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Asymptotic Distribution with Random Indices for Linear Processes

Yu Miao[a], Qinghui Gao[a], Shuili Zhang[b]

[a] *College of Mathematics and Information Science, Henan Normal University*
[b] *College of Mathematics and Statistics, Pingdingshan University*

Abstract. In this paper, we consider the following linear process

$$X_n = \sum_{i=-\infty}^{\infty} c_i \xi_{n-i}, \quad n \in \mathbb{Z},$$

and establish the central limit theorem of the randomly indexed partial sums $S_{\nu_n} := X_1 + \cdots + X_{\nu_n}$, where $\{c_i; i \in \mathbb{Z}\}$ is a sequence of real numbers, $\{\xi_n; n \in \mathbb{Z}\}$ is a stationary m-dependent sequence and $\{\nu_n; n \geq 1\}$ is a sequence of positive integer valued random variables. In addition, in order to show the main result, we prove the central limit theorems for randomly indexed m-dependent random variables, which improve some known results.

1. Introduction and main results

Consider the linear process

$$X_n = \sum_{i \in \mathbb{Z}} c_i \xi_{n-i}, \quad n \in \mathbb{Z}, \tag{1.1}$$

where $\{c_i; i \in \mathbb{Z}\}$ is a sequence of real numbers and $\{\xi_n; n \in \mathbb{Z}\}$ is a stationary sequence of real random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}\xi_1 = 0$ and $\mathbb{E}\xi_1^2 \in (0, \infty)$. For any square summable sequence $\{c_n; n \in \mathbb{Z}\}$, the processes $\{X_n; n \in \mathbb{Z}\}$ can be defined if and only if the stationary sequence $\{\xi_n; n \in \mathbb{Z}\}$ has a bounded spectral density. Define $S_n = X_1 + \cdots + X_n$. If $(\xi_n)_{n \in \mathbb{Z}}$ is a sequence of independent identically distributed random variables with mean zero and finite variance and assume that $\sum_{j \in \mathbb{Z}} |c_j| < \infty$, then Anderson [1] proved that $n^{-1/2} S_n$ converges in distribution to the normal random variable. Hannan [11] proved the asymptotic normality by assuming only that the spectral density

$$f(\lambda) = \mathbb{E}\xi_1^2 (2\pi)^{-1} \left(\sum_{j \in \mathbb{Z}} c_j e^{i\lambda j} \right)^2 \tag{1.2}$$

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Research supported by NSFC (11971154)
Email addresses: yumiao728@gmail.com (Yu Miao), qinghuigaocb@163.com (Qinghui Gao), zhangshuilicong@126.com (Shuili Zhang)

is uniformly bounded and is continuous at $\lambda = 0$ with $f(0) > 0$, which weakened the conditions of Anderson. Hall and Heyde [10] observed that the result in fact continues to hold without the uniform boundedness of spectral density. Recently, Peligrad and Utev [25, 26] studied the central limit theorems of moving average processes generated by dependent sequences $\{\xi_n; n \in \mathbb{Z}\}$, which are weakened by Miao et al. [20] without the assumption of finite second moment for the sequences $\{\xi_n; n \in \mathbb{Z}\}$. In addition, Miao [17, 19] studied some central limit theorems for moving average processes. Other fluctuation results for the moving average processes have been proved by various authors. For example, Yokoyama [35, 36], Li and Zhang [14] considered the law of iterated logarithm; Lin and Li [15], Tyran-Kamińska [33] gave the functional central limit theorems; the large and moderate deviations principle were obtained in [5–7, 12, 16, 18, 21–23, 34].

Let $\{X_n; n \geq 1\}$ denote a sequence of independent and identically distributed random variables with mean value zero and variance one and let $\{\nu_n; n \geq 1\}$ be a sequence of positive integer valued random variables, such that $n^{-1}\nu_n \xrightarrow{\mathbb{P}} \nu$, where ν is a positive constant or positive random variable. When ν is a positive constant, Anscombe [2] proved that

$$\frac{S_{\nu_n}}{\sqrt{\nu_n}} \xrightarrow{\mathfrak{D}} N(0,1) \tag{1.3}$$

where $N(0,1)$ denotes the normal random variable. Rényi [28] obtained the central limit theorem (1.3) provided ν is a positive discrete random variable. Blum et al. [4] weakened the conditions in Anscombe [2] and Rényi [28] by assuming that ν is a positive, not necessarily discrete, random variable. Shang [30] gave the central limit theorem for randomly indexed martingale differences. Shang [31] studied the central limit theorems for functions of random variables under mixing conditions on the differences between the joint cumulative distribution functions and the product of the marginal cumulative distribution functions. Shang [32] and Belloni [3] considered the central limit theorem for randomly indexed m-dependent random variables. These known results did not assume that the random sequence $\{\nu_n; n \geq 1\}$ is independent of the sequence $\{X_n; n \geq 1\}$. Miao and Yang [24] discussed the moderate deviation principle for m-dependent random variables with unbounded m. Fakhre-Zakeri and Farshidi [8] and Fotopoulos and Wang [9] studied the central limit theorem (1.3) of the linear process (1.1) driven by the stationary independent sequences $\{\xi_n; n \in \mathbb{Z}\}$.

Motivated by the these known works, we are interested in the central limit theorem (1.3) of the linear process (1.1) driven by the stationary m-dependent sequences $\{\xi_n; n \in \mathbb{Z}\}$. The following theorem is our main result and its proof will be given in the next section.

Theorem 1.1. *Let $\{\xi_n; n \in \mathbb{Z}\}$ be a sequence of stationary m-dependent random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}\xi_1 = 0$ and $\mathbb{E}\xi_1^2 < \infty$. Consider a linear process of the form*

$$X_n = \sum_{i=-\infty}^{\infty} c_i \xi_{n-i}, \quad n \in \mathbb{Z},$$

with $\sum_{i=-\infty}^{\infty} |c_i| < \infty$. Assume that $\{\nu_n; n \in \mathbb{N}\}$ is a sequence of positive integer valued random variables defined on the same probability space. Let $S_n := X_1 + \cdots + X_n$. If there exists a positive random variable ν, such that

$$\frac{\nu_n}{n} \xrightarrow{\mathbb{P}} \nu, \tag{1.4}$$

then we have

$$\frac{S_{\nu_n}}{\sqrt{\nu_n}} \xrightarrow{\mathfrak{D}} N(0, \sigma^2 c^2),$$

where $c := \sum_{i=-\infty}^{\infty} c_i \neq 0$ and

$$\sigma^2 := \mathbb{E}\xi_1^2 + 2\sum_{k=1}^{m+1} \mathbb{E}\xi_1 \xi_k.$$

2. Proofs

2.1. Some lemmas

In this subsection, we give some lemmas to prove our results.

Lemma 2.1. *[13] Let $\{\xi_n; n \geq 1\}$ be a sequence of stationary m-dependent random variables with $\mathbb{E}\xi_1 = 0$ and $\mathbb{E}|\xi_1|^p < \infty$ for some $p \geq 2$. Then for all $n \geq 1$,*

$$\mathbb{E}\left[\max_{1\leq j\leq n} |\xi_1 + \cdots + \xi_j|^p\right] \leq C_p \mathbb{E}|\xi_1|^p n^{p/2},$$

where the positive constant C_p depends only on p (regardless of m).

Definition 2.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A sequence of events $\{A_n \in \mathcal{F}; n \geq 0\}$ is said to be $\mathbb{P}$-mixing, if for any event $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$, we have*

$$\lim_{n\to\infty}[\mathbb{P}(A_n|B) - \mathbb{P}(A_n)] = 0.$$

Lemma 2.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A sequence of events $\{A_n \in \mathcal{F}; n \geq 0\}$ with $\mathbb{P}(A_n) > 0$ is $\mathbb{P}$-mixing, if and only if for every k, we have*

$$\lim_{n\to\infty}[\mathbb{P}(A_n|A_k) - \mathbb{P}(A_n)] = 0.$$

Proof. Following the method of Theorem 2 in Rényi [27], we can get the lemma. □

Lemma 2.3. *Let $\{\xi_n; n \geq 1\}$ be a sequence of stationary m-dependent random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}\xi_1 = 0$ and $\mathbb{E}|\xi_1|^2 < \infty$. For any $x \in \mathbb{R}$, define*

$$B_n = \left\{\frac{\xi_1 + \xi_2 + \cdots + \xi_n}{\sigma\sqrt{n}} \leq x\right\},$$

then the sequence $\{B_n; n \geq 1\}$ is $\mathbb{P}$-mixing.

Proof. Let $\eta_n := \xi_1 + \xi_2 + \cdots + \xi_n$, then for any $k + m < n$, we have

$$\mathbb{P}(B_n|B_k) = \mathbb{P}\left(\frac{\eta_n - \eta_{k+m} + \eta_{k+m}}{\sigma\sqrt{n}} \leq x \middle| \frac{\eta_k}{\sigma\sqrt{k}} \leq x\right).$$

Based on the central limit theorem of m-dependent random variables, it is easy to check that

$$\mathbb{P}\left(\frac{\eta_n - \eta_{k+m}}{\sigma\sqrt{n}} \leq x \middle| \frac{\eta_k}{\sigma\sqrt{k}} \leq x\right) = \mathbb{P}\left(\frac{\eta_n - \eta_{k+m}}{\sigma\sqrt{n}} \leq x\right) \to \Phi(x)$$

where $\Phi(\cdot)$ is the distribution function of normal random variables, and for any $r > 0$

$$\mathbb{P}\left(\frac{|\eta_{k+m}|}{\sigma\sqrt{n}} > r \middle| \frac{\eta_k}{\sigma\sqrt{k}} \leq x\right) \xrightarrow{\mathbb{P}} 0,$$

as $n \to \infty$. Hence the sequence $\{B_n; n \geq 1\}$ is $\mathbb{P}$-mixing. □

Lemma 2.4. *[4] Let $W_n, X_{l,n}, Y_{l,n}^{(j)}$ and $Z_{l,n}^{(j)}$ be random variables for $l, n = 1, 2, \cdots$ and $j = 1, 2, \cdots, k$. Suppose that*

$$W_n = X_{l,n} + \sum_{j=1}^{k} Y_{l,n}^{(j)} Z_{l,n}^{(j)}$$

and

(a) $\lim_{l\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(|Y_{l,n}^{(j)}|>r\right)=0$ *for every* $r>0$ *and* $j=1,\cdots,k$;

(b) $\lim_{M\to\infty}\limsup_{l\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(|Z_{l,n}^{(j)}|>M\right)=0$ *for* $j=1,\cdots,k$;

(c) *the distributions of* $\{X_{l,n}\}$ *converge to the distribution function* F *for each fixed* l.

Then the distribution functions of $\{W_n\}$ *converge to* F.

Lemma 2.5. *[29] Let* $\{Y_n; n\geq 1\}$ *be a sequence of random variable such that* $Y_n \xrightarrow{a.s.} Y$, *where* Y *is a random variable. Let* ν_n *be a sequence of positive random variables which take on only integral values such that* $\nu_n \xrightarrow{\mathbb{P}} \infty$. *Then we have* $Y_{\nu_n}\xrightarrow{\mathbb{P}} Y$.

2.2. Central limit theorem for m-dependent random variables

In this subsection, we prove the central limit theorems for randomly indexed m-dependent random variables. Firstly, we consider the case that the random variable ν has a discrete distribution.

Proposition 2.1. *Let* $\{\xi_n; n\geq 1\}$ *be a sequence of stationary m-dependent random variables on a probability space* $(\Omega,\mathcal{F},\mathbb{P})$ *with* $\mathbb{E}\xi_1=0$ *and* $\mathbb{E}\xi_1^2<\infty$. *Assume that* $\{\nu_n; n\in\mathbb{N}\}$ *is a sequence of positive integer valued random variables defined on the same probability space. If there exists a positive random variable* ν *having a discrete distribution, such that*

$$\frac{\nu_n}{n}\xrightarrow{\mathbb{P}}\nu, \tag{2.1}$$

then we have

$$\frac{\xi_1+\xi_2+\cdots+\xi_{\nu_n}}{\sigma\sqrt{\nu_n}}\xrightarrow{\mathfrak{D}}N(0,1).$$

Proof. Let $\eta_n:=\xi_1+\xi_2+\cdots+\xi_n$, then it is easy to check that

$$\frac{\eta_{\nu_n}}{\sigma\sqrt{\nu_n}}=\frac{\eta_{[n\nu]}}{\sigma\sqrt{[n\nu]}}+\sqrt{\frac{[n\nu]}{\nu_n}}\left(\frac{\eta_{\nu_n}-\eta_{[n\nu]}}{\sigma\sqrt{[n\nu]}}\right)+\frac{\eta_{[n\nu]}}{\sigma\sqrt{[n\nu]}}\left(\sqrt{\frac{[n\nu]}{\nu_n}}-1\right). \tag{2.2}$$

Let $\{b_k;k\geq 1\}$ $(0<b_1<b_2<\cdots)$ denote the values taken on by ν with positive probability and let A_k denote the event $A_k=\{\nu=b_k\}$. For any $x\in\mathbb{R}$, we have

$$\mathbb{P}\left(\frac{\eta_{[n\nu]}}{\sigma\sqrt{[n\nu]}}\leq x\right)=\sum_{k=1}^{\infty}\mathbb{P}\left(\frac{\eta_{[nb_k]}}{\sigma\sqrt{[nb_k]}}\leq x\middle|A_k\right)\mathbb{P}(A_k). \tag{2.3}$$

From Lemma 2.3, for any fixed k, we have

$$\mathbb{P}\left(\frac{\eta_{[nb_k]}}{\sigma\sqrt{[nb_k]}}\leq x\middle|A_k\right)\to\Phi(x),\ \text{ as }\ n\to\infty,$$

which from (2.3), implies

$$\lim_{n\to\infty}\mathbb{P}\left(\frac{\eta_{[n\nu]}}{\sigma\sqrt{[n\nu]}}\leq x\right)=\Phi(x). \tag{2.4}$$

From the condition (2.1), we have

$$\frac{\eta_{[n\nu]}}{\sigma\sqrt{[n\nu]}}\left(\sqrt{\frac{[n\nu]}{\nu_n}}-1\right)\xrightarrow{\mathbb{P}}0. \tag{2.5}$$

In order to prove the proposition, from (2.2), (2.4) and (2.5), it is enough to see that

$$\frac{\eta_{\nu_n} - \eta_{[n\nu]}}{\sigma\sqrt{[n\nu]}} \xrightarrow{\mathbb{P}} 0. \tag{2.6}$$

For any $r > 0$ and $\rho > 0$, we have

$$\begin{aligned}\mathbb{P}\left(\frac{|\eta_{\nu_n} - \eta_{[n\nu]}|}{\sigma\sqrt{[n\nu]}} > r\right) &= \sum_{k=1}^{\infty}\mathbb{P}\left(\frac{|\eta_{\nu_n} - \eta_{[n\nu]}|}{\sigma\sqrt{[n\nu]}} > r, \nu = b_k\right)\\ \leq \sum_{k=1}^{\infty}\mathbb{P}\left(\frac{|\eta_{\nu_n} - \eta_{[nb_k]}|}{\sigma\sqrt{[nb_k]}} > r, \nu = b_k, |\nu_n - [nb_k]| < n\rho\right) &+ \mathbb{P}\left(|\nu_n - [n\nu]| \geq n\rho\right)\end{aligned} \tag{2.7}$$

Since $\{b_k, k \geq 1\}$ are the values taken on by ν with positive probability, then there exists a positive constant M, such that $\mathbb{P}(\nu > M) \leq \rho$. Furthermore, from Lemma 2.1, for any k, there exists a positive constant C_k such that

$$\mathbb{P}\left(\max_{|l-[nb_k]|<n\rho}\frac{|\eta_l - \eta_{[nb_k]}|}{\sigma\sqrt{[nb_k]}} > r\right) \leq \frac{C_k\rho}{r^2 b_k}.$$

Hence we can get

$$\begin{aligned}&\sum_{k=1}^{\infty}\mathbb{P}\left(\frac{|\eta_{\nu_n} - \eta_{[nb_k]}|}{\sigma\sqrt{[nb_k]}} > r, \nu = b_k, |\nu_n - [nb_k]| < n\rho\right)\\ \leq&\sum_{k=1}^{M}\mathbb{P}\left(\max_{|l-[nb_k]|<n\rho}\frac{|\eta_l - \eta_{[nb_k]}|}{\sigma\sqrt{[nb_k]}} > r\right) + \mathbb{P}(\nu > M)\\ \leq&\rho\left(\sum_{k=1}^{M}\frac{C_k}{r^2 b_k} + 1\right)\end{aligned} \tag{2.8}$$

From (2.1), (2.7), (2.8) and the arbitrariness of ρ, we get the claim (2.6). □

Next we consider the case that the random variable ν is not necessarily discrete.

Proposition 2.2. *Let $\{\xi_n; n \geq 1\}$ be a sequence of stationary m-dependent random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}\xi_1 = 0$ and $\mathbb{E}\xi_1^2 < \infty$. Assume that $\{\nu_n; n \in \mathbb{N}\}$ is a sequence of positive integer valued random variables defined on the same probability space. If there exists a positive (not necessarily discrete) random variable ν, such that*

$$\frac{\nu_n}{n} \xrightarrow{\mathbb{P}} \nu, \tag{2.9}$$

then we have

$$\frac{\xi_1 + \xi_2 + \cdots + \xi_{\nu_n}}{\sigma\sqrt{\nu_n}} \xrightarrow{\mathfrak{D}} N(0,1).$$

Proof. Let us define

$$\mu_l = \frac{k}{2^l} \text{ when } \frac{k-1}{2^l} \leq \nu < \frac{k}{2^l}$$

and

$$\mu_{l,n} = \nu_n + [n(\mu_l - \nu)].$$

It is easy to check that μ_l is a discrete random variable for each l,

$$0 < \mu_l - \nu < \frac{1}{2^l} \text{ and } \frac{\mu_{l,n}}{n} \xrightarrow{\mathbb{P}} \mu_l > \nu.$$

Let $\eta_n := \xi_1 + \xi_2 + \cdots + \xi_n$, then we have

$$\begin{aligned}\frac{\eta_{\nu_n}}{\sqrt{\nu_n}} =& \frac{\eta_{\mu_{l,n}}}{\sqrt{\mu_{l,n}}} + \left(\frac{\eta_{\nu_n} - \eta_{\mu_{l,n}}}{\sqrt{n\mu_l}}\right)\sqrt{\frac{n\mu_l}{\nu_n}} + \left(\frac{\sqrt{\mu_{l,n}} - \sqrt{\nu_n}}{\sqrt{\nu_n}}\right)\frac{\eta_{\mu_{l,n}}}{\sqrt{\mu_{l,n}}} \\ =:& X_{l,n} + Y^{(1)}_{l,n}Z^{(1)}_{l,n} + Y^{(2)}_{l,n}Z^{(2)}_{l,n}.\end{aligned} \tag{2.10}$$

From Proposition 2.1, for each m, we have

$$X_{l,n} \xrightarrow{\mathfrak{D}} N(0,1) \text{ and } Z^{(2)}_{l,n} \xrightarrow{\mathfrak{D}} N(0,1), \tag{2.11}$$

which implies that for any $M > 0$,

$$\limsup_{n\to\infty} \mathbb{P}\left(|X_{l,n}| > M\right) = \limsup_{n\to\infty} \mathbb{P}\left(|Z^{(2)}_{l,n}| > M\right) = 1 - \Phi(M) + \Phi(-M).$$

For any $\varepsilon > 0$, there exists a positive constant l_0, such that for all $l > l_0$, we have

$$\mathbb{P}\left(0 < \nu < \frac{l}{2^l}\right) < \varepsilon.$$

Hence, for any $r > 0$ and all $l > l_0$, we have

$$\begin{aligned}&\limsup_{n\to\infty} \mathbb{P}\left(|Y^{(2)}_{l,n}| > r\right) \\ =&\limsup_{n\to\infty} \mathbb{P}\left(\frac{[n(\mu_l - \nu)]}{\nu_n} > r^2 + 2r\right) \\ =&\limsup_{n\to\infty} \mathbb{P}\left(\frac{[n(\mu_l - \nu)]}{\nu_n} > r^2 + 2r, \left|\frac{\nu_n}{n} - \nu\right| \le \frac{l}{2^{l+1}}\right) \\ \le&\limsup_{n\to\infty} \mathbb{P}\left(\frac{[n(\mu_l - \nu)]}{n\left(\nu - \frac{l}{2^{l+1}}\right)} > r^2 + 2r, \nu \ge \frac{l}{2^l}\right) + \varepsilon \\ \le&\limsup_{n\to\infty} \mathbb{P}\left(\frac{[n/2^l]}{\frac{nl}{2^{l+1}}} > r^2 + 2r, \nu \ge \frac{l}{2^l}\right) + \varepsilon \\ \le&\mathbb{P}\left(\frac{1}{l} > r^2 + 2r, \nu \ge \frac{l}{2^l}\right) + \varepsilon.\end{aligned}$$

While $l \to \infty$ and from the arbitrariness of ε, we have

$$\lim_{l\to\infty}\limsup_{n\to\infty} \mathbb{P}\left(|Y^{(2)}_{l,n}| > r\right) = 0. \tag{2.12}$$

By the similar proof as (2.12), we can get

$$\lim_{M\to\infty}\limsup_{l\to\infty}\limsup_{n\to\infty} \mathbb{P}\left(|Z^{(1)}_{l,n}| > M\right) = 0. \tag{2.13}$$

Furthermore, for any $r > 0$, we have

$$\begin{aligned}\mathbb{P}\left(|Y^{(1)}_{l,n}| > r\right) =& \mathbb{P}\left(\left|\frac{\eta_{\nu_n} - \eta_{\mu_{l,n}}}{\sqrt{n\mu_l}}\right| > r\right) \\ \le&\mathbb{P}\left(\left|\frac{\eta_{\nu_n} - \eta_{\mu_{l,n}}}{\sqrt{n\mu_l}}\right| > r, \left|\frac{\nu_n}{n} - \nu\right| < \frac{1}{2^l}, \left|\frac{\mu_{l,n}}{n} - \mu_l\right| < \frac{1}{2^l}\right) \\ &+ \mathbb{P}\left(\left|\frac{\nu_n}{n} - \nu\right| \ge \frac{1}{2^l}\right) + \mathbb{P}\left(\left|\frac{\mu_{l,n}}{n} - \mu_l\right| \ge \frac{1}{2^l}\right),\end{aligned}$$

which yields

$$
\begin{aligned}
&\limsup_{n\to\infty}\mathbb{P}\left(|Y^{(1)}_{l,n}|>r\right)\\
&\quad\le \limsup_{n\to\infty}\mathbb{P}\left(\left|\frac{\eta_{\nu_n}-\eta_{\mu_{l,n}}}{\sqrt{n\mu_l}}\right|>r,\left|\frac{\nu_n}{n}-\nu\right|<\frac{1}{2^l},\left|\frac{\mu_{l,n}}{n}-\mu_l\right|<\frac{1}{2^l}\right)\\
&\quad\le \limsup_{n\to\infty}\mathbb{P}\left(\max_{\substack{|in^{-1}-\nu|<2^{-l}\\ |jn^{-1}-\mu_l|<2^{-l}}}\left|\frac{\eta_i-\eta_j}{\sqrt{n\mu_l}}\right|>r\right).
\end{aligned}
$$

For any $\varepsilon>0$ and for all l large enough such that

$$\mathbb{P}\left(\nu<\frac{l-1}{2^l}\text{ or }\nu\ge l\right)<\varepsilon,$$

we have

$$
\begin{aligned}
&\mathbb{P}\left(\max_{\substack{|in^{-1}-\nu|<2^{-l}\\ |jn^{-1}-\mu_l|<2^{-l}}}\left|\frac{\eta_i-\eta_j}{\sqrt{n\mu_l}}\right|>r\right)\\
&\quad\le \varepsilon+\sum_{k=l}^{l2^l}\mathbb{P}\left(\max_{\substack{|in^{-1}-\nu|<2^{-l}\\ |jn^{-1}-\nu|<2^{-l+1}}}\left|\frac{\eta_i-\eta_j}{\sqrt{n\mu_l}}\right|>r,\frac{k-1}{2^l}\le\nu<\frac{k}{2^l}\right)\\
&\quad\le \varepsilon+2\sum_{k=l}^{l2^l}\mathbb{P}\left(\max_{|\frac{i}{n}-\frac{k}{2^l}|<\frac{3}{2^l}}\left|\frac{\eta_i-\eta_{[n(k-3)2^{-l}]}}{\sqrt{\frac{nk}{2^l}}}\right|>\frac{r}{2},\frac{k-1}{2^l}\le\nu<\frac{k}{2^l}\right)\\
&\quad\le \varepsilon+2\sum_{k=l}^{l2^l}\mathbb{P}\left(A_{n,l,k}\middle|B_{l,k}\right)\mathbb{P}(B_{l,k})
\end{aligned}
$$

where

$$A_{n,l,k}:=\left\{\max_{|\frac{i}{n}-\frac{k}{2^l}|<\frac{3}{2^l}}\left|\frac{\eta_i-\eta_{[n(k-3)2^{-l}]}}{\sqrt{\frac{nk}{2^l}}}\right|>\frac{r}{2}\right\}\quad\text{and}\quad B_{l,k}:=\left\{\frac{k-1}{2^l}\le\nu<\frac{k}{2^l}\right\}.$$

By the similar proof as Lemma 2.3, for any fixed l,k, we can obtain that the events $\{A_{n,l,k};n\ge1\}$ are $\mathbb{P}$-mixing. Hence we get, from Lemma 2.1,

$$\lim_{n\to\infty}\mathbb{P}\left(A_{n,l,k}\middle|B_{l,k}\right)=\lim_{n\to\infty}\mathbb{P}\left(A_{n,l,k}\right)\le\frac{C}{k},$$

which implies

$$\lim_{l\to\infty}\lim_{n\to\infty}\sum_{k=l}^{l2^l}\mathbb{P}\left(A_{n,l,k}\middle|B_{l,k}\right)\mathbb{P}(B_{l,k})=0.$$

From the above discussions and the arbitrariness of ε, we have

$$\lim_{l\to\infty}\lim_{n\to\infty}\mathbb{P}\left(|Y^{(1)}_{l,n}|>r\right)=0. \tag{2.14}$$

From (2.10)-(2.14) and by using Lemma 2.4, the desired results can be obtained. □

Remark 2.1. *Shang [32] and Belloni [3] considered respectively the central limit theorems for randomly indexed m-dependent random variables based on the cases that ν is a positive (discrete or general) random variable. But they assume the following additive condition: there exist some $k_0 \geq 0$ and $c > 0$ such that for any $\lambda > 0$ and $n > k_0$, we have*

$$\mathbb{P}\left(\max_{k_0<k_1\leq k_2\leq n}\left|S_{k_2}-S_{k_1}-(k_2-k_1)\nu\right|\geq\varepsilon\right)\leq\frac{cVar(S_n-S_{k_0})}{\varepsilon^2}.$$

In Proposition 2.1 and Proposition 2.2, the above condition is removed.

2.3. Proof of Theorem 1.1

Let

$$c=\sum_{j\in\mathbb{Z}}c_j \quad \text{and} \quad \bar{c}=\sum_{j\in\mathbb{Z}}|c_j|.$$

For $l, k \in \mathbb{N}$, define

$$\begin{aligned}
&a_l=\sum_{j=-l}^{l}c_j,\quad X_{k,l}=\sum_{j=-l}^{l}c_j\xi_{k-j},\\
&\tilde{c}_l=0,\quad \tilde{c}_j=\sum_{i=j+1}^{l}c_i,\quad j=0,1,\cdots,l-1,\\
&\tilde{\tilde{c}}_{-l}=0,\quad \tilde{\tilde{c}}_j=\sum_{i=-l}^{j-1}c_i,\quad j=-l+1,-l+2,\cdots,0,\\
&\tilde{\xi}_k=\sum_{j=0}^{l}\tilde{c}_j\xi_{k-j},\quad \tilde{\tilde{\xi}}_k=\sum_{j=-l}^{0}\tilde{\tilde{c}}_j\xi_{k-j}.
\end{aligned}$$

According to the above notations, it is clear that

$$\begin{aligned}
X_{k,l}=&\left(\sum_{j=-l}^{l}c_j\right)\xi_k-\left(\sum_{j=1}^{l}c_j\right)\xi_k+\sum_{j=1}^{l}c_j\xi_{k-j}-\left(\sum_{j=-l}^{-1}c_j\right)\xi_k+\sum_{j=-l}^{-1}c_j\xi_{k-j}\\
=&\left(\sum_{j=-l}^{l}c_j\right)\xi_k-\tilde{c}_0\xi_k+\sum_{j=1}^{l}(\tilde{c}_{j-1}-\tilde{c}_j)\xi_{k-j}-\tilde{\tilde{c}}_0\xi_k+\sum_{j=-l}^{-1}(\tilde{\tilde{c}}_{j+1}-\tilde{\tilde{c}}_j)\xi_{k-j} \qquad (2.15)\\
=&a_l\xi_k+\tilde{\xi}_{k-1}-\tilde{\xi}_k+\tilde{\tilde{\xi}}_{k+1}-\tilde{\tilde{\xi}}_k.
\end{aligned}$$

Based on the decomposition, we get

$$\begin{aligned}
\sum_{k=1}^{\nu_n}X_k=&\sum_{k=1}^{\nu_n}X_{k,l}+\sum_{k=1}^{\nu_n}\sum_{|j|>l}c_j\xi_{k-j}\\
=&a_l\sum_{k=1}^{\nu_n}\xi_k+\tilde{\xi}_0-\tilde{\xi}_{\nu_n}+\tilde{\tilde{\xi}}_{\nu_n+1}-\tilde{\tilde{\xi}}_1+\sum_{k=1}^{\nu_n}\sum_{|j|>l}c_j\xi_{k-j} \qquad (2.16)\\
=:&M_1+\cdots+M_6.
\end{aligned}$$

From Proposition 2.2, for any $x \in \mathbb{R}$, we have

$$\lim_{l\to\infty}\lim_{n\to\infty}\mathbb{P}\left(a_l\frac{\sum_{k=1}^{\nu_n}\xi_k}{\sqrt{\nu_n}}\leq x\right)=\hat{\Phi}(x),$$

where $\hat{\Phi}(\cdot)$ denotes the distribution of the normal random variable $N(0, c^2\sigma^2)$. Hence it is enough to show

$$\frac{M_i}{\sqrt{\nu_n}} \xrightarrow{\mathbb{P}} 0, \quad \text{for} \;\; i = 2, \cdots, 6.$$

Now we consider the term M_3. For any l and n, $\tilde{\xi}_n$ contains only $\{\xi_n, \xi_{n-1}, \cdots, \xi_{n-l}\}$, so it is easy to prove that $n^{-1/2}\tilde{\xi}_n \xrightarrow{a.s.} 0$. From Lemma 2.5, we know that

$$\nu_n^{-1/2} M_3 = \nu_n^{-1/2}\tilde{\xi}_{\nu_n} \xrightarrow{\mathbb{P}} 0.$$

By the same proof, we have

$$\frac{M_i}{\sqrt{\nu_n}} \xrightarrow{\mathbb{P}} 0, \quad \text{for} \;\; i = 2, \cdots, 5.$$

For any $r > 0$ and $t \in \mathbb{N}$, we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\left(\nu_n^{-1/2} \left| \sum_{k=1}^{\nu_n} \sum_{|j|>l} c_j \xi_{k-j} \right| > r \right) \\
\leq &\lim_{n\to\infty} \mathbb{P}\left(\nu_n^{-1/2} \left| \sum_{k=1}^{\nu_n} \sum_{|j|>l} c_j \xi_{k-j} \right| > r, \left| \frac{\nu_n}{n} - \nu \right| < 2^{-t} \right) \\
\leq &\lim_{n\to\infty} \mathbb{P}\left(\frac{\sum_{|j|>l} |c_j|}{\sqrt{n(\nu - 2^{-t})}} \max_{\{q : |qn^{-1}-\nu| < 2^{-t}\}} \left| \sum_{k=1}^{q} \xi_{k-j} \right| > r, \frac{t-1}{2^t} \leq \nu < t \right) \\
&+ \mathbb{P}\left(\nu \leq \frac{t}{2^t} \right) + \mathbb{P}(\nu > t) \\
\leq &\sum_{i=t}^{t2^t} \lim_{n\to\infty} \mathbb{P}\left(\frac{\sum_{|j|>l} |c_j|}{\sqrt{n(i-2)2^{-t}}} \max_{\frac{i-2}{2^t} \leq \frac{q}{n} \leq \frac{i+1}{2^t}} \left| \sum_{k=1}^{q} \xi_{k-j} \right| > r, \frac{i-1}{2^t} \leq \nu < \frac{i}{2^t} \right) \\
&+ \mathbb{P}\left(\nu \leq \frac{t}{2^t} \right) + \mathbb{P}(\nu \geq t).
\end{aligned} \tag{2.17}$$

For any $\varepsilon > 0$, there exists a positive constant $T > 2$, such that for all $t \geq T$,

$$\mathbb{P}\left(\nu \leq \frac{t}{2^t} \right) + \mathbb{P}(\nu \geq t) < \varepsilon. \tag{2.18}$$

By the similar proof of Proposition 2.2, we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\left(\frac{\sum_{|j|>l} |c_j|}{\sqrt{n(i-2)2^{-t}}} \max_{\frac{i-2}{2^t} \leq \frac{q}{n} \leq \frac{i+1}{2^t}} \left| \sum_{k=1}^{q} \xi_{k-j} \right| > r, \frac{i-1}{2^t} \leq \nu < \frac{i}{2^t} \right) \\
= &\lim_{n\to\infty} \mathbb{P}\left(\frac{\sum_{|j|>l} |c_j|}{\sqrt{n(i-2)2^{-t}}} \max_{\frac{i-2}{2^t} \leq \frac{q}{n} \leq \frac{i+1}{2^t}} \left| \sum_{k=1}^{q} \xi_{k-j} \right| > r \right) \mathbb{P}\left(\frac{i-1}{2^t} \leq \nu < \frac{i}{2^t} \right).
\end{aligned} \tag{2.19}$$

From Lemma 2.1, we get

$$\begin{aligned}
&\mathbb{P}\left(\frac{\sum_{|j|>l} |c_j|}{\sqrt{n(i-2)2^{-t}}} \max_{\frac{i-2}{2^t} \leq \frac{q}{n} \leq \frac{i+1}{2^t}} \left| \sum_{k=1}^{q} \xi_{k-j} \right| > r \right) \\
\leq &\frac{\left(\sum_{|j|>l} |c_j|\right)^2}{r^2 n(i-2)2^{-t}} \mathbb{E} \max_{\frac{i-2}{2^t} \leq \frac{q}{n} \leq \frac{i+1}{2^t}} \left| \sum_{k=1}^{q} \xi_{k-j} \right|^2 \leq \frac{C}{i-2} \left(\sum_{|j|>l} |c_j| \right)^2,
\end{aligned} \tag{2.20}$$

where C is a positive constant. From (2.17) to (2.20), we have

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{P}\left(\nu_n^{-1/2}\left|\sum_{k=1}^{\nu_n}\sum_{|j|>l}c_j\xi_{k-j}\right|>r\right)\\
&\le\left(\sum_{|j|>l}|c_j|\right)^2\sum_{i=t}^{t2^t}\frac{C}{i-2}\mathbb{P}\left(\frac{i-1}{2^t}\le\nu<\frac{i}{2^t}\right)+\varepsilon\\
&\le\left(\sum_{|j|>l}|c_j|\right)^2\frac{C}{T-2}\mathbb{P}\left(\frac{t-1}{2^t}\le\nu<t\right)+\varepsilon.
\end{aligned}$$

Letting $l\to\infty$, we can get

$$\nu_n^{-1/2}M_6\xrightarrow{\mathbb{P}}0.$$

So we finish the proof of Theorem 1.1.

3. Conclusion

In the paper, we establish the central limit theorem for randomly indexed linear process driven by m-dependent sequence. The main methods are to use the decomposition (2.16) for the sums S_{ν_n} and the the central limit theorems for randomly indexed m-dependent random variables (see Proposition 2.2). The randomly indexed linear processes driven by the independent random variables and m-dependent sequence have been discussed, so a more general and worth studying question could be: whether the central limit theorem with random indexed linear process driven by other dependent sequence holds?

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References

[1] T. W. Anderson, The statistical analysis of time series, John Wiley & Sons, Inc., New York-London-Sydney, 1971.
[2] F. J. Anscombe, Large-sample theory of sequential estimation, Proceedings of the Cambridge Philosophical Society 48 (1952) 600-607.
[3] S. Belloni, An extension of central limit theorem for randomly indexed m-dependent random variables, Filomat 31 (2017) 4369-4377.
[4] J. R. Blum, D. L. Hanson and J. I. Rosenblatt, On the central limit theorem for the sum of a random number of independent random variables, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 1 (1963) 389-393.
[5] Z. S. Dong, X. L. Tan, and X. Y. Yang, Moderate deviation principles for moving average processes of real stationary sequences, Statistics & Probability Letters 74 (2005) 139-150.
[6] H. Djellout and A. Guillin, Large and moderate deviations for moving average processes, Annales de la Faculté des Sciences de Toulouse. Mathématiques. Série 6. 6 (2001) 23-31.
[7] H. Djellout, A. Guillin, and L. Wu, Moderate deviations of empirical periodogram and non-linear functionals of moving average processes, Annales de l'Institut Henri Poincaré Probabilités et Statistiques 42 (2006) 393-416.
[8] I. Fakhre-Zakeri and J. Farshidi, A central limit theorem with random indices for stationary linear processes, Statistics & Probability Letters 17 (1993) 91-95.
[9] S. B. Fotopoulos and M. C. Wang, Asymptotic normality for random sums of linear processes. ournal of Statistical Planning and Inference 39 (1993) 349–360.
[10] P. Hall and C. C. Heyde, Martingale limit theory and its application, Academic Press, Inc., New York-London, 1980.
[11] E. J. Hannan, Multiple time series, John Wiley and Sons, Inc., New York-London-Sydney, 1970.
[12] T. F. Jiang, M. B. Rao, and X. C. Wang, Large deviations for moving average processes, Stochastic Processes and their Applications 59 (1995) 309-320.
[13] S. Lee, Sequential estimation for the autocorrelations of linear processes, Annals of Statistics 24 (1996) 2233-2249.

[14] Y. X. Li and L. X. Zhang, Precise asymptotics in the law of the iterated logarithm of moving-average processes, Acta Mathematica Sinica (English Series) 22 (2006) 143-156.
[15] Z. Y. Lin and D. G. Li, Functional limit theorem for moving average processes generated by dependent random variables. Progress in Natural Science. English Edition 16 (2006) 266-273.
[16] Y. Miao, Large deviation principles for moving average processes of real stationary sequences, Acta Applicandae Mathematicae 106 (2009) 177-184.
[17] Y. Miao, Asymptotic normality of autoregressive processes, Acta Applicandae Mathematicae 110 (2010) 1077-1085.
[18] Y. Miao, The discounted large deviation principle for autoregressive processes, Communications in Statistics. Theory and Methods 40 (2011) 2455-2461.
[19] Y. Miao, T. Y. Xue and T. Du, The discounted Berry-Esséen analogue for autoregressive processes, Communications in Statistics. Theory and Methods 42 (2013) 2684-2693.
[20] Y. Miao, L. Ge and S. F. Xu, Central limit theorems for moving average processes, Lithuanian Mathematical Journal 53 (2013) 80-90.
[21] Y. Miao, W. Q. Geng, N. Li and Q. Xiao, Moderate deviation principle for the error variance estimator in linear models, Communications in Statistics. Theory and Methods 43 (2014) 4215-4222.
[22] Y. Miao and S. Shen, Moderate deviation principle for autoregressive processes, Journal of Multivariate Analysis 100 (2009) 1952-1961.
[23] Y. Miao, Y. L. Wang and G. Y. Yang, Moderate deviation principles for empirical covariance in the neighbourhood of the unit root, Scandinavian Journal of Statistics 42 (2015) 234-255.
[24] Y. Miao and G. Y. Yang, A moderate deviation principle for m-dependent random variables with unbounded m, Acta Applicandae Mathematicae 104 (2008) 191-199.
[25] M. Peligrad and S. Utev, Central limit theorem for linear processes, Annals of Probability 25 (1997) 443-456.
[26] M. Peligrad and S. Utev, Central limit theorem for stationary linear processes, Annals of Probability 34 (2006) 1608-1622.
[27] A. Rényi, On mixing sequences of sets, Acta Mathematica. Academiae Scientiarum Hungaricae 9 (1958) 215-228.
[28] A. Rényi, On the central limit theorem for the sum of a random number of independent random variables, Acta Mathematica. Academiae Scientiarum Hungaricae 11 (1960) 97–102.
[29] W. Richter, Limit theorems for sequences of random variables with sequences of random indeces, Theory of Probability and its Applications 10 (1965) 74–84.
[30] Y. L. Shang, A martingale central limit theorem with random indices, Azerbaijan Journal of Mathematics 1 (2011) 109-114.
[31] Y. L. Shang, A note on the central limit theorems for dependent random variables, ISRN Probability and Statistics, (2012), Article ID 192427.
[32] Y. L. Shang, A central limit theorem for randomly indexed m-dependent random variables, Filomat 26 (2012) 713-717.
[33] M. Tyran-Kamińska, Functional limit theorems for linear processes in the domain of attraction of stable laws, Statistics & Probability Letters 80 (2010) 975-981.
[34] L. M. Wu, On large deviations for moving average processes, Probability, finance and insurance, 15-49, World Sci. Publ., River Edge, NJ, 2004.
[35] R. Yokoyama, An iterated logarithm result for partial sums of a stationary linear process, Yokohama Mathematical Journal 31 (1983) 139-148.
[36] R. Yokoyama, An iterated logarithm theorem for some stationary linear processes, Yokohama Mathematical Journal 40 (1993) 143-148.