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Generalized Para-Kähler Spaces in Eisenhart's Sense Admitting a Holomorphically Projective Mapping

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Abstract. We relax the conditions related to the almost product structure and in such a way introduce a wider class of generalized para-Kähler spaces. Some properties of the curvature tensors as well as those of the corresponding Ricci tensors of these spaces are pointed out. We consider holomorphically projective mappings between generalized para-Kähler spaces in Eisenhart's sense. Also, we examine some invariant geometric objects with respect to equitorsion holomorphically projective mappings. These geometric objects reduce to the para-holomorphic projective curvature tensor in case of holomorphically projective mappings between usual para-Kähler spaces.

1. Introduction

A *hyperbolic Kähler space* or a *para-Kähler space* is a differentiable manifold M endowed with a pseudo-Riemannian metric g and an almost product structure $F \neq I$ satisfying the conditions [30]

$$F^2 = I,$$
$$g(FX, FY) = -g(X, Y),$$
$$(\nabla_X F)X = 0,$$

where $X, Y \in T_p(M)$ and ∇ is the Levi-Civita connection of the metric g.

A holomorphically planar curve was first introduced in a usual Kähler space by T. Otsuki and Y. Tashiro, see [30]. This curve is defined in the same manner as in a para-Kähler space as follows. A curve $l : I \to M$ in a para-Kähler space (M, g, F) of real dimension $2m \geq 4$ satisfying the regularity condition $\lambda(t) = \frac{\mathrm{d}l(t)}{\mathrm{d}t} \neq 0$, $t \in I$, is said to be a *holomorphically planar curve* if for some functions ρ_1 and ρ_2 of a parameter t the following ordinary differential equation holds [27, 30]

$$\nabla_{\lambda(t)}\lambda(t) = \rho_1(t)\lambda(t) + \rho_2(t)F\lambda(t),$$

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where ∇ denotes the Levi-Civita connection corresponding to the symmetric part $\underline{g}$ of metric g.

A mapping $f : M \to \overline{M}$ is said to be *holomorphically projective* if each holomorphically planar curve of the para-Kähler space (M, g, F) is mapped onto a holomorphically planar curve of the para-Kähler space $(\overline{M}, \overline{g}, \overline{F})$. J. Mikeš [5, 7–11, 24] made some of significant contributions to study of holomorphically projective mappings between Kähler, para-Kähler and parabolic Kähler spaces. Invariant geometric objects with respect to equitorsion holomorphically projective mappings of generalized Kähler spaces were described in [28, 29, 31]. M. Prvanović [25] considered for the first time an analogue of holomorphically projective transformations in locally product spaces and described para-holomorphic projective curvature tensor in these spaces as well as in para-Kähler spaces, particularly. C.-L. Bejan [1, 2] classified almost para-Hermitian spaces and found some examples of spaces with hyperbolic structures. Recently, C.-L. Bejan and G. Nakova [3] studied almost para-Hermitian and almost paracontact metric structures induced by natural Riemann extensions. Some interesting results concerning para-Kähler-like statistical submersions were obtained by G. E. Vîlcu [4].

We should note that investigation of special Eisenhart's generalized Riemannian spaces and their diffeomorphisms is an active research topic [32–36]. A kind of generalized hyperbolic Kähler spaces and holomorphically projective mappings between these spaces were considered in [16]. On the other hand we gave a more general definition of generalized Kähler spaces in Eisenhart's sense [22]. In the same manner generalized m-parabolic Kähler spaces were defined in [17, 18]. A new type of generalized para-Kähler space is given in [23]. In the present paper we provide a more general definition of generalized para-Kähler spaces in Eisenhart's sense than the one given in [16]. These results as well as those concerning F-planar mappings given in [21] are included in the author's Ph.D. thesis [19].

A generalized pseudo-Riemannian space in L.P. Eisenhart's sense [6] is a differentiable manifold M endowed with a non-symmetric metric g. Therefore the metric g can be described as

$$g(X,Y) = \underline{g}(X,Y) + \underset{\vee}{g}(X,Y),$$

where $\underline{g}$ denotes the symmetric part of the metric g and $\underset{\vee}{g}$ denotes the skew-symmetric part of g, i.e.,

$$\underline{g}(X,Y) = \frac{1}{2}\Big(g(X,Y) + g(Y,X)\Big) \quad \text{and} \quad \underset{\vee}{g}(X,Y) = \frac{1}{2}\Big(g(X,Y) - g(Y,X)\Big).$$

A non-symmetric linear connection $\underset{1}{\nabla}$ of a generalized Riemannian manifold with a metric g is explicitly defined by

$$\underline{g}(\underset{1}{\nabla}_X Y, Z) = \frac{1}{2}\Big(Xg(Y,Z) + Yg(Z,X) - Zg(Y,X)\Big),$$

or in local coordinates by

$$\Gamma_{i.jk} = g_{\underline{ip}}\Gamma^p_{jk} = \frac{1}{2}\Big(g_{ji,k} - g_{jk,i} + g_{ik,j}\Big).$$

Here the functions $\Gamma_{i.jk}$ and Γ^i_{jk} are called generalized Christoffel symbols of the first kind and the second kind, respectively.

On a generalized Riemannian space (M, g) another non-symmetric linear connection $\underset{2}{\nabla}$ is defined by [26]

$$\underset{2}{\nabla}_X Y = \underset{1}{\nabla}_Y X + [X,Y], \ X, Y \in T_p(M),$$

where as usual $[\cdot,\cdot]$ denotes the Lie bracket.

Consequently, there exist four kinds of covariant derivatives of tensor fields [13]:

$$\underset{1}{\nabla}_m a^i_j \equiv a^i_{j\underset{1}{|}m} = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{jm}a^i_p, \quad \underset{2}{\nabla}_m a^i_j \equiv a^i_{j\underset{2}{|}m} = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{mj}a^i_p,$$
$$\underset{3}{\nabla}_m a^i_j \equiv a^i_{j\underset{3}{|}m} = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{mj}a^i_p, \quad \underset{4}{\nabla}_m a^i_j \equiv a^i_{j\underset{4}{|}m} = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{jm}a^i_p.$$

Also, we can consider usual covariant differentiation:

$$\nabla_m a^i_j \equiv a^i_{j;m} = a^i_{j,m} + \Gamma^i_{\underline{mp}} a^p_j - \Gamma^p_{\underline{mj}} a^i_p,$$

where $a^i_{j,m}$ denotes the partial derivative of a tensor a^i_j with respect to x^m and $\underline{mp}$ signifies a symmetrization with division, i.e., $\Gamma^i_{\underline{mp}} = \frac{1}{2}\left(\Gamma^i_{mp} + \Gamma^i_{pm}\right)$.

2. Generalized para-Kähler spaces in Eisenhart's sense

The non-symmetric linear connections $\underset{1}{\nabla}$ and $\underset{2}{\nabla}$ can be described thorough theirs symmetric part ∇ and torsion tensor $\underset{1}{T}$ as follows

$$\underset{\theta}{\nabla}_X Y = \nabla_X Y + \frac{(-1)^{\theta-1}}{2} \underset{1}{T}(X,Y), \quad \theta = 1, 2. \tag{1}$$

Here ∇ denotes the symmetric part of the non-symmetric linear connections $\underset{1}{\nabla}$ and $\underset{2}{\nabla}$ and it is given by

$$\nabla_X Y = \frac{1}{2}\Big(\underset{1}{\nabla}_X Y + \underset{1}{\nabla}_Y X\Big) = \frac{1}{2}\Big(\underset{2}{\nabla}_X Y + \underset{2}{\nabla}_Y X\Big),$$

and the torsion tensor $\underset{1}{T}$ is defined by

$$\underset{1}{T}(X,Y) = \underset{1}{\nabla}_X Y - \underset{1}{\nabla}_Y X.$$

Note that for a $(1,1)$ tensor field F the condition

$$\underset{1}{\nabla} F = 0 \ \text{ and } \ \underset{2}{\nabla} F = 0, \tag{2}$$

is stronger than the condition

$$\nabla F = 0, \tag{3}$$

where ∇ denotes the symmetric part of the non-symmetric linear connection $\underset{1}{\nabla}$. Indeed, if we assume that condition (2) holds, then we have that

$$\nabla_X F Y = \frac{1}{2}\Big(\underset{1}{\nabla}_X F Y + \underset{1}{\nabla}_{FY} X\Big) = \frac{1}{2}\Big(\underset{1}{\nabla}_X F Y + \underset{2}{\nabla}_X F Y\Big) = 0,$$

for arbitrary vector fields X and Y, i.e., condition (3) is fulfilled.

The previous discussion leads to the more general definition of generalized para-Kähler spaces than the one given in [16].

Definition 2.1. *A generalized Riemannian space* (M, g) *is called a* generalized para-Kähler space in Eisenhart's sense *if there exists a* $(1,1)$ *tensor field* F *on* M *such that*

$$F^2 = I, \tag{4}$$

$$\underline{g}(FX, FY) = -\underline{g}(X, Y), \tag{5}$$

$$\nabla F = 0, \tag{6}$$

where ∇ *is the Levi-Civita connection of the symmetric part* $\underline{g}$ *of the metric* g *and* I *is the identity operator.*

Let us consider the following five linearly independent curvature tensors [14] in generalized para-Kähler spaces in Eisenhart's sense:

$$\underset{\theta}{R}(X,Y)Z = \underset{\theta}{\nabla}_X \underset{\theta}{\nabla}_Y Z - \underset{\theta}{\nabla}_Y \underset{\theta}{\nabla}_X Z - \underset{\theta}{\nabla}_{[X,Y]} Z, \quad \theta = 1,2;$$
$$\underset{3}{R}(X,Y)Z = \underset{2}{\nabla}_X \underset{1}{\nabla}_Y Z - \underset{1}{\nabla}_Y \underset{2}{\nabla}_X Z + \underset{2}{\nabla}_{\underset{1}{\nabla}_Y X} Z - \underset{1}{\nabla}_{\underset{2}{\nabla}_X Y} Z;$$
$$\underset{4}{R}(X,Y)Z = \underset{2}{\nabla}_X \underset{1}{\nabla}_Y Z - \underset{1}{\nabla}_Y \underset{2}{\nabla}_X Z + \underset{2}{\nabla}_{\underset{2}{\nabla}_Y X} Z - \underset{1}{\nabla}_{\underset{1}{\nabla}_X Y} Z;$$
$$\underset{5}{R}(X,Y)Z = \frac{1}{2}\Big(\underset{1}{\nabla}_X \underset{1}{\nabla}_Y Z - \underset{2}{\nabla}_Y \underset{1}{\nabla}_X Z + \underset{2}{\nabla}_X \underset{2}{\nabla}_Y Z - \underset{1}{\nabla}_Y \underset{2}{\nabla}_X Z + \underset{1}{\nabla}_{[Y,X]} Z + \underset{2}{\nabla}_{[Y,X]} Z\Big).$$

Theorem 2.1. *Let (M, g, F) be a generalized para-Kähler space in Eisenhart's sense, then the curvature tensors $\underset{\theta}{R}$, $\theta = 1, \ldots, 4$ and the torsion tensor $\underset{1}{T}$ of this space satisfy*

$$\underset{1}{R}(X,Y)FZ = F(\underset{1}{R}(X,Y)Z) + \frac{1}{2}\underset{1}{T}(FZ, \underset{1}{T}(Y,X)) + \frac{1}{2}F(\underset{1}{T}(Z, \underset{1}{T}(Y,X))) + \underset{1}{S}(X,Y,Z), \tag{7}$$

or locally

$$\underset{1}{R}{}^h_{pjk} F^p_i = F^h_p \underset{1}{R}{}^p_{ijk} + \frac{1}{2} T^h_{pq} F^p_i T^q_{jk} + \frac{1}{2} F^h_p T^p_{iq} T^q_{jk} + \underset{1}{S}{}^h_{ijk},$$

where $\underset{1}{S}$ is a $(1,3)$ tensor field determined in local components by

$$\underset{1}{S}{}^h_{ijk} = \Big(\frac{1}{2}\underset{1}{T}{}^h_{pj\underset{1}{|}k} F^p_i + \frac{1}{4}\underset{1}{T}{}^h_{pj}\Big(\underset{1}{T}{}^p_{qk} F^q_i - \underset{1}{T}{}^q_{ik} F^p_q\Big) - \frac{1}{2}\underset{1}{T}{}^p_{ij\underset{1}{|}k} F^h_p - \frac{1}{4}\underset{1}{T}{}^p_{ij}\Big(\underset{1}{T}{}^h_{qk} F^q_p - \underset{1}{T}{}^q_{pk} F^h_q\Big)\Big)_{[jk]};$$

$$\underset{2}{R}(X,Y)FZ = F(\underset{2}{R}(X,Y)Z) + \frac{1}{2}\underset{1}{T}(FZ, \underset{1}{T}(Y,X)) + \frac{1}{2}F(\underset{1}{T}(Z, \underset{1}{T}(Y,X))) + \underset{2}{S}(X,Y,Z), \tag{8}$$

or locally

$$\underset{2}{R}{}^h_{pjk} F^p_i = F^h_p \underset{2}{R}{}^p_{ijk} + \frac{1}{2} T^h_{pq} F^p_i T^q_{jk} + F^h_p T^p_{iq} T^q_{jk} + \underset{2}{S}{}^h_{ijk},$$

where $\underset{2}{S}$ is a $(1,3)$ tensor field determined in local components by

$$\underset{2}{S}{}^h_{ijk} = \Big(\frac{1}{2}\underset{1}{T}{}^h_{jp\underset{1}{|}k} F^p_i + \frac{1}{4}\underset{1}{T}{}^h_{jp}\Big(\underset{1}{T}{}^p_{kq} F^q_i - \underset{1}{T}{}^q_{ki} F^p_q\Big) - \frac{1}{2}\underset{1}{T}{}^p_{ji\underset{1}{|}k} F^h_p - \frac{1}{4}\underset{1}{T}{}^p_{ji}\Big(\underset{1}{T}{}^h_{kq} F^q_p - \underset{1}{T}{}^q_{kp} F^h_q\Big)\Big)_{[jk]}$$

$$\underset{3}{R}(X,Y)FZ = F(\underset{3}{R}(X,Y)Z) + \underset{3}{S}(X,Y,Z), \tag{9}$$

or locally

$$\underset{3}{R}{}^h_{pjk} F^p_i = F^h_p \underset{3}{R}{}^p_{ijk} + \underset{3}{S}{}^h_{ijk},$$

where $\underset{3}{S}$ is a $(1,3)$ tensor field determined in local components by

$$\underset{3}{S}{}^h_{ijk} = \Big(\frac{1}{2}\underset{1}{T}{}^h_{pj\underset{2}{|}k} F^p_i + \frac{1}{4}\underset{1}{T}{}^h_{pj}\Big(\underset{1}{T}{}^p_{kq} F^q_i - \underset{1}{T}{}^q_{ki} F^p_q\Big) - \frac{1}{2}\underset{1}{T}{}^p_{ij\underset{2}{|}k} F^h_p - \frac{1}{4}\underset{1}{T}{}^p_{ij}\Big(\underset{1}{T}{}^h_{kq} F^q_p - \underset{1}{T}{}^q_{kp} F^p_q\Big)$$
$$- \frac{1}{2}\underset{1}{T}{}^h_{kp\underset{1}{|}j} F^p_i - \frac{1}{4}\underset{1}{T}{}^h_{kp}\Big(\underset{1}{T}{}^p_{qj} F^q_i - \underset{1}{T}{}^q_{ij} F^p_q\Big) + \frac{1}{2}\underset{1}{T}{}^p_{ki\underset{1}{|}j} F^h_p + \frac{1}{4}\underset{1}{T}{}^p_{ki}\Big(\underset{1}{T}{}^h_{qj} F^q_p - \underset{1}{T}{}^q_{pj} F^h_q\Big)\Big)_{[jk]},$$

$$\underset{4}{R}(X,Y)FZ + F(\underset{3}{R}(Y,X)Z) = \underset{4}{S}(X,Y,Z), \tag{10}$$

or locally

$$\underset{4}{R}{}^h_{pjk}F^p_i = F^h_p \underset{3}{R}{}^p_{ikj} + \underset{4}{S}{}^h_{ijk},$$

where $\underset{4}{S}$ *is a* $(1,3)$ *tensor field determined in local components by*

$$\begin{aligned}\underset{4}{S}{}^h_{ijk} =&\Big(\frac{1}{2}\underset{1}{T}{}^h_{pj\underset{4}{|}k}F^p_i + \frac{1}{4}\underset{1}{T}{}^h_{pj}\Big(\underset{1}{T}{}^p_{kq}F^q_i - \underset{1}{T}{}^q_{ik}F^p_q\Big) - \frac{1}{2}\underset{1}{T}{}^p_{ji\underset{4}{|}k}F^h_p - \frac{1}{4}\underset{1}{T}{}^p_{ji}\Big(\underset{1}{T}{}^h_{kq}F^q_p - \underset{1}{T}{}^q_{pk}F^h_q\Big)\\ &+\frac{1}{2}\underset{1}{T}{}^h_{kp\underset{3}{|}j}F^p_i + \frac{1}{4}\underset{1}{T}{}^h_{kp}\Big(\underset{1}{T}{}^p_{qj}F^q_i - \underset{1}{T}{}^q_{ji}F^p_q\Big) - \frac{1}{2}\underset{1}{T}{}^p_{ik\underset{3}{|}j}F^h_p - \frac{1}{4}\underset{1}{T}{}^p_{ik}\Big(\underset{1}{T}{}^h_{qj}F^q_p - \underset{1}{T}{}^q_{jp}F^h_q\Big)\Big)_{[jk]}.\end{aligned}$$

Proof. By using (1) and (6) in the first Ricci type identity (Eq. (9) in [12])

$$-F(\underset{1}{R}(X,Y)Z) + \underset{1}{R}(X,Y)FZ - \underset{1}{\nabla}_{\underset{1}{T}(Y,X)}FZ = 0,$$

we obtain the proof of relation (7).

We use (1) and (6) in the second Ricci type identity (Eq. (13) in [12])

$$\underset{2}{\nabla}_Z\underset{2}{\nabla}_Y FX - \underset{2}{\nabla}_Y\underset{2}{\nabla}_Z FX = \underset{2}{R}(Z,Y)FX - F(\underset{2}{R}(Z,Y)X) + \underset{2}{\nabla}_{\underset{1}{T}(Y,Z)}FX,$$

which completes the proof of relation (8).

The Ricci type identity (Eq. (58′) from [12]) reads

$$\underset{2}{\nabla}_Z\underset{1}{\nabla}_Y FX - \underset{1}{\nabla}_Y\underset{2}{\nabla}_Z FX = \underset{3}{R}(Z,Y)FX - F(\underset{3}{R}(Z,Y)X),$$

together with (1) and (6) leads to the proof of (9).

To prove (10) we first observe that

$$F^h_{i\underset{3}{|}j} = \frac{1}{2}\underset{1}{T}{}^h_{pj}F^p_i - \frac{1}{2}\underset{1}{T}{}^p_{ji}F^h_p, \tag{11}$$

and

$$F^h_{i\underset{4}{|}j} = \frac{1}{2}\underset{1}{T}{}^h_{jp}F^p_i - \frac{1}{2}\underset{1}{T}{}^p_{ij}F^h_p, \tag{12}$$

where we used (6).

After taking the covariant derivative of the fourth and third kind in (11) and (12) we respectively obtain that

$$F^h_{i\underset{3}{|}j\underset{4}{|}k} =\frac{1}{2}\underset{1}{T}{}^h_{pj\underset{4}{|}k}F^p_i + \frac{1}{4}\underset{1}{T}{}^h_{pj}\Big(\underset{1}{T}{}^p_{kq}F^q_i - \underset{1}{T}{}^q_{ik}F^p_q\Big) - \frac{1}{2}\underset{1}{T}{}^p_{ji\underset{4}{|}k}F^h_p - \frac{1}{4}\underset{1}{T}{}^p_{ji}\Big(\underset{1}{T}{}^h_{kq}F^q_p - \underset{1}{T}{}^q_{pk}F^h_q\Big)$$

and

$$F^h_{i\underset{4}{|}k\underset{3}{|}j} =\frac{1}{2}\underset{1}{T}{}^h_{kp\underset{3}{|}j}F^p_i + \frac{1}{4}\underset{1}{T}{}^h_{kp}\Big(\underset{1}{T}{}^p_{qj}F^q_i - \underset{1}{T}{}^q_{ji}F^p_q\Big) - \frac{1}{2}\underset{1}{T}{}^p_{ik\underset{3}{|}j}F^h_p - \frac{1}{4}\underset{1}{T}{}^p_{ik}\Big(\underset{1}{T}{}^h_{qj}F^q_p - \underset{1}{T}{}^q_{jp}F^h_q\Big).$$

Taking into account the last two relations and the Ricci type identity (Eq. (56′) from [13])

$$\underset{4}{\nabla}_Z\underset{3}{\nabla}_Y FX - \underset{3}{\nabla}_Y\underset{4}{\nabla}_Z FX = \underset{4}{R}(Z,Y)FX + F(\underset{3}{R}(Y,Z)X),$$

we get (8) which completes the proof. □

We denote the curvature tensor of type $(0,4)$ by

$$\underset{\theta}{R}(X,Y,Z,W) := \underline{g}(\underset{\theta}{R}(X,Y)Z,W), \quad \theta = 1,\dots,4,$$

and the torsion tensor of type $(0,3)$ by

$$\underset{1}{T}(X,Y,Z) := \underline{g}(X,\underset{1}{T}(Y,Z)).$$

Also, we will use the same symbols for the $(0,4)$ tensor fields corresponding to the tensor fields $\underset{\theta}{S}$, $\theta = 2,\dots,4$, that were given in Theorem 2.1, i.e.,

$$\underset{\theta}{S}(X,Y,Z,W) := \underline{g}(\underset{\theta}{S}(X,Y)Z,W), \quad \theta = 2,\dots,4.$$

Corollary 2.1. *The curvature* $(0,4)$ *tensor fields* $\underset{\theta}{R}(X,Y,Z,V)$, $\theta = 1,\dots,4$, *the torsion tensors of type* $(1,2)$ *and* $(0,3)$, *and the* $(0,4)$ *tensor fields* $\underset{\theta}{S}(X,Y,Z,V)$, $\theta = 1,\dots,4$ *of a generalized para-Kähler space in Eisenhart's sense* (M,g,F) *satisfy*

$$\begin{aligned}
&\underset{1}{R}(X,Y,FZ,W) + \underset{1}{R}(X,Y,Z,FW) = \frac{1}{2}\Big(\underset{1}{T}(W,FZ,\underset{1}{T}(Y,X)) - \underset{1}{T}(FW,Z,\underset{1}{T}(Y,X))\Big) + \underset{1}{S}(W,Z,Y,X),\\
&\underset{2}{R}(X,Y,FZ,W) + \underset{2}{R}(X,Y,Z,FW) = -\frac{1}{2}\Big(\underset{1}{T}(W,\underset{1}{T}(Y,X),FZ) + \underset{1}{T}(FW,Z,\underset{1}{T}(Y,X))\Big) + \underset{2}{S}(W,Z,Y,X),\\
&\underset{3}{R}(X,Y,FZ,W) + \underset{3}{R}(X,Y,Z,FW) = \underset{3}{S}(W,Z,Y,X),\\
&\underset{4}{R}(X,Y,FZ,W) - \underset{3}{R}(Y,X,Z,FW) = \underset{4}{S}(W,Z,Y,X).
\end{aligned}$$

Proof. The proof directly follows from Theorem 2.1 by using the symmetry properties of the curvature tensors $\underset{\theta}{R}(X,Y,Z,V)$, $\theta = 1,\dots,4$ and equations (4) and (5). □

3. Equitorsion holomorphically projective mappings

In this subsection we shall consider holomorphically projective mappings between generalized para-Kähler spaces preserving the torsion tensor, i.e., so called *equitorsion holomorphically projective mappings* [16]. Equitorsion holomorphically projective mappings were firstly considered between generalized Kähler spaces [15, 28, 29] and later between generalized hyperbolic and m-parabolic Kähler spaces [16, 20].

Theorem 3.1. *Let* (M,g,F) *and* $(\overline{M},\overline{g},\overline{F})$ *be two generalized para-Kähler spaces in Eisenhart's sense of dimension* $n > 2$ *and* $f : M \to \overline{M}$ *be an equitorsion holomorphically projective mapping, then the geometric object given by*

$$\begin{aligned}
\underset{1}{P}{}^h_{ijk} = \underset{1}{R}{}^h_{ijk} + \frac{1}{n+2}\Big[&\delta^h_j \underset{1}{Q}_{ik} - \delta^h_k \underset{1}{Q}_{ij} + \delta^h_i \underset{1}{Q}_{[jk]} - F^h_k \underset{1}{Q}_{pj} F^p_i + F^h_j \underset{1}{Q}_{pk} F^p_i - F^h_i\Big(\underset{1}{Q}_{pj}F^p_k - \underset{1}{Q}_{pk}F^p_j\Big)\\
&- \frac{1}{2}\Big(F^h_j \Gamma^r_{pr}(\underset{1}{T}{}^p_{qk}F^q_i - \underset{1}{T}{}^q_{ik}F^p_q)\Big)_{[ij]} - \frac{1}{2}F^h_i\Gamma^r_{pr}(\underset{1}{T}{}^p_{qk}F^q_j - \underset{1}{T}{}^p_{qj}F^q_k + 2\underset{1}{T}{}^q_{kj}F^p_q)\\
&- \frac{1}{2}\Gamma^r_{pr}F^p_i(\underset{1}{T}{}^h_{qk}F^q_j - \underset{1}{T}{}^h_{qj}F^q_k + 2\underset{1}{T}{}^q_{kj}F^h_q) - \frac{1}{2}\Big(\Gamma^r_{pr}F^p_j(\underset{1}{T}{}^h_{qk}F^q_i - \underset{1}{T}{}^q_{ik}F^h_q)\Big)_{[ij]}\\
&- \underset{1}{T}{}^h_{jk}\Gamma^p_{ip} - \delta^h_i \Gamma^q_{pq}\underset{1}{T}{}^p_{jk} - F^h_p \underset{1}{T}{}^p_{jk}\Gamma^r_{qr}F^q_i - \Gamma^r_{pr}F^p_q\underset{1}{T}{}^q_{jk}F^h_i\Big],
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
\underset{1}{Q}_{ij} = \underset{1}{R}_{ij} - \frac{1}{n+2}\Big(&\Gamma^q_{pq}\underset{1}{T}{}^p_{ji} + \frac{n}{2(n-2)}\Big(\Gamma^s_{ps}F^p_i\underset{1}{T}{}^q_{rj}F^r_q\Big) + \frac{n-1}{n-2}F^q_p\underset{1}{T}{}^p_{jq}\Gamma^s_{rs}F^r_i - \frac{1}{n-2}F^q_p\underset{1}{T}{}^p_{iq}\Gamma^s_{rs}F^r_j\\
&+ \frac{1}{n-2}\Big(\Gamma^s_{is}\underset{1}{T}{}^p_{qr}F^q_pF^r_j + F^q_p\underset{1}{T}{}^p_{rq}F^r_j\Gamma^s_{is}\Big)_{(ij)} + \frac{1}{2}\Big(\Gamma^s_{ps}F^p_q\underset{1}{T}{}^q_{jr}F^r_i\Big)_{[ij]}\Big),
\end{aligned}$$

is invariant with respect to the mapping f.

Proof. We follow the steps of the proof of related theorem from [16]. The deformation tensor $\underset{1}{P}(X,Y)$ with respect to an equitorsion holomorphically projective mapping $f : M \to \overline{M}$ between generalized para-Kähler spaces in Eisenhart's sense is a symmetric bilinear form given by

$$\underset{1}{P}(X,Y) = \psi(X)Y + \psi(Y)X + \psi(FX)FY + \psi(FY)FX. \tag{14}$$

The curvature tensors $\underset{1}{R}$ and $\underset{1}{\overline{R}}$ of generalized para-Kähler spaces in Eisenhart's sense (M, g, F) and $(\overline{M}, \overline{g}, \overline{F})$, respectively, satisfy the relation

$$\underset{1}{\overline{R}}(X,Y)Z = \underset{1}{R}(X,Y)Z + (\underset{1}{\nabla}_X \underset{1}{P})(Z,Y) - (\underset{1}{\nabla}_Y \underset{1}{P})(Z,X) + \underset{1}{P}(\underset{1}{P}(Z,Y),X) - \underset{1}{P}(\underset{1}{P}(Z,X),Y) + \underset{1}{P}(Z,\underset{1}{T}(Y,X)). \tag{15}$$

Let us denote

$$\underset{1}{\psi}(X,Y) = (\underset{1}{\nabla}_Y \psi)(X) - \psi(X)\psi(Y) - \psi(FX)\psi(FY),$$

which in local components reads

$$\underset{1}{\psi}_{ij} = \psi_{i\underset{1}{|}j} - \psi_i \psi_j - \psi_p F^p_i \psi_q F^q_j.$$

Substituting (14) into (15) we obtain that

$$\begin{aligned}\underset{1}{\overline{R}}{}^h_{ijk} = \underset{1}{R}{}^h_{ijk} &+ \delta^h_j \underset{1}{\psi}_{ik} - \delta^h_k \underset{1}{\psi}_{ij} + \delta^h_i \underset{1}{\psi}_{[jk]} - F^h_k \underset{1}{\psi}_{pj} F^p_i + F^h_j \underset{1}{\psi}_{pk} F^p_i - F^h_i (\underset{1}{\psi}_{pj} F^p_k - \underset{1}{\psi}_{pk} F^p_j)\\ &+ \frac{1}{2} F^h_j \psi_p (\underset{1}{T}{}^p_{qk} F^q_i - \underset{1}{T}{}^q_{ik} F^p_q) - \frac{1}{2} F^h_k \psi_p (\underset{1}{T}{}^p_{qj} F^q_i - \underset{1}{T}{}^q_{ij} F^p_q) + \frac{1}{2} F^h_i \psi_p (\underset{1}{T}{}^p_{qk} F^q_j - \underset{1}{T}{}^p_{qj} F^q_k + 2\underset{1}{T}{}^q_{kj} F^p_q)\\ &+ \frac{1}{2} \psi_p F^p_i (\underset{1}{T}{}^h_{qk} F^q_j - \underset{1}{T}{}^h_{qj} F^q_k + 2\underset{1}{T}{}^q_{kj} F^h_q) + \frac{1}{2} \psi_p F^p_j (\underset{1}{T}{}^h_{qk} F^q_i - \underset{1}{T}{}^q_{ik} F^h_q) - \frac{1}{2} \psi_p F^p_k (\underset{1}{T}{}^h_{qj} F^q_i - \underset{1}{T}{}^q_{ij} F^h_q)\\ &+ \underset{1}{T}{}^h_{jk} \psi_i + \delta^h_i \psi_p \underset{1}{T}{}^p_{jk} + F^h_p \underset{1}{T}{}^p_{jk} \psi_q F^q_i + \psi_p F^p_q \underset{1}{T}{}^q_{jk} F^h_i.\end{aligned} \tag{16}$$

Contracting on the indices h and k in (16) and using $\underset{1}{T}{}^p_{ip} = 0$ we get

$$\begin{aligned}\underset{1}{\overline{R}}_{ij} =& \underset{1}{R}_{ij} - n\underset{1}{\psi}_{ij} + \underset{1}{\psi}_{[ji]} + \underset{1}{\psi}_{(pq)} F^p_i F^q_j + \frac{1}{2}\psi_p F^p_q \underset{1}{T}{}^q_{ri} F^r_j + \frac{1}{2}\psi_p F^p_q \underset{1}{T}{}^q_{rj} F^r_i\\ &+ \frac{1}{2}\psi_p F^p_i \underset{1}{T}{}^q_{rj} F^r_q + \frac{1}{2}\psi_p F^p_j \underset{1}{T}{}^q_{ri} F^r_q + \psi_p \underset{1}{T}{}^p_{ji} + F^q_p \underset{1}{T}{}^p_{jq} \psi_r F^r_i + \psi_p F^p_q \underset{1}{T}{}^q_{jr} F^r_i,\end{aligned} \tag{17}$$

where $\underset{1}{R}_{ij} = \underset{1}{R}{}^p_{ijp}$ and $\underset{1}{\overline{R}}_{ij} = \underset{1}{\overline{R}}{}^p_{ijp}$ are components of the Ricci tensors $\underset{1}{\mathrm{Ric}}(X,Y)$ and $\underset{1}{\overline{\mathrm{R}}\mathrm{ic}}(X,Y)$, respectively.

Anti-symmetrization without division in (17) with respect to the indices i and j gives

$$(n+2)\underset{1}{\psi}_{[ij]} = -\underset{1}{\overline{R}}_{[ij]} + \underset{1}{R}_{[ij]} + 2\psi_p \underset{1}{T}{}^p_{ji} + F^q_p \underset{1}{T}{}^p_{jq} \psi_r F^r_i - F^q_p \underset{1}{T}{}^p_{iq} \psi_r F^r_j + \psi_p F^p_q \underset{1}{T}{}^q_{jr} F^r_i - \psi_p F^p_q \underset{1}{T}{}^q_{ir} F^r_j. \tag{18}$$

By symmetrization without division in (17) with respect to i and j we obtain that

$$\underset{1}{\overline{R}}_{(ij)} = \underset{1}{R}_{(ij)} - n\underset{1}{\psi}_{(ij)} + 2\underset{1}{\psi}_{(pq)} F^p_i F^q_j + \psi_p F^p_i \underset{1}{T}{}^q_{rj} F^r_q + \psi_p F^p_j \underset{1}{T}{}^q_{ri} F^r_q + F^q_p \underset{1}{T}{}^p_{jq} \psi_r F^r_i + F^q_p \underset{1}{T}{}^p_{iq} \psi_r F^r_j, \tag{19}$$

and by composing with F^i_p and F^j_q in the last relation we obtain that

$$\underset{1}{\overline{R}}_{(pq)} F^p_i F^q_j = \underset{1}{R}_{(pq)} F^p_i F^q_j - n\underset{1}{\psi}_{(pq)} F^p_i F^q_j + 2\underset{1}{\psi}_{(ij)} + \psi_i \underset{1}{T}{}^p_{qr} F^q_p F^r_j + \psi_j \underset{1}{T}{}^p_{qr} F^q_p F^r_i + F^q_p \underset{1}{T}{}^p_{rq} F^r_j \psi_i + F^q_p \underset{1}{T}{}^p_{rq} F^r_i \psi_j. \tag{20}$$

The Ricci tensors $\underset{1}{\mathrm{Ric}}(X,Y) = \mathrm{Tr}\Big(U \to \underset{1}{R}(U,X)Y\Big)$ on a generalized para-Kähler space in Eisenhart's sense (M,g,F) satisfy

$$\underset{1}{R}_{(pq)}F^p_iF^q_j = -\underset{1}{R}_{(ij)} - \frac{1}{2}\underset{1}{T}^p_{rq}\underset{1}{T}^q_{ps}F^r_iF^s_j - \frac{1}{2}\underset{1}{T}^p_{iq}\underset{1}{T}^q_{pj}, \tag{21}$$

and the same relation is valid on the space $(\overline{M},\overline{g},\overline{F})$, that is,

$$\underset{1}{\overline{R}}_{(pq)}\overline{F}^p_i\overline{F}^q_j = -\underset{1}{\overline{R}}_{(ij)} - \frac{1}{2}\underset{1}{\overline{T}}^p_{rq}\underset{1}{\overline{T}}^q_{ps}\overline{F}^r_i\overline{F}^s_j - \frac{1}{2}\underset{1}{\overline{T}}^p_{iq}\underset{1}{\overline{T}}^q_{pj}. \tag{22}$$

By using the fact that the torsion tensor $\underset{1}{T}$ and the structure F are preserved under an equitorsion holomorphically projective mapping and by substituting (21) and (22) into (20) we obtain that

$$-\underset{1}{\overline{R}}_{(ij)} = -\underset{1}{R}_{(ij)} - n\underset{1}{\psi}_{(pq)}F^p_iF^q_j + 2\underset{1}{\psi}_{(ij)} + \psi_i\underset{1}{T}^p_{qr}F^q_pF^r_j + \psi_j\underset{1}{T}^p_{qr}F^q_pF^r_i + F^q_p\underset{1}{T}^p_{rq}F^r_j\psi_i + F^q_p\underset{1}{T}^p_{rq}F^r_i\psi_j. \tag{23}$$

Summing (19) and (23) we obtain that

$$\underset{1}{\psi}_{(pq)}F^p_iF^q_j = -\underset{1}{\psi}_{(ij)} + \frac{1}{n-2}\Big(\psi_pF^p_i\underset{1}{T}^q_{rj}F^r_q + F^q_p\underset{1}{T}^p_{jq}\psi_rF^r_i + \psi_i\underset{1}{T}^p_{qr}F^q_pF^r_j + F^q_p\underset{1}{T}^p_{rq}F^r_j\psi_i\Big)_{(ij)}. \tag{24}$$

Plugging (24) in (23) we get

$$(n+2)\underset{1}{\psi}_{(ij)} = -\underset{1}{\overline{R}}_{(ij)} + \underset{1}{R}_{(ij)} + \frac{n}{n-2}\Big(\psi_pF^p_i\underset{1}{T}^q_{rj}F^r_q + F^q_p\underset{1}{T}^p_{jq}\psi_rF^r_i\Big)_{(ij)} + \frac{2}{n-2}\Big(\psi_i\underset{1}{T}^p_{qr}F^q_pF^r_j + F^q_p\underset{1}{T}^p_{rq}F^r_j\psi_i\Big)_{(ij)}. \tag{25}$$

Now, by summing (18) and (25) we get

$$\begin{aligned}(n+2)\underset{1}{\psi}_{ij} = &-\underset{1}{\overline{R}}_{ij} + \underset{1}{R}_{ij} + \psi_p\underset{1}{T}^p_{ji} + \frac{n}{2(n-2)}\Big(\psi_pF^p_i\underset{1}{T}^q_{rj}F^r_q + \psi_pF^p_j\underset{1}{T}^q_{ri}F^r_q\Big) + \frac{n-1}{n-2}F^q_p\underset{1}{T}^p_{jq}\psi_rF^r_i\\ &+ \frac{1}{n-2}F^q_p\underset{1}{T}^p_{iq}\psi_rF^r_j + \frac{1}{n-2}\Big(\psi_i\underset{1}{T}^p_{qr}F^q_pF^r_j + F^q_p\underset{1}{T}^p_{rq}F^r_j\psi_i\Big)_{(ij)} + \frac{1}{2}\psi_pF^p_q\underset{1}{T}^q_{jr}F^r_i - \frac{1}{2}\psi_pF^p_q\underset{1}{T}^q_{ir}F^r_j.\end{aligned}$$

It is a simple matter to verify that

$$\overline{\Gamma}^p_{ip} - \Gamma^p_{ip} = (n+2)\psi_i. \tag{26}$$

By using (26) the last relation can be written in form

$$(n+2)\underset{1}{\psi}_{ij} = -\underset{1}{\overline{Q}}_{ij} + \underset{1}{Q}_{ij}, \tag{27}$$

where $\underset{1}{Q}_{ij}$ is defined by

$$\begin{aligned}\underset{1}{Q}_{ij} = \underset{1}{R}_{ij} - \frac{1}{n+2}\Big(&2\overline{\Gamma}^q_{pq}\underset{1}{T}^p_{ji} + \frac{n}{2(n-2)}\Big(\Gamma^s_{ps}F^p_i\underset{1}{T}^q_{rj}F^r_q\Big)_{(ij)} + \frac{n-1}{n-2}F^q_p\underset{1}{T}^p_{jq}\overline{\Gamma}^s_{rs}F^r_i + \frac{1}{n-2}F^q_p\underset{1}{T}^p_{iq}\overline{\Gamma}^s_{rs}F^r_j\\ &+ \frac{1}{n-2}\Big(\Gamma^s_{is}\underset{1}{T}^p_{qr}F^q_pF^r_j + F^q_p\underset{1}{T}^p_{rq}F^r_j\Gamma^s_{is}\Big)_{(ij)} + \frac{1}{2}\Gamma^s_{ps}F^p_q\underset{1}{T}^q_{jr}F^r_i - \frac{1}{2}\Gamma^s_{ps}F^p_q\underset{1}{T}^q_{ir}F^r_j\Big),\end{aligned}$$

and $\underset{1}{\overline{Q}}_{ij}$ is defined in the same manner for the space $(\overline{M},\overline{g},\overline{F})$.

Finally, plugging (26) and (27) into (16) we get

$$\underset{1}{P}^h_{ijk} = \underset{1}{\overline{P}}^h_{ijk},$$

where the geometric object $\underset{1}{P}^h_{ijk}$ is defined by (13) and $\underset{1}{\overline{P}}^h_{ijk}$ is defined in the same manner. Since the generalized Christoffel symbols are not tensors, the geometric object $\underset{1}{P}^h_{ijk}$ is not a tensor. □

In the same manner as in [16] we can take into account the curvature tensor $\underset{2}{R}{}^h_{ijk}$ and find another invariant geometric objects of the equitorsion holomorphically projective mappings between generalized para-Kähler spaces in Eisenhart's sense.

Theorem 3.2. *Let* (M, g, F) *and* $(\overline{M}, \overline{g}, \overline{F})$ *be two generalized para-Kähler spaces in Eisenhart's sense of dimension* $n > 2$ *and* $f : M \to \overline{M}$ *be an equitorsion holomorphically projective mapping, then the geometric object given by*

$$\begin{aligned}\underset{2}{P}{}^h_{ijk} = \underset{2}{R}{}^h_{ijk} + \frac{1}{n+2}\Big[&\delta^h_j \underset{2}{Q}_{ik} - \delta^h_k \underset{2}{Q}_{ij} + \delta^h_i \underset{2}{Q}_{[jk]} - F^h_k \underset{2}{Q}_{pj} F^p_i + F^h_j \underset{2}{Q}_{pk} F^p_i - F^h_i\Big(\underset{2}{Q}_{pj} F^p_k - \underset{2}{Q}_{pk} F^p_j\Big)\\
&+ \frac{1}{2}\Big(F^h_j \Gamma^r_{pr}\Big(\underset{1}{T}{}^p_{qk} F^q_i - \underset{1}{T}{}^q_{ik} F^p_q\Big)\Big)_{[ij]} + \frac{1}{2} F^h_i \Gamma^r_{pr}\Big(\underset{1}{T}{}^p_{qk} F^q_j - \underset{1}{T}{}^p_{qj} F^q_k + 2\underset{1}{T}{}^q_{kj} F^p_q\Big)\\
&+ \frac{1}{2}\Gamma^r_{pr} F^p_i\Big(\underset{1}{T}{}^h_{qk} F^q_j - \underset{1}{T}{}^h_{qj} F^q_k + 2\underset{1}{T}{}^q_{kj} F^h_q\Big) + \frac{1}{2}\Big(\Gamma^r_{pr} F^p_j\Big(\underset{1}{T}{}^h_{qk} F^q_i - \underset{1}{T}{}^q_{ik} F^h_q\Big)\Big)_{[ij]}\\
&+ \underset{1}{T}{}^h_{jk} \Gamma^p_{ip} + \delta^h_i \Gamma^q_{pq} \underset{1}{T}{}^p_{jk} + F^h_p \underset{1}{T}{}^p_{jk} \Gamma^r_{qr} F^q_i + \Gamma^r_{pr} F^p_q \underset{1}{T}{}^q_{jk} F^h_i\Big],\end{aligned} \tag{28}$$

where

$$\begin{aligned}\underset{2}{Q}_{ij} = \underset{2}{R}_{ij} + \frac{1}{n+2}\Big(&\Gamma^q_{pq} \underset{1}{T}{}^p_{ji} - \frac{n-1}{2(n-1)}\Big(\Gamma^s_{ps} F^p_q \underset{1}{T}{}^q_{jr} F^r_i\Big)_{[ij]} + \frac{n}{2(n-1)}\Big(\Gamma^s_{is} \underset{1}{T}{}^p_{qr} F^q_p F^r_j - F^q_p \underset{1}{T}{}^p_{rq} F^r_j \Gamma^s_{is} + \Gamma^s_{ps} F^p_i \underset{1}{T}{}^q_{rj} F^r_q\\
&+ \Gamma^s_{is} \underset{1}{T}{}^p_{qr} F^q_p F^r_j - F^q_p \underset{1}{T}{}^p_{rq} F^r_j \Gamma^s_{is}\Big)_{(ij)} + \frac{n-2}{2(n-1)}\Big(F^q_p \underset{1}{T}{}^p_{jq} \psi_r F^r_i\Big)_{(ij)}\Big),\end{aligned}$$

is invariant with respect to the mapping f.

The technique from the previous theorems can be applied for the curvature tensors $\underset{3}{R}{}^h_{ijk}$ and $\underset{4}{R}{}^h_{ijk}$. In such a way we obtain some other invariant geometric objects.

Theorem 3.3. *Let* (M, g, F) *and* $(\overline{M}, \overline{g}, \overline{F})$ *be two generalized para-Kähler spaces in Eisenhart's sense of dimension* $n > 2$ *and* $f : M \to \overline{M}$ *be an equitorsion holomorphically projective mapping, then the geometric objects given by*

$$\begin{aligned}\underset{\theta}{P}{}^h_{ijk} = \underset{\theta}{R}{}^h_{ijk} + \frac{1}{n+2}\Big[&\delta^h_j \underset{\theta}{Q}_{ik} - \delta^h_j \underset{1}{T}{}^p_{ik} \Gamma^s_{ps} + \delta^h_i \underset{\theta}{Q}_{[jk]} - \delta^h_i \underset{1}{T}{}^p_{jk} \Gamma^s_{ps} - \delta^h_k \underset{\theta}{Q}_{ij} - F^h_k \underset{\theta}{Q}_{pj} F^p_i + F^h_j \underset{\theta}{Q}_{pk} F^p_i - F^h_j \underset{1}{T}{}^p_{qk} F^q_i \Gamma^s_{ps}\\
&- F^h_i \underset{\theta}{Q}_{pj} F^p_k + F^h_i \underset{\theta}{Q}_{pk} F^p_j - F^h_i \underset{1}{T}{}^p_{qk} F^q_j \Gamma^s_{ps} - \frac{1}{2}\Big(F^h_j \Gamma^s_{ps}\Big(\underset{1}{T}{}^p_{kq} F^q_i - \underset{1}{T}{}^q_{ki} F^p_q\Big)\Big)_{[jk]}\\
&- \frac{1}{2}\Gamma^s_{ps} F^p_i\Big(\underset{1}{T}{}^h_{kq} F^q_j - \underset{1}{T}{}^h_{qj} F^q_k\Big) - \frac{1}{2} F^h_i \Gamma^s_{ps}\Big(\underset{1}{T}{}^p_{kq} F^q_j - \underset{1}{T}{}^p_{qj} F^q_k\Big)\\
&- \frac{1}{2}\Gamma^s_{ps} F^p_j\Big(\underset{1}{T}{}^h_{kq} F^q_i - \underset{1}{T}{}^q_{ki} F^h_q\Big) + \frac{1}{2}\Gamma^s_{ps} F^p_k\Big(\underset{1}{T}{}^h_{qj} F^q_i - \underset{1}{T}{}^q_{ij} F^h_q\Big)\\
&- \underset{1}{T}{}^h_{ji} \Gamma^p_{kp} - \underset{1}{T}{}^h_{ki} \Gamma^p_{jp} - \underset{1}{T}{}^h_{pi} F^p_j \Gamma^r_{qr} F^q_k - \underset{1}{T}{}^h_{pi} F^p_k \Gamma^r_{qr} F^q_j\Big], \quad \theta = 3, 4,\end{aligned}$$

where

$$\underset{\theta}{Q}_{ij} = \underset{\theta}{R}_{ij} + \frac{n-4}{4(n^2-4)}\Big(\Gamma^s_{ps} F^p_q \underset{1}{T}{}^q_{rj} F^r_i\Big)_{[ij]} - \frac{1}{n+2}\underset{1}{T}{}^p_{ji} \Gamma^s_{ps} - \frac{1}{4(n+2)}\underset{1}{T}{}^r_{pi} F^p_j \Gamma^s_{qs} F^q_r - \frac{1}{2(n+2)}\Big(\underset{1}{T}{}^r_{pi} F^p_r \Gamma^s_{qs} F^q_j\Big)_{[ij]}, \quad \theta = 3, 4,$$

are invariant with respect to the mapping f.

Proof. We first observe that plugging (14) in the relations between the curvature tensors $\underset{\theta}{R}{}^h_{ijk}$ and $\underset{\theta}{\overline{R}}{}^h_{ijk}$

($\theta = 3, 4$) of the generalized para-Kähler spaces in Eisenhart's sense yields

$$\begin{aligned}\underset{\theta}{\overline{R}}{}^h_{ijk} =& \underset{\theta}{R}{}^h_{ijk} + \delta^h_j \underset{2}{\psi}_{ik} + \delta^h_i (\underset{2}{\psi}_{jk} - \underset{1}{\psi}_{kj}) - \delta^h_k \underset{1}{\psi}_{ij} - F^h_k \underset{1}{\psi}_{pj} F^p_i + F^h_j \underset{2}{\psi}_{pk} F^p_i - F^h_i\Big(\underset{1}{\psi}_{pj} F^p_k - \underset{2}{\psi}_{pk} F^p_j\Big)\\ &+ \frac{1}{2} F^h_j \psi_p (\underset{1}{T}{}^p_{kq} F^q_i - \underset{1}{T}{}^q_{ki} F^p_q) - \frac{1}{2} F^h_k \psi_p (\underset{1}{T}{}^p_{qj} F^q_i - \underset{1}{T}{}^q_{ij} F^p_q) + \frac{1}{2} \psi_p F^p_i (\underset{1}{T}{}^h_{kq} F^q_j - \underset{1}{T}{}^h_{qj} F^q_k) + \frac{1}{2} F^h_i \psi_p (\underset{1}{T}{}^p_{kq} F^q_j - \underset{1}{T}{}^p_{qj} F^q_k)\\ &+ \frac{1}{2} \psi_p F^p_j (\underset{1}{T}{}^h_{kq} F^q_i - \underset{1}{T}{}^q_{ki} F^h_q) - \frac{1}{2} \psi_p F^p_k (\underset{1}{T}{}^h_{qj} F^q_i - \underset{1}{T}{}^q_{ij} F^h_q) + \underset{1}{T}{}^h_{ji} \psi_k + \underset{1}{T}{}^h_{ki} \psi_j + \underset{1}{T}{}^h_{pi} F^p_j \psi_q F^q_k + \underset{1}{T}{}^h_{pi} F^p_k \psi_q F^q_j, \ \theta = 3, 4,\end{aligned} \tag{29}$$

where

$$\underset{\eta}{\psi}_{ij} = \psi_{i\underset{\eta}{|}j} - \psi_i \psi_j - \psi_p F^p_i \psi_q F^q_j, \ \eta = 1, 2.$$

According to the definition of covariant derivatives of the first and second kind we conclude that

$$\underset{2}{\psi}_{ij} = \underset{1}{\psi}_{ij} + \underset{1}{T}{}^p_{ij} \psi_p,$$

which implies that relation (29) becomes

$$\begin{aligned}\underset{\theta}{\overline{R}}{}^h_{ijk} =& \underset{\theta}{R}{}^h_{ijk} + \delta^h_j \underset{1}{\psi}_{ik} + \delta^h_j \underset{1}{T}{}^p_{ik} \psi_p + \delta^h_i (\underset{1}{\psi}_{jk} - \underset{1}{\psi}_{kj}) + \delta^h_i \underset{1}{T}{}^p_{jk} \psi_p\\ &- \delta^h_k \underset{1}{\psi}_{ij} - F^h_k \underset{1}{\psi}_{pj} F^p_i + F^h_j \underset{2}{\psi}_{pk} F^p_i - F^h_i\Big(\underset{1}{\psi}_{pj} F^p_k - \underset{2}{\psi}_{pk} F^p_j\Big)\\ &+ \frac{1}{2} F^h_j \psi_p (\underset{1}{T}{}^p_{kq} F^q_i - \underset{1}{T}{}^q_{ki} F^p_q) - \frac{1}{2} F^h_k \psi_p (\underset{1}{T}{}^p_{qj} F^q_i - \underset{1}{T}{}^q_{ij} F^p_q)\\ &+ \frac{1}{2} \psi_p F^p_i (\underset{1}{T}{}^h_{kq} F^q_j - \underset{1}{T}{}^h_{qj} F^q_k) + \frac{1}{2} F^h_i \psi_p (\underset{1}{T}{}^p_{kq} F^q_j - \underset{1}{T}{}^p_{qj} F^q_k)\\ &+ \frac{1}{2} \psi_p F^p_j (\underset{1}{T}{}^h_{kq} F^q_i - \underset{1}{T}{}^q_{ki} F^h_q) - \frac{1}{2} \psi_p F^p_k (\underset{1}{T}{}^h_{qj} F^q_i - \underset{1}{T}{}^q_{ij} F^h_q)\\ &+ \underset{1}{T}{}^h_{ji} \psi_k + \underset{1}{T}{}^h_{ki} \psi_j + \underset{1}{T}{}^h_{pi} F^p_j \psi_q F^q_k + \underset{1}{T}{}^h_{pi} F^p_k \psi_q F^q_j, \ \theta = 3, 4.\end{aligned}$$

The rest of the proof is analogous to the proof of Theorem 3.1. □

Finally, we take into account the curvature tensor $\underset{5}{R}{}^h_{ijk}$ and obtain the fifth invariant geometric object which coincides with the tensor $\underset{5}{P}{}^h_{ijk}$ from [16].

Proposition 3.1. *Let (M, g, F) and $(\overline{M}, \overline{g}, \overline{F})$ be two generalized para-Kähler spaces in Eisenhart's sense of dimension $n > 2$ and $f : M \to \overline{M}$ be an equitorsion holomorphically projective mapping, then the tensor given by*

$$\underset{5}{P}{}^h_{ijk} = \underset{5}{R}{}^h_{ijk} + \frac{1}{n+2}\Big[\underset{5}{R}_{ki}\delta^h_j - \underset{5}{R}_{ji}\delta^h_k + \underset{5}{R}_{kp}F^p_i F^h_j - \underset{5}{R}_{jp}F^p_i F^h_k + 2\underset{5}{R}_{kp}F^p_j F^h_i\Big],$$

is invariant with respect to the mapping f.

4. Conclusion

As it was stated in [16] in case when a generalized (non-symmetric) Riemannian metric g is symmetric, i.e., has vanishing the skew-symmetric part $\underset{\vee}{g}$, a generalized para-Kähler space in Eisenhart's sense reduces to a usual para-Kähler space. The geometric objects $\underset{\theta}{P}{}^h_{ijk}$, $\theta = 1, \dots, 4$ are invariant with respect to an equitorsion holomorphically projective mapping between more general generalized hyperbolic Kähler spaces than those defined in [16]. It was surprising that the tensor $\underset{5}{P}{}^h_{ijk}$ did not change the shape in the more general class of spaces, i.e., it coincides with the tensor $\underset{5}{P}{}^h_{ijk}$ that was described in [16]. All these invariant geometric objects can be quite interesting for further investigations.

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