wiener index*

$$W_\lambda(\overline{G}) \geq \frac{(n^2-5n+14)(n-1)^\lambda + 4n - 14}{2},$$

for $\lambda > 0$, or

$$W_\lambda(\overline{G}) \leq \frac{(n^2-5n+14)(n-1)^\lambda + 4n - 14}{2},$$

for $\lambda < 0$, then G is Hamiltonian unless $G \in \mathbb{NC}_1$.

Next, we will give the sufficient condition for a bipartite graph to be Hamiltonian in terms of the Wiener-type index of its quasi-complement.

Lemma 3.9. *([18]) Let $G := G[X, Y]$ be a bipartite graph with minimum degree $\delta \geq 2$ and m edges, where $|X| = |Y| = n \geq 4$. If*

$$m \geq n^2 - 2n + 4,$$

then G is Hamiltonian unless $G = K_{n,n-2} + 4e$.

Theorem 3.10. *Let $G := G[X, Y]$ be a bipartite graph with minimum degree $\delta \geq 2$ and m edges, where $|X| = |Y| = n \geq 3$ and G^* be the quasi-complement of G. If*

$$W_f(G^*) \geq [f(2n-2) + f(2n-1)]n^2 + [2f(1) - f(2n-2) - 2f(2n-1)]n - 4[f(1) - f(2n-1)],$$

for a monotonically increasing function $f(x)$ on $x \in [1, 2n-1]$, or

$$W_f(G^*) \leq [f(2n-2) + f(2n-1)]n^2 + [2f(1) - f(2n-2) - 2f(2n-1)]n + 4[f(2n-1) - f(1)],$$

for a monotonically decreasing function $f(x)$ on $x \in [1, 2n-1]$, then G is Hamiltonian unless $G = K_{n,n-2} + 4e$.

Proof. If $f(x)$ is a monotonically increasing function for $x \in [1, 2n-1]$, then

$$\begin{aligned}
W_f(G^*) &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(d_{G^*}(v_i, v_j)) \\
&\leq \frac{1}{2}\sum_{i=1}^{2n}[f(1)d_{G^*}(v_i) + f(2n-1)(n - d_{G^*}(v_i)) + f(2n-2)(n-1)] \\
&= \frac{1}{2}\sum_{i=1}^{2n}[(f(1) - f(2n-1))d_{G^*}(v_i) + f(2n-1)n + f(2n-2)(n-1)] \\
&= \frac{1}{2}f(2n-1)n \cdot 2n + \frac{1}{2}f(2n-2)(n-1)2n + \frac{1}{2}[f(1) - f(2n-1)]\sum_{i=1}^{2n}[n - d_G(v_i)] \\
&= f(2n-1)n^2 + f(2n-2)(n-1)n + \frac{1}{2}[f(1) - f(2n-1)]n \cdot 2n - \frac{1}{2}[f(1) - f(2n-1)] \cdot 2m \\
&= [f(2n-2) + f(1)]n^2 - f(2n-2)n + [f(2n-1) - f(1)]m.
\end{aligned}$$

Since $W_f(G^*) \geq [f(2n-2) + f(2n-1)]n^2 + [2f(1) - f(2n-2) - 2f(2n-1)]n - 4[f(1) - f(2n-1)]$, we have $m \geq n^2 - 2n + 4$. By Lemma 3.9, we get G is Hamiltonian unless $G = K_{n,n-2} + 4e$.

If $f(x)$ is a monotonically decreasing function on $[1, 2n-1]$, we can prove the result by the similar method. □

Under the condition that a bipartite graph G and its quasi-complement G^* are both connected, the previous work (see Theorem 4.8 in [20]) is the direct corollary of Theorem 3.10 when $f(x) = x$. So there seems to have some flaws in their original theorem. Also, Theorem 4.8 in [19] is the direct corollary of Theorem 3.10 when $f(x) = \frac{1}{x}$. Moreover, we also have the following two corollaries.

Corollary 3.11. *Let $G := G[X, Y]$ be a bipartite graph with minimum degree $\delta \geq 2$ and m edges, where $|X| = |Y| = n \geq 3$, G^* be the quasi-complement of G and G^* is connected. If its hyper-Wiener index*

$$WW(G^*) \geq 4n^4 - 10n^3 + 14n^2 - 3n - 4,$$

then G is Hamiltonian unless $G = K_{n,n-2} + 4e$.

Corollary 3.12. *Let $G := G[X, Y]$ be a bipartite graph with minimum degree $\delta \geq 2$ and m edges, where $|X| = |Y| = n \geq 3$, G^* be the quasi-complement of G and G^* is connected. If its modified Wiener index*

$$W_\lambda(G^*) \geq (2n-2)^\lambda(n^2-n) + (2n-1)^\lambda(n^2-2n+4) + 2n-4,$$

for $\lambda > 0$, or

$$W_\lambda(G^*) \leq (2n-2)^\lambda(n^2-n) + (2n-1)^\lambda(n^2-2n+4) + 2n-4,$$

for $\lambda < 0$, then G is Hamiltonian unless $G = K_{n,n-2} + 4e$.

Finally, we will give some sufficient conditions for a connected graph to be Hamilton-connected or traceable from every vertex in term of the Wiener-type index of its complement. At first, we present two sufficient conditions in terms of the edge number for a connected graph to be Hamilton-connected or traceable from every vertex.

Lemma 3.13. *([32]) Let G be a connected graph on $n \geq 6$ vertices and m edges with minimum degree $\delta \geq 3$. If*

$$m \geq \binom{n-2}{2} + 6,$$

then G is Hamilton-connected unless $G \in \mathbb{NC}_2 = \{K_3 \vee (K_{n-5} + 2K_1), K_6 \vee 6K_1, K_4 \vee (K_2 + 3K_1), 5K_1 \vee K_5, K_4 \vee (K_{1,4} + K_1), K_4 \vee (K_{1,3} + K_2), K_3 \vee K_{2,5}, K_4 \vee 4K_1, K_3 \vee (K_1 + K_{1,3}), K_3 \vee (K_{1,2} + K_2), K_2 \vee K_{2,4}\}$.

Lemma 3.14. *([32]) Let G be a connected graph on $n \geq 5$ vertices and m edges with minimum degree $\delta \geq 2$. If*

$$m \geq \binom{n-2}{2} + 4,$$

then G is traceable from every vertex unless $G \in \mathbb{NP}_2 = \{K_2 \vee (K_{n-4} + 2K_1), K_5 \vee 6K_1, K_3 \vee (K_2 + 3K_1), 5K_1 \vee K_4, K_3 \vee (K_{1,4} + K_1), K_3 \vee (K_{1,3} + K_2), K_2 \vee K_{2,5}, K_3 \vee 4K_1, K_2 \vee (K_1 + K_{1,3}), K_2 \vee (K_{1,2} + K_2), K_1 \vee K_{2,4}\}$

Theorem 3.15. *Let G be a connected graph of order $n \geq 6$ with minimum degree $\delta \geq 3$ and edge number m, and $\overline{G}$ be its complement graph. If*

$$W_f(\overline{G}) \geq \frac{f(n-1)n^2 - [5f(n-1) - 4f(1)]n + 18[f(n-1) - f(1)]}{2},$$

for a monotonically increasing function $f(x)$ on $x \in [1, n-1]$, or

$$W_f(\overline{G}) \leq \frac{f(n-1)n^2 - [5f(n-1) - 4f(1)]n + 18[f(n-1) - f(1)]}{2},$$

for a monotonically decreasing function $f(x)$ on $x \in [1, n-1]$, then G is Hamilton-connected unless $G \in \mathbb{NC}_2$.

Proof. If $f(x)$ is a monotonically increasing function for $x \in [1, n-1]$. From the proof of Theorem 3.3, we have

$$W_f(\overline{G}) \leq \frac{f(1)}{2}n(n-1) + [f(n-1) - f(1)]m.$$

Since $W_f(\overline{G}) \geq \dfrac{f(n-1)n^2 - [5f(n-1) - 4f(1)]n + 18[f(n-1) - f(1)]}{2}$, we have $m \geq \binom{n-2}{2} + 6$. By Lemma 3.13, we get G is Hamilton-connected unless $G \in \mathbb{NC}_2$.

If $f(x)$ is a monotonically decreasing function on $[1, n-1]$, we can prove the result by the similar method. □

Under the condition that a graph G and its complement $\overline{G}$ are both connected, the previous work (see Theorem 2.6 in [12]) is the direct corollary of Theorem 3.15 when $f(x) = x$. So there seems to have some flaws in their original theorem. Also, Theorem 2.8 in [12] is the direct corollary of Theorem 3.15 when $f(x) = \dfrac{1}{x}$. Moreover, we also have the following two corollaries.

Corollary 3.16. *Let G be a connected simple graph of order $n \geq 6$ with minimum degree $\delta \geq 3$, $\overline{G}$ be its complement graph and $\overline{G}$ is connected. If its hyper-Wiener index*

$$WW(\overline{G}) \geq \frac{n^4 - 6n^3 + 23n^2 - 10n - 36}{4},$$

then G is Hamilton-connected unless $G \in \mathbb{NC}_2$.

Corollary 3.17. *Let G be a connected simple graph of order $n \geq 6$ with minimum degree $\delta \geq 3$, $\overline{G}$ be its complement graph and $\overline{G}$ is connected. If its modified Wiener index*

$$W_\lambda(\overline{G}) \geq \frac{(n^2 - 5n + 18)(n-1)^\lambda + 4n - 18}{2},$$

for $\lambda > 0$, or

$$W_\lambda(\overline{G}) \leq \frac{(n^2 - 5n + 18)(n-1)^\lambda + 4n - 18}{2},$$

for $\lambda < 0$, then G is Hamilton-connected unless $G \in \mathbb{NC}_2$.

Theorem 3.18. *Let G be a connected graph of order $n \geq 5$ with minimum degree $\delta \geq 2$ and edge number m, and $\overline{G}$ be its complement graph. If*

$$W_f(\overline{G}) \geq \frac{f(n-1)n^2 - [5f(n-1) - 4f(1)]n + 14[f(n-1) - f(1)]}{2},$$

for a monotonically increasing function $f(x)$ on $x \in [1, n-1]$, or

$$W_f(\overline{G}) \leq \frac{f(n-1)n^2 - [5f(n-1) - 4f(1)]n + 14[f(n-1) - f(1)]}{2},$$

for a monotonically decreasing function $f(x)$ on $x \in [1, n-1]$, then G is traceable from every vertex unless $G \in \mathbb{NP}_2$.

Proof. If $f(x)$ is a monotonically increasing function for $x \in [1, n-1]$. From the proof of Theorem 3.3, we have

$$W_f(\overline{G}) \leq \frac{f(1)}{2}n(n-1) + [f(n-1) - f(1)]m.$$

Since $W_f(\bar{G}) \geq \dfrac{f(n-1)n^2 - [5f(n-1) - 4f(1)]n + 14[f(n-1) - f(1)]}{2}$, we have $m \geq \binom{n-2}{2} + 4$. By Lemma 3.14, we can get G is traceable from every vertex unless $G \in \mathbb{NP}_2$.

If $f(x)$ is a monotonically decreasing function on $[1, n-1]$, we can prove the result by the similar method. □

Under the condition of a graph G and its complement $\overline{G}$ are both connected, the previous work (see Theorem 3.3 in [12]) is the direct corollary of Theorem 3.18 when $f(x) = x$. So there seems to have some flaws in their original theorem. Also, Theorem 3.5 in [12] is the direct corollary of Theorem 3.18 when $f(x) = \frac{1}{x}$. Moreover, we also have the following two corollaries.

Corollary 3.19. *Let G be a connected simple graph of order $n \geq 5$ with minimum degree $\delta \geq 2$, $\overline{G}$ be its complement graph and $\overline{G}$ is connected. If its hyper-Wiener index*

$$WW(\overline{G}) \geq \frac{n^4 - 6n^3 + 19n^2 - 6n - 28}{4},$$

then G is traceable from every vertex unless $G \in \mathbb{NP}_2$.

Corollary 3.20. *Let G be a connected simple graph of order $n \geq 5$ with minimum degree $\delta \geq 2$, $\overline{G}$ be its complement graph and $\overline{G}$ is connected. If its modified Wiener index*

$$W_\lambda(\overline{G}) \geq \frac{(n^2 - 5n + 14)(n-1)^\lambda + 4n - 14}{2},$$

for $\lambda > 0$, or

$$W_\lambda(\overline{G}) \leq \frac{(n^2 - 5n + 14)(n-1)^\lambda + 4n - 14}{2},$$

for $\lambda < 0$, then G is traceable from every vertex unless $G \in \mathbb{NP}_2$.

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