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Characterization Theorems on Warped Product Semi-Slant Submanifolds in Kenmotsu Manifolds

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Abstract. In this paper, we deal with the study of warped product semi-slant submanifolds isometrically immersed into a Kenmotsu manifold. We prove two characterization theorems for a warped product semi-slant submanifold in Kenmotsu manifolds in terms of the tensor fields.

1. Introduction and Motivation

Warped product semi-slant submanifolds are important classes to study in differential geometry. Every structure on a manifold may not be admited warped product semi-slant submanifolds (for instance, see [11, 17]). In 1969, R.L. Bishop and B. O'Neil [4] introduced the idea of warped product manifolds to study manifolds of negative curvature. These manifolds are the generalization of Riemannian product manifolds. Afterwards, these manifolds were studied by many mathematicians and geometers. In early 20th century B.-Y. Chen in [7] introduced the notion of warped product submanifolds. Motivated form his idea many researchers studied warped product submanifolds for different structures on manifolds. Recently, M. Atceken [1] proved the non-existence of warped product semi-slant submanifolds of Kenmotsu manifold such that the spherical submanifold is tangential to the characteristic vector field. While, S. Uddin in [18] showed that the warped product semi-slant submanifold of Kenmotsu manifold exists except in the case when the structure vector field ξ is tangent to the fiber of warped products. Further, he also obtained a characterization result in term of shape operator and establish an inequality for the second fundamental form in terms of warping functions (cf. [23]). On the other hand, V. A. Khan in [13, 14] obtained characterizations for contact CR-warped product submanifolds of Kenmotsu manifolds and warped product semi-slant submanifolds of nearly Kaehler manifolds in terms of canonical tensor fields T and F. In this paper, we extend this study to the warped product semi-slant submanifolds of Kenmotsu manifolds. We obtain two main theorems for the characterization of warped product semi-slant submanifolds of Kenmotsu manifolds in terms of the endomorphisms T and F and the projections P_1 and P_2. We find that the results obtained in [13] for contact CR-warped products in Kenmotsu manifolds are particular cases of our Theorems 4.6 and 4.9.

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2. Preliminaries

Let $\tilde{M}$ be a $(2m+1)$-dimensional almost contact manifold with almost contact structure (φ, ξ, η) where φ is a $(1,1)$ tensor field, ξ a structure vector field, η is a dual 1-form satisfying the following property:

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0. \tag{1}$$

On an almost contact manifold there exists a Riemannian metric g which is satisfying the following

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \quad \eta(U) = g(U, \xi), \tag{2}$$

for any U, V tangent to $\tilde{M}$. Then an almost contact manifold $\tilde{M}$ equipped with Riemannian metric g is called an almost contact metric manifold $(\tilde{M}, g)$. Furthermore, an almost contact metric manifold is known to be a *Kenmotsu manifold* [10] if

$$(\tilde{\nabla}_U \varphi)V = g(\varphi U, V)\xi - \eta(V)\varphi U \tag{3}$$

and

$$\tilde{\nabla}_U \xi = U - \eta(U)\xi, \tag{4}$$

for any vector fields U, V on $\tilde{M}$, where $\tilde{\nabla}$ denotes the Riemannian connection with respect to g and we shall use the symbol $\Gamma(T\tilde{M})$ to denote Lie algebra of vector fields on a manifold $\tilde{M}$.

Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ with induced metric g and if ∇ and $\nabla^\perp$ are induced connections on the tangent bundle TM and normal bundle $T^\perp M$ of M, respectively. Then the Gauss and Weingarten formulas are given by

$$(i)\ \tilde{\nabla}_U V = \nabla_U V + h(U, V), \quad (ii)\ \tilde{\nabla}_U N = -A_N U + \nabla^\perp_U N, \tag{5}$$

for each $U, V \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where h and A_N are the second fundamental form and shape operator (corresponding to the normal vector field N) respectively for the immersion of M into $\tilde{M}$. They are related as

$$g(h(U, V), N) = g(A_N U, V). \tag{6}$$

In this paper we assume that the structure vector field ξ is tangential to the submanifold M. For any $U \in \Gamma(TM)$, we have

$$\nabla_U \xi = U - \eta(X)\xi, \quad h(U, \xi) = 0. \tag{7}$$

For any $U \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, we may write

$$\varphi U = TU + FU, \quad \varphi N = tN + fN, \tag{8}$$

where $TX(tN)$ and $FX(fN)$ are the tangential and normal components of φX (φN), respectively. From (1) and (8), it is easy to observe that

$$g(TU, V) = -g(U, TV) \tag{9}$$

for each $U, V \in \Gamma(TM)$. The covariant derivatives of endomorphisms φ, T and F are respectively defined as

$$(\tilde{\nabla}_U \varphi)V = \tilde{\nabla}_U \varphi V - \varphi \bar{\nabla}_U V, \quad \forall\ U, V \in \Gamma(T\bar{M}), \tag{10}$$

$$(\tilde{\nabla}_U T)V = \nabla_U TV - T\nabla_U V, \quad \forall\ U, V \in \Gamma(TM), \tag{11}$$

$$(\tilde{\nabla}_U F)V = \nabla^\perp_U FV - F\nabla_U V, \quad \forall\ U, V \in \Gamma(TM). \tag{12}$$

On using (3), (5), (8) and (10)-(12), we obtain

$$(\tilde{\nabla}_U T)V = g(TU, V)\xi - \eta(V)TU + A_{FV}U + th(U, V), \tag{13}$$

$$(\tilde{\nabla}_U F)U = fh(U, V) - h(U, TV) - \eta(V)FU. \tag{14}$$

A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be *totally umbilical* and *totally geodesic* if $h(U, V) = g(U, V)H$ and $h(U, V) = 0$, for any $U, V \in \Gamma(TM)$, respectively, where H is the mean curvature vector of M. Furthermore, if $H = 0$, then M is *minimal* in $\tilde{M}$.

Let M be a submanifold tangent to the structure vector field ξ of an almost contact metric manifold $\tilde{M}$. Then the angle $\theta(U)$ between φU and T_pM for each non zero vector U tangent to M at a point p, is called Wirtinger angle of U. Thus M is said to be a *slant submanifold* [5], if the angle $\theta(U)$ is constant which is independent the choice of $U \in (T_pM) - < \xi >$ and $p \in M$. Invariant and anti-invariant submanifolds are *slant submanifolds* with the slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A *slant submanifold* is *proper slant* if it is neither invariant nor anti-invariant. More generally, a distribution $\mathfrak{D}$ on M is called *slant distribution* if the angle $\theta(X)$ between φX and $\mathfrak{D}_x$ has the same value θ for each $x \in M$ and a non zero vector $X \in \mathfrak{D}_x$. Thus, for slant submanifold M, tangent bundle TM as well as normal bundle $T^{\perp}M$ are respectively decomposed by

$$(i)\ TM = \mathfrak{D} \oplus < \xi >, \qquad (ii)\ T^{\perp}M = F(TM) \oplus \nu, \tag{15}$$

where ν is an invariant normal bundle with respect to φ orthogonal to $F(TM)$.

Recently, Cabrerizo et.al [5] gave the following characterization result for a slant submanifold in a contact metric manifold.

Theorem 2.1. *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = \delta(-I + \eta \otimes \xi). \tag{16}$$

Furthermore, in such a case, if θ is slant angle, then it satisfies that $\delta = \cos^2\theta$.

Hence, for a slant submanifold M of an almost contact metric manifold $\tilde{M}$, the following relations are consequences of Theorem 2.1, we have

$$g(TU, TV) = \cos^2\theta\Big(g(U, V) - \eta(U)\eta(V)\Big), \tag{17}$$

$$g(FU, FV) = \sin^2\theta\Big(g(U, V) - \eta(U)\eta(V)\Big), \tag{18}$$

for any $U, V \in \Gamma(TM)$.

3. Semi-slant submanifolds of a Kenmotsu manifold

Semi-slant submanifolds were defined and studied by N. Papaghiuc [16] as a natural generalization of CR-submanifolds of almost Hermitian manifolds in terms of slant distribution and were later extended to the setting of contact manifolds by Cabrerizo et al. [6]. They defined these submanifolds as follows:

Definition 3.1. *A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be a semi-slant submanifold if there exists a pair of orthogonal distributions $\mathfrak{D}$ and $\mathfrak{D}^{\theta}$ such that*

(i) $TM = \mathfrak{D} \oplus \mathfrak{D}^{\theta} \oplus < \xi >$ *where* $< \xi >$ *is a 1-dimensional distribution spanned by* ξ,

(ii) $\mathfrak{D}$ *is invariant, i.e.,* $\varphi(\mathfrak{D}) \subseteq \mathfrak{D}$,

(iii) $\mathfrak{D}^{\theta}$ *is a proper slant distribution with slant angle* $\theta \neq 0, \pi/2$.

Let d_1 and d_2 are the dimensions of *invariant* distribution $\mathfrak{D}$ and *slant* distribution $\mathfrak{D}^\theta$ of semi-slant submanifold of an almost contact metric manifold $\tilde{M}$. Then M is *invariant* if $d_2 = 0$ and it is *slant* if $d_1 = 0$. It is proper semi-slant if $d_1 \neq 0$ and the slant angle is different from 0 and $\pi/2$.

Moreover, if ν is an *invariant* normal subbundle under φ in the normal bundle $T^\perp M$, then in case of semi-slant submanifold, the normal bundle $T^\perp M$ can be decomposed as $T^\perp M = F\mathfrak{D}^\theta \oplus \nu$. Furthermore, let us denotes the orthogonal projections on $\mathfrak{D}$ and $\mathfrak{D}^\theta$ by P_1 and P_2 respectively. then we can write

$$U = P_1U + P_2U + \eta(U)\xi, \tag{19}$$

for any $U \in \Gamma(TM)$, where $P_1U \in \Gamma(\mathfrak{D})$ and $P_2U \in \Gamma(\mathfrak{D}^\theta)$. From (8) and (19), we obtain

$$\varphi P_1U \in \Gamma(\mathfrak{D}), \quad FP_1U = 0, \tag{20}$$

and

$$TP_2U \in \Gamma(\mathfrak{D}^\theta), \quad FP_2U \in \Gamma(T^\perp M). \tag{21}$$

Then (20) and (21) imply that

$$TU = \varphi P_1U + TP_2U,$$

for all $U \in \Gamma(TM)$. On a semi-slant submanifold M of an almost contact metric manifold $\tilde{M}$, the following are straightforward observations

$$\left.\begin{array}{ll} (i)\ F\mathfrak{D} = 0, & (ii)\ T\mathfrak{D} = \mathfrak{D}, \\ (iii)\ t(T^\perp M) = \mathfrak{D}^\theta, & (iv)\ T\mathfrak{D}^\theta \subset \mathfrak{D}^\theta. \end{array}\right\} \tag{22}$$

It is easy to deduce from the (13) and (14) of semi-slant submanifold M of a Kenmotsu manifold $\tilde{M}$,

$$\left.\begin{array}{ll} (i)\ (\tilde{\nabla}_\xi T)U = 0, & (ii)\ (\tilde{\nabla}_U T)\xi = -TU, \\ (iii)\ (\tilde{\nabla}_\xi F)U = 0, & (iv)\ (\tilde{\nabla}_U F)\xi = -FU. \end{array}\right\} \tag{23}$$

We refer [23] for the integrability conditions of distributions involved in the definition semi-slant submanifold and the examples of semi-slant submanufold in a Kenmotsu manifold. Now, we give some useful results for later use.

Theorem 3.2. *Let M be a semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$. Then the invariant distribution $\mathfrak{D} \oplus < \xi >$ defines a totally geodesic foliation if and only if*

$$h(X, \varphi Y) \in \Gamma(\nu)$$

for all $X, Y \in \Gamma(\mathfrak{D} \oplus < \xi >)$.

Proof. Let for any $X, Y \in \Gamma(\mathfrak{D} \oplus < \xi >)$ and $Z \in \Gamma(\mathfrak{D}^\theta)$, we have

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X Y, Z) = g(\varphi \tilde{\nabla}_X Y, \varphi Z).$$

Using (10) and (8), we obtain

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X \varphi Y, TZ) + g(\tilde{\nabla}_X \varphi Y, FZ) - g((\tilde{\nabla}_X \varphi)Y, \varphi Z)$$

From (5), (4) and the definition of totally geodesic foliation, we achieve the required result. □

Theorem 3.3. *On a semi-slant submanifold M of a Kenmotsu manifold $\tilde{M}$, the slant distribution $\mathfrak{D}^\theta$ defines a totally geodesic foliation if and only if*

$$g(h(X,Z),FTW) = g(h(Z,\varphi X),FW) + \eta(X)g(Z,W)$$

for any $X \in \Gamma(\mathfrak{D}^\oplus < \xi >)$ and $Z \in \Gamma(\mathfrak{D}^\theta)$.

Proof. For any $Z \in \Gamma(\mathfrak{D}^\theta)$ and $X \in \Gamma(\mathfrak{D}\oplus < \xi >)$, we have

$$g(\nabla_Z W, X) = g(\varphi\bar{\nabla}_Z W, \varphi X) - \eta(X)g(Z,W).$$

Thus, on using (10) and (8), we derive

$$g(\nabla_Z W, X) = g(\tilde{\nabla}_Z TW, \varphi X) + g(\tilde{\nabla}_Z FW, \varphi X) - g((\tilde{\nabla}_Z\varphi)W, \varphi X) - \eta(X)g(Z,W).$$

From (4), (8) and (5) (ii), we obtain

$$\sin^2\theta g(\nabla_Z W, X) = g(h(X,Z),FTW) - g(h(Z,\varphi X),FW) - \eta(X)g(Z,W).$$

Hence, by definition $\mathfrak{D}^\theta$ defines a totally geodesic foliation if and only if the given condition holds, which proves the lemma completely. □

4. Warped product semi-slant submanifolds

In 1969, R. L. Bishop and B. O'Neill [4] initiated the idea of warped product manifolds to construct examples of Riemannian manifolds with negative curvature. These manifolds are natural generalizations of Riemannian product manifolds. They defined these manifolds as follows: Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and $f : M_1 \to (0,\infty)$, a positive differentiable function on M_1. Consider the product manifold $M_1 \times M_2$ with it's canonical projections $\pi_1 : M_1 \times M_2 \to M_1$, $\pi_2 : M_1 \times M_2 \to M_2$ and the projection maps given by $\pi_1(p,q) = p.$ and $\pi_2(p,q) = q$ for every $t = (p,q) \in M_1 \times M_2$. The warped product $M = M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ equipped with the Riemannian structure such that

$$\|U\|^2 = \|\pi_{1*}(U)\|^2 + (f \circ \pi_1(p))^2\|\pi_{2*}(U)\|^2.$$

for any tangent vector $U \in \Gamma(T_tM)$, where $*$ is the symbol for the tangent maps and we have $g = g_1 + f^2g_2$. Thus the function f is called the warping function on M. We recall the following result of [4] for later use:

Lemma 4.1. *Let $M = M_1 \times_f M_2$ be a warped product manifold. Then for any $X, Y \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$, we have*

(i) $\nabla_X Y \in \Gamma(TM_1)$,
(ii) $\nabla_Z X = \nabla_X Z = (X \ln f)Z$,
(iii) $nor(\nabla_Z W) = -g(Z,W)\nabla \ln f$,

where ∇ denotes the Levi-Civita connection on M and $\nabla \ln f$ is the gradient of $\ln f$ which is defined as $g(\nabla \ln f, U) = U \ln f$.

A warped product manifold $M = M_1 \times_f M_2$ is said to be *trivial* if the warping function f is constant.

Moreover, if $M = M_1 \times_f M_2$ is a warped product manifold, then M_1 is totally geodesic and M_2 is totally umbilical in M, respectively [4, 7].

In this paper, we study warped product semi-slant submanifold of the form $M_T \times_f M_\theta$, where M_T and M_θ are $\varphi-$invariant and proper slant submanifold of a Kenmotsu manifold $\tilde{M}$, respectively.

First, we recall the following lemma of [18]] for late use.

Lemma 4.2. *Let $M = M_T \times_f M_\theta$ be a warped product semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$ such that ξ is tangent to M_T, where M_T and M_θ are invariant and proper slant submanifolds of $\tilde{M}$, respectively. Then, we have*

(i) $\xi \ln f = 1$,
(ii) $g(h(X,Z),FTZ) = g(h(X,TZ),FZ) = (X \ln f - \eta(X)) \cos^2 \theta \|Z\|^2$,
(iii) $g(h(X,Z),FZ) = -(\varphi X \ln f)\|Z\|^2$,

for any $X \in \Gamma(TM_T)$ *and* $Z \in \Gamma(TM_\theta)$.

Further, we derive the following results which are useful to our main theorems.

Lemma 4.3. *On a warped product semi-slant submanifold* $M = M_T \times_f M_\theta$ *of a Kenmotsu manifold* $\tilde{M}$*, we have*

(i) $(\tilde{\nabla}_X T)Z = 0$,
(ii) $(\tilde{\nabla}_Z T)X = (\varphi X \ln f)Z - (X \ln f)TZ$,
(iii) $(\tilde{\nabla}_{TZ} T)X = (\varphi X \ln f)TZ + \cos^2 \theta (X \ln f)Z$,

for all $X \in \Gamma(TM_T)$ *and* $Z \in \Gamma(TM_\theta)$.

Proof. For any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$, from (11) and Lemma 4.1 (ii), it is easy to see that

$$(\tilde{\nabla}_X T)Z = \nabla_X TZ - T\nabla_X Z = (X \ln f)TZ - (X \ln f)TZ = 0,$$

which is the first part of the lemma. Again, from (11) and Lemma 4.1 (ii), we obtain

$$(\tilde{\nabla}_Z T)X = \nabla_Z TX - T\nabla_Z X = (\varphi X \ln f)Z - (X \ln f)TZ,$$

which proves second part of the lemma. Interchanging Z by TZ in (ii) and using (16), we get (iii). Hence, the proof is complete. □

Lemma 4.4. *Let* $M = M_T \times_f M_\theta$ *be a warped product semi-slant submanifold of a Kenmotsu manifold* $\tilde{M}$*. Then*

(i) $(\tilde{\nabla}_U T)X = (\varphi X \ln f)P_2 U - (X \ln f)TP_2 U + g(TP_1 U, X)\xi - \eta(X)TU$,
(ii) $(\tilde{\nabla}_U T)Z = g(P_2 U, Z)\varphi \nabla \ln f - g(P_2 U, TZ)\nabla \ln f$,
(iii) $(\tilde{\nabla}_U T)TZ = g(P_2 U, TZ)\varphi \nabla \ln f + \cos^2 \theta g(P_2 U, Z)\nabla \ln f$,

for any $U \in \Gamma(TM)$, $X \in \Gamma(TM_T)$ *and* $Z, W \in \Gamma(TM_\theta)$.

Proof. For any $X, Y \in \Gamma(TM_T)$, using (13), it follows that

$$(\tilde{\nabla}_X T)Y = th(X,Y) + g(TX,Y)\xi - \eta(Y)TX. \tag{24}$$

Since for a warped product manifold $M = M_T \times_f M_\theta$, M_T is totally geodesic in M, using this fact and then equating the tangential components to M_T, we get $th(X,Y) = 0$, which implies that $h(X,Y) \in \Gamma(\nu)$. Thus the equation (24) can be modified as:

$$(\tilde{\nabla}_X T)Y = g(TX,Y)\xi - \eta(Y)TX. \tag{25}$$

Now, from the relation (19), for any $U \in \Gamma(TM)$, we derive

$$(\tilde{\nabla}_U T)X = (\tilde{\nabla}_{P_1 U} T)X + (\tilde{\nabla}_{P_2 U} T)X + \eta(U)(\tilde{\nabla}_\xi T)X.$$

The third term of right hand side in above equation is identically zero by using (23) (i). Thus, using (25) in first term and Lemma 4.3 (ii) in the second term, we get first part of the lemma. Again from using (19), we arrive at

$$(\tilde{\nabla}_U T)Z = (\tilde{\nabla}_{P_1 U} T)Z + (\tilde{\nabla}_{P_2 U} T)Z + \eta(U)(\tilde{\nabla}_\xi T)Z,$$

for any $Z \in \Gamma(TM_\theta)$ and $U \in \Gamma(TM)$. The first and last terms of above equation are identically zero by using (23)(i) and Lemma 4.3 (i). Taking the inner product with $X \in \Gamma(TM_T)$ and using (11), we obtain

$$g((\tilde{\nabla}_U T)Z, X) = g(\nabla_{P_2 U} TZ, X) - g(T\nabla_{P_2 U} Z, X).$$

As X and Z are orthogonal, then by the property of Riemannian connection, we obtain

$$g((\tilde{\nabla}_U T)Z, X) = g(\nabla_{P_2U}Z, \varphi X) - g(\nabla_{P_2U}X, TZ).$$

Again by the property of Riemannian connection, we derive

$$g((\tilde{\nabla}_U T)Z, X) = -g(\nabla_{P_2U}\varphi X, Z) - g(\nabla_{P_2U}X, TZ).$$

From Lemma 4.1 (ii), we get

$$g((\tilde{\nabla}_U T)Z, X) = -(\varphi X \ln f)g(P_2U, Z) - (X \ln f)g(P_2U, TZ) = g(P_2U, Z)g(\varphi\nabla \ln f, X) - g(P_2U, TZ)g(\nabla \ln f, X),$$

which implies that

$$(\tilde{\nabla}_U T)Z = g(P_2U, Z)\varphi\nabla \ln f - g(P_2U, TZ)\nabla \ln f.$$

This is the second result of lemma. Interchanging Z by TZ and using relation (16), thus above equation takes the form

$$(\tilde{\nabla}_U T)TZ = g(P_2U, TZ)\varphi\nabla \ln f + \cos^2\theta g(P_2U, Z)\nabla \ln f,$$

which is the last relation. Hence, the lemma is proved completely. □

Lemma 4.5. *Assume that M be a warped product semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$. Then*

(i) $(\tilde{\nabla}_U F)X = -(X \ln f)FP_2U$,
(ii) $(\tilde{\nabla}_U F)Z = fh(U, Z) - h(U, TZ)$,
(iii) $(\tilde{\nabla}_U F)TZ = fh(U, TZ) + \cos^2\theta h(U, Z)$,

for any $U \in \Gamma(TM)$, $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$.

Proof. By virtue of (19) with $(\tilde{\nabla}_U F)X$, we have

$$(\tilde{\nabla}_U F)X = (\tilde{\nabla}_{P_1U}F)X + (\tilde{\nabla}_{P_2U}F)X + \eta(U)(\tilde{\nabla}_\xi F)X,$$

for any $U \in \Gamma(TM)$ and $X \in \Gamma(TM_T)$. The first term of above equation is identically zero by using the fact that M_T is totally geodesic on M and last term is zero by using the relation (23) (iii). Then, from (12) and Lemma 4.1 (ii), we obtain

$$(\tilde{\nabla}_U F)X = -F\nabla_{P_2U}X = -(X \ln f)FP_2U,$$

which is first part of lemma. Since $\eta(Z) = 0$, for any $Z \in \Gamma(\mathfrak{D}^\theta)$, then by relation (14), we derive

$$(\tilde{\nabla}_U F)Z = fh(U, Z) - h(U, TZ).$$

This is the second part of lemma. Now replacing Z by TZ in above equation and using (16), we get the third relation, which proves the lemma completely. □

Now, we prove the main theorem as a characterization theorem in term of ∇T for warped product semi-slant submanifold.

Theorem 4.6. *Let M be a semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$ with slant distribution $\mathfrak{D}^\theta$ is integrable. Then M is locally a warped product submanifold if and only if*

$$\begin{aligned}(\tilde{\nabla}_U T)V =&(\varphi P_1V\lambda)P_2U - (P_1V\lambda)TP_2U + g(P_2U, P_2V)\varphi\nabla\lambda - g(P_2U, TP_2V)\nabla\lambda \\ &+ g(P_1V, TP_1U)\xi - \eta(V)TP_1U,\end{aligned} \tag{26}$$

for each $U, V \in \Gamma(TM)$ and a C^∞-function μ on M with $Z\lambda = 0$, for each $Z \in \Gamma(\mathfrak{D}^\theta)$, where P_1 and P_2 are orthogonal projections on $\mathfrak{D}$ and $\mathfrak{D}^\theta$, respectively.

Proof. Let M be a warped product semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$. Then for any $U, V \in \Gamma(TM)$ and using (19), we obtain

$$(\tilde{\nabla}_U T)V = (\tilde{\nabla}_U T)P_1 V + (\tilde{\nabla}_U T)P_2 V + \eta(V)(\tilde{\nabla}_U T)\xi.$$

Thus, by Lemma 4.4 (i)-(ii) and the relation (23) (ii), we get the desired result (26) with $\lambda = \ln f$.

Conversely, if M is a semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$ such that the given condition (26) holds, then we have

$$(\tilde{\nabla}_X T)Y = g(TX, Y)\xi - \eta(Y)TX,$$

for any $X, Y \in \Gamma(\mathfrak{D} \oplus < \xi >)$. Taking the inner product with $Z \in \Gamma(\mathfrak{D}^\theta)$ and the fact that ξ is tangent to $\mathfrak{D}$, we derive

$$g(\tilde{\nabla}_X \varphi Y, Z) = -g(\nabla_X Y, TZ).$$

From the covariant derivative property of φ, we find

$$g((\tilde{\nabla}_X \varphi)Y, Z) - g(\tilde{\nabla}_X Y, \varphi Z) = -g(\nabla_X Y, TZ).$$

Using the structure equation of a Kenmotsu manifold, we get

$$g(\tilde{\nabla}_X Y, \varphi Z) = g(\nabla_X Y, TZ).$$

Then, from (8) and (5) (i), we obtain $g(h(X, Y), FZ) = 0$, which implies that $h(X, Y) \in \Gamma(\nu)$ for all $X, Y \in \Gamma(\mathfrak{D} \oplus < \xi >)$. Then, by Theorem 3.2 it leads to that $\mathfrak{D} \oplus < \xi >$ defines a totally geodesic foliations and its leaves are totally geodesic in M. As we have considered that the slant distribution is integrable, then if M_θ be a leaf of $\mathfrak{D}^\theta$ in M and h^θ be the second fundamental form of M_θ in M, then (26) implies that

$$(\tilde{\nabla}_Z T)W = g(Z, W)\varphi\nabla\lambda + g(TZ, W)\nabla\lambda,$$

for any $Z, W \in \Gamma(\mathfrak{D}^\theta)$. Taking the inner product with X for any $X \in \Gamma(\mathfrak{D} \oplus < \xi >)$ and using (11), we arrive at

$$g(h^\theta(Z, TW), X) + g(h^\theta(Z, W), \varphi X) = -(\varphi X\lambda)g(Z, W) - (X\lambda)g(Z, TW). \tag{27}$$

Interchanging W by TW and X by φX, the above equation takes the form

$$\begin{aligned} -\cos^2\theta g(h^\theta(Z, W), \varphi X) - g(h^\theta(Z, TW), X) &= (X\lambda)g(Z, TW) + \cos^2\theta(\varphi X\lambda)g(Z, W) \\ &\quad - \eta(X)(\xi\lambda)g(Z, TW) + \eta(X)g(Z, TW). \end{aligned} \tag{28}$$

Then (27) and (28), we derive

$$\sin^2\theta g(h^\theta(Z, W), \varphi X) = \eta(X)g(Z, TW) - \sin^2\theta(\varphi X\lambda)g(Z, W) - \eta(X)(\xi\lambda)g(Z, TW). \tag{29}$$

By polarization identity, we find that

$$\sin^2\theta g(h^\theta(Z, W), \varphi X) = \eta(X)g(TZ, W) - \sin^2\theta(\varphi X\lambda)g(Z, W) - \eta(X)(\xi\lambda)g(TZ, W). \tag{30}$$

Since, $\xi \ln f = 1$, then again from relations (29) and (30), we obtain

$$g(h^\theta(Z, W), \varphi X) = -g(Z, W)g(\nabla\lambda, \varphi X),$$

or equivalently,

$$h^\theta(Z, W) = -g(Z, W)\nabla\lambda,$$

which means that M_θ is a totally umbilical submanifold of M with mean curvature vector field $H^\theta = -\nabla\lambda$. Now, we can easily show that the mean curvature vector H^θ is parallel corresponding to the normal connection ∇' of M_θ in M, which means that M_θ is an extrinsic sphere in M. Hence, a result of Hiepko (cf. [9]), M is locally a warped product submanifold. Hence, the theorem is proved completely. □

Remark 4.7. *As an immediate consequence of Theorem 4.6 is that if we consider the slant angle $\theta = \frac{\pi}{2}$, then $TP_2U = TP_2V = 0$, for anti-invariant submanifold $M_\perp$. Thus a warped product semi-slant submanifold $M = M_T \times_f M_\theta$ turns into contact CR-warped product submanifold $M = M_T \times_f M_\perp$ such that M_T and $M_\perp$ are invariant and anti-invariant submanifolds, respectively.*

In other words, Theorem 4.6 is generalizing the characterization theorem for contact CR-warped products as follows.

Theorem 4.8. *Let $\tilde{M}$ be a Kenmotsu manifold and a CR-submanifold M of $\tilde{M}$ is locally a contact CR-warped products if and only if*

$$(\tilde{\nabla}_U T)V = (\varphi P_1 V\lambda)P_2U + g(P_2U, P_2V)\varphi\nabla\lambda + g(P_1V, TP_1U)\xi - \eta(V)TP_1U, \tag{31}$$

for each $U, V \in \Gamma(TM)$ and a C^∞-function λ on M with $Z\lambda = 0$, for each $Z \in \Gamma(\mathfrak{D}^\perp)$. Furthermore, P_1 and P_2 are orthogonal projections on $\mathfrak{D}$ and $\mathfrak{D}^\perp$, respectively .

In this sense, Theorem 4.4 of [13] is a special case of Theorem 4.6.

Now, we have another characterization theorem in terms of ∇F.

Theorem 4.9. *Every proper semi-slant submanifold M of a Kenmotsu manifold $\tilde{M}$ with integrable slant distribution $\mathfrak{D}^\theta$ is locally a warped product submanifold if and only if*

$$(\tilde{\nabla}_U F)V = fh(U, P_2V) - h(U, TP_2V) - (P_1V\lambda)FP_2U - \eta(V)FU, \tag{32}$$

for every $U, V \in \Gamma(TM)$ and a C^∞-function λ on M with $Z\lambda = 0$, for each $Z \in \Gamma(\mathfrak{D}^\theta)$, where P_1 and P_2 are the orthogonal projections on $\mathfrak{D}$ and $\mathfrak{D}^\theta$, respectively.

Proof. If M is a warped product semi-slant submanifold in a Kenmotsu manifold $\tilde{M}$, then from (19), we obtain

$$(\tilde{\nabla}_U F)V = (\tilde{\nabla}_U F)P_1V + (\tilde{\nabla}_U F)P_2V + \eta(V)(\tilde{\nabla}_U F)\xi. \tag{33}$$

Hence, from Lemma 4.5 (i)-(ii) and (23) (iv), we get desired result (32).

Conversely, if M is a semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$ such that (32) holds, then for any $X, Y \in \Gamma(\mathfrak{D} \oplus < \xi >)$, it follows from (32) that $-F\nabla_X Y = 0$, which implies that $\nabla_X Y \in \Gamma(\mathfrak{D} \oplus < \xi >)$, which means that, the invariant distribution $(\mathfrak{D} \oplus < \xi >)$ is integrable and its leaves are totally geodesic in M. Furthermore, by the hypothesis of the theorem that $\mathfrak{D}^\theta$ assumed to be integrable, thus we can consider M_θ be a leaf of $\mathfrak{D}^\theta$ and h^θ be the second fundamental form of M_θ in M. Then from (32), we derive

$$(\tilde{\nabla}_Z F)X = -(X\lambda)FZ - \eta(X)FZ. \tag{34}$$

for any $Z \in \Gamma(\mathfrak{D}^\theta)$ and $X \in \Gamma(\mathfrak{D} \oplus < \xi >)$. Taking the inner product in (34) with FW for any $W \in \Gamma(\mathfrak{D}^\theta)$ and by virtue of (18), the above equation takes the form

$$g((\tilde{\nabla}_Z F)X, FW) = -\sin^2\theta\Big((X\lambda) + \eta(X)\Big)g(Z, W). \tag{35}$$

Using (12), we obtain

$$(F\nabla_Z X, FW) = -\sin^2\theta\Big((X\lambda) + \eta(X)\Big)g(Z, W).$$

Thus from (18), we arrive at

$$-\sin^2\theta g(\nabla_Z X, W) = -\sin^2\theta\Big(g(\nabla\lambda, X) + g(\xi, X)\Big)g(Z, W),$$

which implies that

$$g(h^\theta(Z,W),X) = -g(X,\nabla\lambda + \xi)g(Z,W). \tag{36}$$

Since, in a Kenmotsu manifold $\xi \ln f = 1$, which implies that $g(\nabla \ln f, \xi) = 1$, i.e., $\xi = \nabla \ln f$, using this relation in (36), we get

$$g(h^\theta(Z,W),X) = -2g(Z,W)g(\nabla\lambda, X),$$

equivalently, we find

$$h^\theta(Z,W) = -2g(Z,W)\nabla\lambda.$$

The above relation shows that M_θ is totally umbilical in M with mean curvature vector $H^\theta = -2\nabla\lambda$. Moreover, the condition $Z\lambda = 0$, for any $Z \in \Gamma(\mathfrak{D}^\theta)$ implies that the leaves of $\mathfrak{D}^\theta$ are extrinsic spheres in M. Hence , by a result of Hiepko (cf. [9]) $M = M_T \times_f M_\theta$ is locally a warped product submanifold, where M_T and M_θ are integral manifolds of $\mathfrak{D} \oplus < \xi >$ and $\mathfrak{D}^\theta$, respectively. Hence, the proof is complete. □

Remark 4.10. *In particular if M is a contact CR-submanifold, then $\mathfrak{D}^\theta$ turn into anti-invariant distribution, i.e, $\theta = \frac{\pi}{2}$, in this case, $TP_2V = 0$, thus characterization Theorem 4.9 is generalized the characterization theorem obtained for contact CR-warped product submanifolds in Kenmotsu manifold which was proved by V. A. Khan for contact CR-warped products in [13].*

Theorem 4.11. *Every proper CR-submanifold M of a Kenmotsu manifold $\tilde{M}$ is locally a warped product submanifold if and only if*

$$(\tilde{\nabla}_U F)V = fh(U,P_2V) - (P_1V\lambda)FP_2U - \eta(V)FU, \tag{37}$$

for every $U, V \in \Gamma(TM)$ and a C^∞-function λ on M with $Z\lambda = 0$, for each $Z \in \Gamma(\mathfrak{D}^\perp)$. Moreover, P_1 and P_2 are orthogonal projections on $\mathfrak{D}$ and $\mathfrak{D}^\perp$, respectively (see [13]).

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