Filomat 33:12 (2019), 3901–3912
https://doi.org/10.2298/FIL1912901C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On a Projective Conformal Semi-Symmetric Connection

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Abstract. In this paper we propose a projective conformal semi-symmetric connection and study its geometric and physical properties. The Schur's theorem of this connection is obtained.

1. Introduction

The concept of a semi-symmetric connection in a Riemannian manifold was firstly introduced in [9]. K.Yano in [21] introduced firstly and investigated a semi-symmetric metric connection, and T. Imai in [15] studied its properties. Afterwards some kinds of semi-symmetric connections were studied in [10, 20, 24]. A semi-symmetric connection that is projectively equivalent to the Levi-Civita connection was defined as a projective semi-symmetric connection and some of its properties were investigated ([22, 23]). In [13, 14] a projective conformal semi-symmetric connection was considered. In [7] a curvature copy problem of a non-metric connection was considered. In [16] the Amari-Chentsov connection as a geometrical structure of a statistical manifold was regarded and in [18] a conjugate symmetry condition of the statistical manifold was regarded and in [18] a conjugate symmetry condition of the statistical manifold discovered. In [1, 2, 4, 5] and [17] the properties of a semi-symmetric non- metric connection were investigated. In [6, 8], a physical model of a semi-symmetric non-metric connection was studied. In [10] the Schur's theorem of the Levi-Civita connection was considered and in [11] the Schur's theorem of the semi-symmetric non-metric connection was studied. Recently, Han, Fu and Zhao [12] studied a projective connection and its properties on Sub-Riemannian manifolds.

In this paper we will propose a new projective conformal semi-symmetric connection and study its properties.

The paper is organized as follows. Section 1 introduces one type of semi-symmetric connection and studied its properties. Section 2 studies the property of projective semi-symmetric connection and Section 3 studies conformal semi-symmetric connection. Finally in Section 4 a new projective conformal semi-symmetric connection is defined and its properties and conjugate symmetry condition are studied. And we study the Schur's theorem of the projective conformal semi-symmetric connection.

2010 *Mathematics Subject Classification*. Primary 58E20; Secondary 53C21
Keywords. Semi-symmetric connection; Projective conformal semi-symmetric connection; Conjugate symmetry; Constant curvature.
Received: 18 April 2018; Accepted: 19 July 2018
Communicated by Mića S. Stanković
Supported by NNSF of China(No.11871275; No.11371194) and by Postgraduate Research & Practice Innovation Program of Jiangsu Province and by Found for the Doctoral Program of Anhui Normal University (901-751841).
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2. Semi-symmetric Connections

Definition 2.1. *In a Riemannian manifold (M, g), a connection $\overset{t}{\nabla}$ is called a semi-symmetric connection if it satisfies the relation*

$$\overset{t}{\nabla}_Z g(X,Y) = (1-t)\pi(Z)g(X,Y) - \frac{1-t}{2}\pi(X)g(Y,Z) - \frac{1-t}{2}\pi(Y)g(X,Z), \quad (2.1)$$

$$T(X,Y) = \pi(Y)X - \pi(X)Y \quad (2.2)$$

for any $X, Y, Z \in \mathcal{T}(M)$ and a 1-form π.

A semi-symmetric connection $\overset{t}{\nabla}$ is a non-metric connection and it is can be written as

$$\overset{t}{\nabla}_X Y = \overset{0}{\nabla}_X Y + \frac{1+t}{2}\pi(Y)X - \frac{1-t}{2}\pi(X)Y, \quad (2.3)$$

for any $X, Y \in \mathcal{T}(M)$, where $\overset{0}{\nabla}$ is the Levi-Civita connection.

The coefficient of the semi-symmetric connection $\overset{t}{\nabla}$ is

$$\overset{t}{\Gamma}{}^k_{ij} = \left\{ {k \atop ij} \right\} + \frac{1+t}{2}\pi_j\delta^k_i - \frac{1-t}{2}\pi_i\delta^k_j, \quad (2.4)$$

where $\{ {k \atop ij} \}$ is the coefficient of the Levi-Civita connection $\overset{0}{\nabla}$ and π_i is a component of the 1-form π(This is called a semi-symmetric component). Using expression (2.4), the curvature tensor $\overset{t}{R}{}^l{}_{ijk}$ of $\overset{t}{\nabla}$ is

$$\overset{t}{R}{}^l{}_{ijk} = K^l_{ijk} + \delta^l_j\tau_{jk} - \delta^l_i\tau_{jk} - \delta^l_k\beta_{ij}, \quad (2.5)$$

where K^l_{ijk} is the curvature tensor of the Levi-Civita connection and $\tau_{ik} = \frac{1+t}{2}(\overset{0}{\nabla}_i\pi_k - \frac{1+t}{2}\pi_i\pi_k)$, $\beta_{ij} = \frac{1-t}{2}(\overset{0}{\nabla}_i\pi_j - \overset{0}{\nabla}_j\pi_i)$.

The coefficient of dual connection $\overset{t}{\nabla}{}^*$ of the connection $\overset{t}{\nabla}$ is

$$\overset{t}{\Gamma}{}^{*k}_{ij} = \left\{ {k \atop ij} \right\} + \frac{1-t}{2}\pi_j\delta^k_i - \frac{1+t}{2}\pi^k g_{ij},$$

and the curvature tensor of $\overset{t}{\nabla}{}^*$ is

$$\overset{t}{R}{}^{*l}{}_{ijk} = K^l_{ijk} + g_{ik}\tau^l_j - g_{jk}\tau^l_i + \delta^l_k\beta_{ij}, \quad (2.6)$$

The coefficient of mutual connection $\overset{tm}{\nabla}$ of the connection $\overset{t}{\nabla}$ is

$$\overset{tm}{\Gamma}{}^k{}_{ij} = \left\{ {k \atop ij} \right\} + \frac{1+t}{2}\pi_i\delta^k_j - \frac{1-t}{2}\pi^j\delta^k_i,$$

and the curvature tensor of $\overset{tm}{\nabla}$ is

$$\overset{tm}{R}{}^l{}_{ijk} = K^l_{ijk} + \delta^l_i\bar{\tau}jk - \delta^l_j\bar{\tau}_{ik} + \delta^l_k\bar{\beta}_{ij}, \quad (2.7)$$

where $\bar{\tau}_{ik} = \frac{1-t}{2}(\overset{0}{\nabla}_i\pi_k + \frac{1-t}{2}\pi_i\pi_k)$, $\bar{\beta}_{ij} = \frac{1+t}{2}(\overset{0}{\nabla}_i\pi_j - \overset{0}{\nabla}_j\pi_i)$.

Lemma 2.2. *In the Riemannian manifold $(M, g)(dimM > 3)$, the tensor*

$$\begin{aligned}\overset{t}{W}{}^{l}{}_{ijk} &= \overset{t}{R}{}^{l}{}_{ijk} - \frac{1}{n-1}(\delta^l_i \overset{t}{R}_{jk} - \delta^l_j \overset{t}{R}_{ik}) \\ &+ \frac{t-1}{(n-1)[(n+1)t+n-3]}\Big[\delta^l_i(\overset{t}{R}_{jk} - \overset{t}{R}_{kj}) + \delta^l_j(\overset{t}{R}_{ik} - \overset{t}{R}_{ki}) + (n-1)\delta^l_k(\overset{t}{R}_{ji} - \overset{t}{R}_{ij})\Big],\end{aligned} \tag{2.8}$$

is an invariant under the connection transformation $\overset{0}{\nabla} \longrightarrow \overset{t}{\nabla}$.

Proof. By using contraction of the indices i and l of (2.5), we find

$$\overset{*}{R}_{jk} = K_{jk} - (n-1)\tau_{jk} + \beta_{jk}, \tag{2.9}$$

Alternating the indices j and k, using $\tau_{jk} - \tau_{kj} = \frac{1+t}{1-t}\beta_{jk}$, we find

$$\beta_{jk} = \frac{1-t}{(n+1)t+n-3}\Big[(K_{jk} - K_{kj}) - (\overset{t}{R}_{jk} - \overset{t}{R}_{kj})\Big], \tag{2.10}$$

Substituting (2.10) into (2.9), we have

$$\tau_{jk} = \frac{1}{n-1}\Big\{(K_{jk} - \overset{t}{R}_{jk}) + \frac{t-1}{(n+1)t+n-3}\Big[(K_{jk} - K_{kj}) - (\overset{t}{R}_{jk} - \overset{t}{R}_{kj})\Big]\Big\}, \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.5), by a direct computation, we arrive at

$$\overset{t}{W}{}^{l}_{ijk} = \overset{0}{W}{}^{l}_{ijk} \tag{2.12}$$

where the tensor $\overset{0}{W}{}^{l}_{ijk}$ is

$$\begin{aligned}\overset{0}{W}{}^{l}{}_{ijk} &= K^l_{ijk} - \frac{1}{n-1}(\delta^l_i K_{jk} - \delta^l_j K_{ik}) \\ &+ \frac{t-1}{(n-1)[(n+1)t+n-3]}\Big[\delta^l_i(K_{jk} - K_{kj}) + \delta^l_j(K_{ik} - K_{ki}) + (n-1)\delta^l_k(K_{ji} - K_{ij})\Big] \\ &= K^l_{ijk} - \frac{1}{n-1}(\delta^l_i K_{jk} - \delta^l_j K_{ik}).\end{aligned}$$

This ends the proof of Lemma 2.2. □

Remark 2.3. *The tensor $\overset{0}{W}{}^{l}_{ijk}$ is a Weyl projective curvature tensor of the connection $\overset{0}{\nabla}$.*

Using Lemma 2.2, the following theorem is proved without difficulty.

Theorem 2.4. *If a Riemannian metric admits the semi-symmetric connection $\overset{t}{\nabla}$ with vanishing curveture tensor, then the Riemannian metric is projective flat.*

Remark 2.5. *In [21], it is proved that if a Riemannian metric admits the semi-symmetric metric connection with vanishing curvature tensor, then the Riemannian metric is conformal flat.*

For the Riemannian manifold, if $R^l_{ijk} = \overset{*}{R}{}^l_{ijk}$, then the connection ∇ is called a conjugate symmetry and if $R_{jk} = \overset{*}{R}_{jk}$, then the connection ∇ is called a conjugate Ricci symmetry and if $\overset{(s)}{R}_{jk} = \overset{*(s)}{R}_{jk}$, then the connection ∇ is a conjugate semi-Ricci symmetry (see [20]).

Lemma 2.6. *In the Riemannian manifold (M, g), the tensor $\overset{t}{V}{}^l{}_{ijk}$ of the connection $\overset{t}{\nabla}$*

$$\begin{aligned}\overset{t}{V}{}^l{}_{ijk} &= \overset{t}{R}{}^l{}_{ijk} - \frac{1}{n-1}\Big(\delta^l_i \overset{t}{R}_{jk} - \delta^l_j \overset{t}{R}_{ik} + g_{ik}\overset{t}{R}{}^l_j - g_{jk}\overset{t}{R}{}^l_i\Big)\\ &- \frac{2(1-t)}{n[(n+4)t+(n-4)]}\Big[\delta^l_i(\overset{t}{R}_{jk} - \overset{t}{R}_{kj}) - \delta^l_j(\overset{t}{R}_{ik} - \overset{t}{R}_{ki}) + g_{ik}(\overset{t}{R}{}^l_j - \overset{t}{R}{}^l{}_{.j}) - g_{jk}(\overset{t}{R}{}^l_i - \overset{t}{R}{}^l{}_{.i})\\ &+ n\delta^l_k(\overset{t}{R}_{ij} - \overset{t}{R}_{ji})\Big]\end{aligned} \tag{2.13}$$

is an invariant under the connection transformation $\overset{t}{\nabla} \longrightarrow \overset{t*}{\nabla}$, where $\overset{t}{R}{}^l{}_{.j} = \overset{t}{R}_{sj}g^{sl}$.

Proof. From (2.5) and (2.6), we get

$$\overset{t*}{R}{}^l{}_{ijk} = \overset{*}{R}{}^l{}_{ijk} + \delta^l_i\tau_{jk} - \delta^l_j\tau_{ik} + g_{ik}\tau^l_j - g_{jk}\tau^l_i + 2\delta^l_k\beta_{ij} \tag{2.14}$$

Contracting the indices i and l of (2.14), then we find

$$\overset{t*}{R}_{jk} = \overset{*}{R}_{jk} + n\tau_{jk} - g_{jk}\tau^i_i - 2\beta_{jk} \tag{2.15}$$

Alternating the indices j and k of (2.15) and using $\tau_{jk} - \tau_{kj} = \frac{1+t}{1-t}\beta_{jk}$, we have

$$\beta_{jk} = \frac{1-t}{(n+4)t+n-4}\Big[(\overset{t*}{R}_{jk} - \overset{t*}{R}_{kj}) - (R_{jk} - R_{kj})\Big] \tag{2.16}$$

Substituting (2.16) into (2.15), we obtain

$$\tau_{jk} = \frac{1}{n}\Big\{\overset{t*}{R}_{jk} - \overset{t}{R}_{jk} + g_{jk}\tau^i_i + \frac{2(1-t)}{(n+4)t+n-4}\Big[(\overset{t*}{R}_{jk} - \overset{t*}{R}_{kj}) - (R_{jk} - R_{kj})\Big]\Big\} \tag{2.17}$$

Substituting (2.16) and (2.17) into (2.14), by a direct computation, we have

$$\overset{*}{V}{}^l{}_{ijk} = \overset{t*}{V}{}^l{}_{ijk} \tag{2.18}$$

where the tensor $\overset{t*}{V}{}^l{}_{ijk}$ is given as

$$\begin{aligned}\overset{t*}{V}{}^l{}_{ijk} &= \overset{t*}{R}{}^l{}_{ijk} - \frac{1}{n-1}\Big(\delta^l_i \overset{t*}{R}_{jk} - \delta^l_j \overset{t*}{R}_{ik} + g_{ik}\overset{t*}{R}{}^l_j - g_{jk}\overset{t*}{R}{}^l_i\Big)\\ &- \frac{2(1-t)}{n[(n+4)t+(n-4)]}\Big[\delta^l_i(\overset{t*}{R}_{jk} - \overset{t*}{R}_{kj}) - \delta^l_j(\overset{t*}{R}_{ik} - \overset{t*}{R}_{ki}) + g_{ik}(\overset{t*}{R}{}^l_j - \overset{t*}{R}{}^l{}_{.j}) - g_{jk}(\overset{t*}{R}{}^l_i - \overset{t*}{R}{}^l{}_{.i})\\ &+ n\delta^l_k(\overset{t*}{R}_{ij} - \overset{t*}{R}_{ji})\Big]\end{aligned} \tag{2.19}$$

This completes the proof of Lemma 2.6. □

Using Lemma 2.6, it is not hard to show that the following theorem is tenable.

Theorem 2.7. *A manifold associated with a semi-symmetric connection $\overset{t}{\nabla}$ is conjugate symmetric if and only if it is conjugate Ricci symmetric.*

Lemma 2.8. *In the Riemannian manifold $(M, g)(dim M > 3)$, the tensor*

$$\begin{aligned}\overset{t}{W}{}^l{}_{ijk} &= \overset{t}{R}{}^l{}_{ijk} - \frac{1}{n-1}\Big(\delta^l_i \overset{t}{R}_{jk} - \delta^l_j \overset{t}{R}_{ik}\Big) \\ &- \frac{t-1}{(n-1)(n-3)}\Big[\delta^l_i(\overset{t}{R}_{jk} - \overset{t}{R}_{kj}) + \delta^l_j(\overset{t}{R}_{ik} - \overset{t}{R}_{ki}) + (n-1)\delta^l_k(\overset{t}{R}_{ij} - \overset{t}{R}_{ji})\Big]\end{aligned} \tag{2.20}$$

is an invariant under the transformation of the connection $\overset{t}{\nabla} \longrightarrow \overset{tm}{\nabla}$.

Proof. Let $\alpha_{ik} = \tau_{ik}\bar{\tau}_{ik}$ and $\pi_{ik} = \beta_{ik} + \bar{\beta}_{ik}$. From the expressions (2.5) and (2.6), we find

$$\overset{tm}{R}{}^l{}_{ijk} = \overset{t}{R}{}^l{}_{ijk} + \delta^l_i\alpha_{jk} - \delta^l_j\alpha_{ik} + \delta^l_k\pi_{ij} \tag{2.21}$$

Contracting the indices i and l of (2.21), then there holds the following

$$\overset{tm}{R}{}^l{}_{jk} = \overset{t}{R}{}^l{}_{jk} + (n-1)\delta_{jk} - \pi_{jk} \tag{2.22}$$

Alternating the indices j and k of this expression, using $\alpha_{jk} - \alpha_{kj} = \pi_{jk}$, we then obtain

$$\pi_{ij} = \frac{1}{n-3}\Big[(\overset{tm}{R}_{ij} - \overset{tm}{R}_{ji}) - (\overset{t}{R}_{ij} - \overset{t}{R}_{ji})\Big] \tag{2.23}$$

From (2.23) and (2.22), we find

$$\alpha_{ij} = \frac{1}{n-1}\Big\{\overset{tm}{R}_{ij} - \overset{t}{R}_{ij}) + \frac{1}{n-3}\Big[(\overset{tm}{R}_{ij} - \overset{tm}{R}_{ji}) - (\overset{t}{R}_{ij} - \overset{t}{R}_{ji})\Big]\Big\} \tag{2.24}$$

Substituting (2.23) and (2.24) into (2.21), by a direct computation, we have

$$\overset{t}{W}{}^l{}_{ijk} = \overset{tm}{W}{}^l{}_{ijk},$$

where the tensor $\overset{tm}{W}{}^l{}_{ijk}$ is written as

$$\begin{aligned}\overset{tm}{W}{}^l{}_{ijk} &= \overset{tm}{R}{}^l{}_{ijk} - \frac{1}{n-1}\Big(\delta^l_i \overset{tm}{R}_{jk} - \delta^l_j \overset{tm}{R}_{ik}\Big) \\ &- \frac{t-1}{(n-1)(n-3)}\Big[\delta^l_i(\overset{tm}{R}_{jk} - \overset{tm}{R}_{kj}) + \delta^l_j(\overset{tm}{R}_{ik} - \overset{tm}{R}_{ki}) + (n-1)\delta^l_k(\overset{tm}{R}_{ij} - \overset{tm}{R}_{ji})\Big]\end{aligned} \tag{2.25}$$

This completes the proof of Lemma 2.8. □

Remark 2.9. *The tensor $\overset{t}{W}{}^l{}_{ijk}$ is independent of the parameter t. This tensor is called a generalized Weyl projective curvature tensor with respect to connection $\overset{t}{\nabla}$.*

Lemma 2.8 implies that the following theoremark is true.

Theorem 2.10. *In order that $\overset{t}{R}{}^l{}_{ijk} = \overset{tm}{R}{}^l{}_{ijk}$, it is necessary and sufficient that $\overset{t}{R}_{jk} = \overset{tm}{R}_{jk}$.*

Theorem 2.11. *In a Riemannian manifold (M, g), the Weyl conformal curvature tensors $\overset{t}{C}{}^l{}_{ijk}, \overset{t*}{C}{}^l{}_{ijk}$ w. r. t. connections $\overset{t}{\nabla}, \overset{t*}{\nabla}$ are invariants under the transformations of $\overset{0}{\nabla} \longrightarrow \overset{t}{\nabla}$ and $\overset{0}{\nabla} \longrightarrow \overset{t*}{\nabla}$. Furthermore, there holds*

$$\overset{t}{C}{}^l{}_{ijk} + \overset{t*}{C}{}^l{}_{ijk} = 2\overset{0}{C}{}^l{}_{ijk} \tag{2.26}$$

where $\overset{0}{C}{}^{l}{}_{ijk}$ *is the Weyl conformal curvature tensor with respect to Levi-Civita connection* $\overset{0}{\nabla}$*, namely,*

$$\overset{0}{C}{}^{l}{}_{ijk} = K^{l}{}_{ijk} - \frac{1}{n-2}(\delta^l_j K_{jk} - \delta^l_j K_{ik} + g_{jk}K^l_i - g_{ik}K^l_j) - \frac{K}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_i g_{jk}) \tag{2.27}$$

Proof. Adding (2.5) and (2.6), we have

$$\overset{t}{R}{}^{l}{}_{ijk} + \overset{t*}{R}{}^{l}{}_{ijk} = 2K^{l}{}_{ijk} + \delta^l_j \tau_{ik} - \delta^l_i \tau_{jk} - g_{jk}\tau^l_i + g_{ik}\tau^l_j \tag{2.28}$$

Contracting the indices i and l for (2.28), then we find

$$\overset{t}{R}_{jk} + \overset{t*}{R}_{jk} = 2K_{jk} - (n-2)\tau_{jk} - g_{jk}\tau^i_i \tag{2.29}$$

Multiplying both sides of the expression (2.29) by g^{jk}, we obtain

$$\overset{t}{R} + \overset{t*}{R} = 2K - 2(n-1)\tau^i_i$$

From this expression we find

$$\tau^i_i = \frac{1}{2(n-1)}\Big[2K - (\overset{t}{R} + \overset{t*}{R})\Big] \tag{2.30}$$

Using this expression and from the expression (2.29), we find

$$\tau_{ik} = \frac{1}{n-2}\Big\{2K_{ik} - (\overset{t}{R}_{ik} + \overset{t*}{R}_{ik}) - \frac{1}{2(n-1)}[2K - (\overset{t}{R} + \overset{t*}{R})]g_{ik}\Big\} \tag{2.31}$$

Substituting this expression into the expression (2.28), by a direct computation and using (2.27), we have the formula (2.26). □

Theorem 2.12. *In a Riemannian manifold* (M, g)*, the Weyl projective curvature tensors* $\overset{t}{W}{}^{l}{}_{ijk}, \overset{tm}{W}{}^{l}{}_{ijk}$ *w. r. t. connections* $\overset{t}{\nabla}, \overset{tm}{\nabla}$ *are invariants under the transformations of* $\overset{0}{\nabla} \longrightarrow \overset{t}{\nabla}$ *and* $\overset{0}{\nabla} \longrightarrow \overset{tm}{\nabla}$*. Furthermore, there holds*

$$\overset{t}{W}{}^{l}{}_{ijk} + \overset{tm}{W}{}^{l}{}_{ijk} = 2\overset{0}{W}{}^{l}{}_{ijk} \tag{2.32}$$

where $\overset{0}{W}{}^{l}{}_{ijk}$ *is the Weyl projective curvature tensor with respect to Levi-Civita connection* $\overset{0}{\nabla}$*,*

Proof. Adding the expressions (2.5) and (2.7), since π is a closed form we find

$$\overset{t}{R}{}^{l}{}_{ijk} + \overset{tm}{R}{}^{l}{}_{ijk} = 2K^{l}{}_{ijk} - \delta^l_j \gamma_{ik} + \delta^l_i \gamma_{jk} \tag{2.33}$$

where $\gamma_{jk} = \bar{\tau}_{jk} - \tau_{jk}$.

Contracting the indices i and l of (2.33), then we find

$$\overset{t}{R}_{jk} + \overset{tm}{R}_{jk} = 2K_{jk} + (n-1)\gamma_{jk} \tag{2.34}$$

From the expression (2.34), we have

$$\gamma_{jk} = \frac{1}{n-1}(\overset{t}{R}_{jk} + \overset{tm}{R}_{jk} - 2K_{jk}) \tag{2.35}$$

Substituting (2.35) into (2.33), by a direct computation, then we have the expression (2.32). □

3. Projective Semi-symmetric Connections

Definition 3.1. *A connection $\overset{p}{\nabla}$ is called a projective semi-symmetric connection if $\overset{p}{\nabla}$ is projective equivalent to $\overset{t}{\nabla}$ in a Riemannian manifold.*

The coefficient of the projective semi-symmetric connection $\overset{p}{\nabla}$ is

$$\overset{p}{\Gamma}{}^k{}_{ij} = \left\{{}^{k}_{ij}\right\} + (\psi_j + \frac{1+t}{2}\pi_j)\delta_i^k + (\psi_i - \frac{1-t}{2}\pi_i)\delta_j^k, \tag{3.1}$$

where ψ_j is a projective component. The curvature tensor of $\overset{p}{\nabla}$ is

$$\overset{p}{R}{}^l{}_{ijk} = K^l_{ijk} + \delta^l_j\tau_{ik} - \delta^l_i\tau_{jk} + \delta^l_k\beta_{ij}, \tag{3.2}$$

where $\tau_{ik} = \overset{0}{\nabla}_i(\psi_k + \frac{1+t}{2}\pi_k) - (\psi_i + \frac{1+t}{2}\pi_i)(\psi_k + \frac{1+t}{2}\pi_k)$, $\beta_{ij} = \overset{0}{\nabla}_i(\psi_j - \frac{1-t}{2}\pi_j) - \overset{0}{\nabla}_j(\psi_i - \frac{1-t}{2}\pi_i)$. The coefficient of the dual projective connection $\overset{p*}{\nabla}$ with respect to the connection $\overset{p}{\nabla}$ is written as

$$\overset{p*}{\Gamma}{}^k{}_{ij} = \left\{{}^{k}_{ij}\right\} + (\psi_i - \frac{1-t}{2}\pi_i)\delta_j^k - g_{ij}(\psi^k + \frac{1+t}{2}\pi^k),$$

and the curvature tensor of $\overset{p*}{\nabla}$ is

$$\overset{p*}{R}{}^l{}_{ijk} = K^l_{ijk} + g_{ik}\tau^l_j - g_{jk}\tau^l_i - \delta^l_k\beta_{ij}. \tag{3.3}$$

The coefficient of the mutual projective connection $\overset{pm}{\nabla}$ with respect to the connection $\overset{p}{\nabla}$ is

$$\overset{pm}{\Gamma}{}^k{}_{ij} = \left\{{}^{k}_{ij}\right\} + (\psi_j - \frac{1-t}{2}\pi_j)\delta_i^k + (\psi_i + \frac{1+t}{2}\pi_i)\delta_j^k,$$

and the curvature tensor of $\overset{pm}{\nabla}$ is

$$\overset{pm}{R}{}^l{}_{ijk} = K^l_{ijk} + \delta^l_j\bar{\tau}_{ik} - \delta^l_i\bar{\tau}_{jk} + \delta^l_k\bar{\beta}_{ij}, \tag{3.4}$$

where $\bar{\tau}_{ik} = \overset{0}{\nabla}_i(\psi_k - \frac{1-t}{2}\pi_k) - (\psi_i - \frac{1-t}{2}\pi_i)(\psi_k - \frac{1-t}{2}\pi_k)$, $\bar{\beta}_{ij} = \overset{0}{\nabla}_i(\psi_j + \frac{1+t}{2}\pi_j) - \overset{0}{\nabla}_j(\psi_i + \frac{1+t}{2}\pi_i)$.

Lemma 3.2. *In a Riemannian manifold $(M, g)(dim M > 2)$, the tensor*

$$\begin{aligned}\overset{p}{W}{}^l{}_{ijk} &= \overset{p}{R}{}^l{}_{ijk} - \frac{1}{n-1}(\delta^l_i\overset{p}{R}_{jk} - \delta^l_j\overset{p}{R}_{ik}) - \frac{1}{(n+1)(n-2)}\Big[\delta^l_i(\overset{p}{R}_{jk} - \overset{p}{R}_{kj}) - \delta^l_j(\overset{p}{R}_{ik} - \overset{p}{R}_{ki}) \\ &- (n-1)\delta^l_k(\overset{p}{R}_{ij} - \overset{p}{R}_{ji})\Big] - \frac{1}{(n-1)(n-2)}\Big[\delta^l_i\overset{p(s)}{R}{}_{jk} - \delta^l_j\overset{p(s)}{R}{}_{ik} + (n+1)\delta^l_k\overset{p(s)}{R}{}_{ji}\Big],\end{aligned} \tag{3.5}$$

is an invariant under the transformation of the connection $\overset{0}{\nabla} \longrightarrow \overset{p}{\nabla}$, and $\overset{p(s)}{R}{}_{jk}$ is the corresponding projective semi-Ricci curvature tensor of $\overset{p}{\nabla}$ defined by $\overset{p(s)}{R}{}_{jk} = \overset{p}{R}_{jkhl}g^{hl}$(see [20]).

Proof. Contracting the indices i and l of the expression (3.2), then we find

$$\overset{p}{R}_{jk} = K_{jk} - (n-1)\tau_{jk} - \beta_{jk}, \tag{3.6}$$

Alternating the indices j and k of this expression, we get

$$\overset{p}{R}_{jk} - \overset{p}{R}_{kj} = K_{jk} - K_{kj} - (n-1)(\tau_{jk} - \tau_{kj}), \tag{3.7}$$

On the hand, one can contract the indices k and l of (3.2) and arrive at

$$\overset{(s)}{R}_{jk} = \overset{0(s)}{R}_{jk} + \tau_{jk} - \tau_{kj} + n\beta_{jk}, \tag{3.8}$$

From the expressions (3.7) and (3.8), we have

$$\beta_{jk} = \frac{1}{(n-1)(n-2)}\Big[(\overset{p}{R}_{jk} - \overset{p}{R}_{kj}) - (K_{jk} - K_{kj}) + (n-1)(\overset{(s)}{R}_{jk} - \overset{0(s)}{R}_{jk})\Big], \tag{3.9}$$

Substituting (3.9) into (3.8), we obtain

$$\begin{aligned} \tau_{jk} &= \frac{1}{n-1}\Big\{K_{jk} - \overset{p}{R}_{jk} - \frac{1}{(n-1)(n-2)}[(\overset{p}{R}_{jk} - \overset{p}{R}_{kj}) - (K_{jk} - K_{kj}) \\ &+ (n-1)(\overset{(s)}{R}_{jk} - \overset{0(s)}{R}_{jk})]\Big\}, \end{aligned} \tag{3.10}$$

Substituting (3.9) and (3.8) into (3.2), by a direct computation, then from the expression (3.5), we have

$$\overset{p}{W}{}^{l}{}_{ijk} = \overset{0}{W}{}^{l}{}_{ijk}, \tag{3.11}$$

where $\overset{0}{W}{}^{l}{}_{ijk}$ is the Weyl projective curvature tensor of $\overset{0}{\nabla}$, namely,

$$\begin{aligned} \overset{0}{W}{}^{l}{}_{ijk} &= K^{l}{}_{ijk} - \frac{1}{n-1}(\delta^l_i K_{jk} - \delta^l_j R_{ik}) - \frac{1}{(n+1)(n-2)}\Big[\delta^l_i(K_{jk} - K_{kj}) - \delta^l_j(K_{ik} - K_{ki}) \\ &- (n-1)\delta^l_k(K_{ij} - K_{ji})\Big] - \frac{1}{(n-1)(n-2)}\Big[\delta^l_i \overset{0(s)}{R}_{jk} - \delta^l_j \overset{0(s)}{R}_{ik} + (n-1)\delta^l_k \overset{0(s)}{R}_{ji}\Big] \\ &= K^l_{ijk} - \frac{1}{n-1}(\delta^l_i K_{jk} - \delta^l_j K_{ik}) \end{aligned} \tag{3.12}$$

This ends the proof of Lemma 3.2. □

Remark 3.3. *$\overset{0(s)}{R}_{ik} = K_{ijkl}g^{kl}$ is a semi-Ricci curvature tensor of the Levi-Civita connection $\overset{0}{\nabla}$.*

Lemma 3.2 implies that the following theorem is tenable.

Theorem 3.4. *If a Riemannian metric admits the projective semi-symmetric connection $\overset{p}{\nabla}$ with vanishing curvature tensor, then the Riemannian metric is projective flat ($dimM > 2$).*

Lemma 3.5. *In a Riemannian manifold the tensor*

$$\begin{aligned} \overset{p}{V}{}^{l}{}_{ijk} &= \overset{p}{R}{}^{l}{}_{ijk} - \frac{1}{n}(\delta^l_i \overset{p}{R}_{jk} - \delta^l_j \overset{p}{R}_{ik} + g_{ik}\overset{p}{R}{}^{l}{}_j - g_{jk}\overset{p}{R}{}^{l}{}_i) - \frac{1}{n(n^2-4)}\Big[\delta^l_i(\overset{p}{R}_{jk} - \overset{p}{R}_{kj}) - \delta^l_j(\overset{p}{R}_{ik} - \overset{p}{R}_{ki}) \\ &+ g_{ik}(\overset{p}{R}{}^{l}{}_j - \overset{p}{R}{}^{l}{}_{.j}) - g_{jk}(\overset{p}{R}{}^{l}{}_i - \overset{p}{R}{}^{l}{}_{.i}) - n\delta^l_k(\overset{p}{R}_{ij} - \overset{p}{R}_{ji})\Big] \\ &- \frac{1}{n^2-4}\Big(\delta^l_i \overset{p(s)}{R}_{jk} - \delta^l_j \overset{p(s)}{R}_{ik} + g_{ik}\overset{p(s)}{R}{}^{l}{}_j - g_{jk}\overset{p(s)}{R}{}^{l}{}_i + n\delta^l_k \overset{p(s)}{R}_{ji}\Big) \end{aligned} \tag{3.13}$$

is an invariant under the connection transformation $\overset{p}{\nabla} \longrightarrow \overset{p*}{\nabla}$, where $\overset{p(s)}{R}_{jk} = \overset{p}{R}_{ijkl}g^{kl}$.

Proof. From (3.2) and (3.3), we have

$$\overset{p*}{R}{}^{l}{}_{ijk} = \overset{p}{R}{}^{l}{}_{ijk} - \delta^l_j \tau_{ik} + \delta^l_i \tau_{jk} + g_{ik}\tau^l_j - g_{jk}\tau^l_i - 2\delta^l_k \beta_{ij}, \tag{3.14}$$

By using contracting of the indices i and l of the expression (3.14), we find

$$\overset{p*}{R}_{jk} = \overset{p}{R}_{jk} + n\tau_{jk} - g_{jk}\tau^l_l - 2\beta_{jk}, \tag{3.15}$$

Alternating the indices j and k of this expression, we get

$$\overset{p*}{R}_{jk} - \overset{p*}{R}_{kj} = \overset{p}{R}_{jk} - \overset{p}{R}_{kj} + n(\tau_{jk} - \tau_{kj}) - 4\beta_{jk},$$

On the other hand, by contracting the indices k and l of the expression (3.14), we have

$$\overset{p*(s)}{R}{}_{ij} = \overset{p(s)}{R}{}_{ij} - 2(\tau_{ij} - \tau_{ji}) - 2n\beta_{ij},$$

From these expressions we find

$$\beta_{jk} = \frac{1}{2(n^2-4)}\Big[2(\overset{p*(s)}{R}{}_{jk} - \overset{p*(s)}{R}{}_{kj}) - 2(\overset{p(s)}{R}{}_{jk} - \overset{p(s)}{R}{}_{kj}) - n(\overset{p*(s)}{R}{}_{jk} - \overset{p(s)}{R}{}_{jk})\Big], \tag{3.16}$$

Substituting the expression (3.16) into the expression (3.15), we find

$$\begin{aligned}\tau_{jk} &= \frac{1}{n}\Big\{\overset{p*(s)}{R}{}_{jk} - \overset{p(s)}{R}{}_{jk} + g_{jk}\tau^i_i - \frac{1}{(n^2-4)}\Big[2(\overset{p*(s)}{R}{}_{jk} - \overset{p*(s)}{R}{}_{kj}) - 2(\overset{p(s)}{R}{}_{jk} - \overset{p(s)}{R}{}_{kj}) \\ &- n(\overset{p*(s)}{R}{}_{jk} - \overset{p(s)}{R}{}_{jk})\Big]\Big\},\end{aligned} \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.14), by a direct computation and the expression (3.13), we have

$$\overset{p}{V}{}^{l}{}_{ijk} = \overset{p*}{V}{}^{l}{}_{ijk} \tag{3.18}$$

where the tensor $\overset{p*}{V}{}^{l}{}_{ijk}$ is

$$\begin{aligned}\overset{p*}{V}{}^{l}{}_{ijk} &= \overset{p*}{R}{}^{l}{}_{ijk} - \frac{1}{n}(\delta^l_i\overset{p*}{R}_{jk} - \delta^l_j\overset{p*}{R}_{ik} + g_{ik}\overset{p*}{R}{}^{l}{}_{j} - g_{jk}\overset{p*}{R}{}^{l}{}_{i}) - \frac{1}{n(n^2-4)}\Big[\delta^l_i(\overset{p*}{R}_{jk} - \overset{p*}{R}_{kj}) - \delta^l_j(\overset{p*}{R}_{ik} - \overset{p*}{R}_{ki}) \\ &+ g_{ik}(\overset{p*}{R}{}^{l}{}_{j} - \overset{p*}{R}{}^{l}{}_{.j}) - g_{jk}(\overset{p*}{R}{}^{l}{}_{i} - \overset{p*}{R}{}^{l}{}_{.i}) - n\delta^l_k(\overset{p*}{R}_{ij} - \overset{p*}{R}_{ji})\Big] \\ &- \frac{1}{n^2-4}\Big(\delta^l_i\overset{p*(s)}{R}{}_{jk} - \delta^l_j\overset{p*(s)}{R}{}_{ik} + g_{ik}\overset{p*(s)}{R}{}^{l}{}_{j} - g_{jk}\overset{p*(s)}{R}{}^{l}{}_{i} + n\delta^l_k\overset{p*(s)}{R}{}_{ji}\Big)\end{aligned} \tag{3.19}$$

This completes the proof of Lemma 3.5. □

4. Projective Conformal Semi-symmetric Connections

Definition 4.1. *A connection* ∇ *is called a projective conformal semi-symmetric connection, if* ∇ *is projective equivalent to* $\overset{c}{\nabla}$ *in a Riemannian manifold, where* $\overset{c}{\nabla}$ *is conformal equivalent to* $\overset{t}{\nabla}$*.*

It is not hard to write down the coefficient of the conformal semi-symmetric connection $\overset{c}{\nabla}$ is as

$$\overset{c}{\Gamma}{}^k{}_{ij} = \left\{ {k \atop ij} \right\} + (\delta_i - \frac{1-t}{2}\pi_i)\delta^k_j + (\delta_j + \frac{1+t}{2}\pi_j)\delta^k_i - g_{ij}\delta^k,$$

where δ_i is a conformal component w.r.t. the conformal transformation of metric g_{ji}, namely, $g_{ji} \longrightarrow \bar{g}_{ji} = e^{2\delta}g_{ji}$. Furthermore, the projective conformal semi-symmetric connection ∇ satisfies relations

$$\begin{aligned} \nabla_k g_{ij} &= -2(\psi_k + \sigma_k - \frac{1-t}{2}\pi_k)g_{ij} - (\psi_i + \frac{1+t}{2}\pi_i)g_{jk} - (\psi_j + \sigma_j + \frac{1+t}{2}\pi_j)g_{ik} \\ T^k_{ij} &= \pi_j\delta^k_i - \pi_i\delta^k_j \end{aligned} \tag{4.1}$$

and its connection coefficient is

$$\Gamma^k{}_{ij} = \left\{ {k \atop ij} \right\} + (\psi_i + \sigma_i - \frac{1-t}{2}\pi_i)\delta^k_j + (\psi_j + \sigma_j + \frac{1+t}{2}\pi_j)\delta^k_i - g_{ij}\sigma^k, \tag{4.2}$$

and the curvature tensor of is

$$R^l{}_{ijk} = K^l{}_{ijk} + \delta^l_j c_{ik} - \delta^l_i c_{jk} + g_{ik}b^l_j - g_{jk}b^l_i + \delta^l_k\beta_{ij}, \tag{4.3}$$

where

$$\begin{aligned} b_{ji} &= \overset{0}{\nabla}_j\delta_i - \delta_j\delta_i, \\ c_{ji} &= \overset{0}{\nabla}_j(\psi_i + \sigma_i + \frac{1+t}{2}\pi_i) - (\psi_j + \sigma_j + \frac{1+t}{2}\pi_j)(\psi_i + \sigma_i + \frac{1+t}{2}\pi_i) \\ &+ g_{ji}(\psi_l + \sigma_l + \frac{1+t}{2}\pi_l)\sigma^l. \end{aligned}$$

The coefficient of dual connection $\overset{*}{\nabla}$ with respect to the connection ∇ is

$$\overset{*}{\Gamma}{}^k{}_{ij} = \left\{ {k \atop ij} \right\} + (\psi_i + \sigma_i - \frac{1-t}{2}\pi_i)\delta^k_j - g_{ij}(\psi^k + \sigma^k + \frac{1+t}{2}\pi^k) + \sigma_j\delta^k_i,$$

and the curvature tensor of the connection is

$$\overset{*}{R}{}_{ijk}{}^l = K_{ijk}{}^l + \delta^l_j b_{ik} - \delta^l_i b_{jk} + g_{ik}c^l_j - g_{jk}c^l_i - \delta^l_k\beta_{ij}, \tag{4.4}$$

The coefficient of the mutual connection $\overset{m}{\nabla}$ with respect to the connection ∇ is

$$\overset{m}{\Gamma}{}^k{}_{ij} = \left\{ {k \atop ij} \right\} + (\psi_i + \sigma_i + \frac{1+t}{2}\pi_i)\delta^k_j + (\psi_j + \sigma_j - \frac{1-t}{2}\pi_j)\delta^k_i - g_{ij}\sigma^k,$$

and the curvature tensor of the connection $\overset{m}{\nabla}$ is given as

$$\overset{m}{R}{}^l{}_{ijk} = K^l{}_{ijk} + \delta^l_j\bar{c}_{ik} - \delta^l_i\bar{c}_{jk} + g_{ik}b^l_j - g_{jk}b^l_i + \delta^l_k\bar{\beta}_{ij}, \tag{4.5}$$

where

$$\begin{aligned} \bar{c}_{ji} &= \overset{0}{\nabla}_j(\psi_i + \sigma_i - \frac{1-t}{2}\pi_i) - (\psi_j + \sigma_j - \frac{1-t}{2}\pi_j)(\psi_i + \sigma_i - \frac{1-t}{2}\pi_i) \\ &+ g_{ji}(\psi_l + \sigma_l - \frac{1-t}{2}\pi_l)\sigma^l. \end{aligned} \tag{4.6}$$

Theorem 4.2. *(Schur's Theorem) Suppose* $(M^n, g)(n > 3)$ *is a connected Riemannian manifold associated with a projective conformal semi-symmetric connection* ∇ *that is isotropic. If it satisfies the relation*

$$\phi_h + 2\sigma_h + \frac{1+t}{2}\pi_h = 0. \tag{4.7}$$

Then (M^n, g) *is a constant curvature manifold.*

Proof. In a Riemannian manifold the second Bianchi identify of the curvature tensor of the projective conformal semi-symmetric connection ∇ is

$$\nabla_h R_{ijk}{}^l + \nabla_i R_{jhk}{}^l + \nabla_j R_{hik}{}^l = 2(\pi_h R_{ijk}{}^l + \pi_i R_{jhk}{}^l + \pi_j R_{hik}{}^l), \tag{4.8}$$

From the fact that (M^n, g) is of isotropic, this implies that the curvature is

$$R_{ijk}{}^l = k(p)(\delta^l_j g_{ik} - \delta^l_i g_{jk}), \tag{4.9}$$

Substituting the expression (4.9) into the expression (4.8), then we obtain

$$\begin{aligned}&\nabla_h k(\delta^l_j g_{ik} - \delta^l_i g_{jk}) + \nabla_i k(\delta^l_h g_{jk} - \delta^l_j g_{hk}) + \nabla_j k(\delta^l_i g_{hk} - \delta^l_h g_{ik})\\ &+k(\delta^l_j \nabla_h g_{ik} - \delta^l_i \nabla_h g_{jk} + \delta^l_h \nabla_i g_{jk} - \delta^l_j \nabla_i g_{hk} + \delta^l_i \nabla_j g_{hk} - \delta^l_h \nabla_j g_{ik})\\ &= 2[\pi_h(\delta^l_j g_{ik} - \delta^l_i g_{jk}) + \pi_i(\delta^l_h g_{jk} - \delta^l_j g_{hk}) + \pi_j(\delta^l_i g_{hk} - \delta^l_h g_{ik})].\end{aligned}$$

Using the expression (4.1) we have

$$\begin{aligned}&[\nabla_h k - (\psi_h + 2\delta_h + \frac{1+t}{2}\pi_h)k](\delta^l_k g_{ij} - \delta^l_i g_{jk}) + [\nabla_h k - (\psi_h + 2\delta_h + \frac{1+t}{2}\pi_h)k](\delta^l_h g_{jk} - \delta^l_j g_{kh})\\ &+[\nabla_h k - (\psi_h + 2\delta_h + \frac{1+t}{2}\pi_h)k](\delta^l_i g_{kh} - \delta^l_k g_{hi}) = 0.\end{aligned}$$

Contracting the indices i and l of this expression, then we find

$$(n-2)\Big\{[\nabla_j k - (\psi_j + 2\sigma_j + \frac{1+t}{2}\pi_j)k]g_{hk} - [\nabla_h k - (\psi_h + 2\sigma_h + \frac{1+t}{2}\pi_h)k]g_{jk}\Big\} = 0,$$

multiplying both sides of this expression by g^{ik}, then we obtain

$$(n-1)(n-2)\Big[\nabla_h k - (\psi_h + 2\sigma_h + \frac{1+t}{2}\pi_h)k\Big] = 0.$$

Consequently, from $dimM > 2$, it is known that $k = const$ if and only if

$$\phi_h + 2\sigma_h + \frac{1+t}{2}\pi_h = 0.$$

This completes the proof of Theorem 4.2. □

Remark 4.3. *If* $\sigma_i = 0$*, we see from Theorem 4.2 that* $\Gamma_{ij}{}^k = \left\{ {k \atop ij} \right\} - \pi_i \delta^k_j$*, this connection is studied in [6]. If* $\psi_i = \sigma_i = 0$*, then the projective conformal semi-symmetric connection* ∇ *is exactly the semi-symmetric connection* $\overset{t}{\nabla}$*. By virtue of the expressions in [4, 7], the semi-symmetric connection* $\overset{t}{\nabla}$ *with constant curvature is just that the connection* $\overset{t}{\nabla}$ *with* $t = -1$.

5. Acknowledgements

The authors wish to thank Professor X. Yang for his guidance and help! The authors are very grateful to the referee(s) for valuable suggestions.

References

[1] N. S. Agache and M. R. Chafle, A semi-symmetric non-metric connection on a Riemannian manifold, *Indian J. Pure Appl. Math.*, 1992, **23**, 399-409

[2] S. K. Chaubey and R. H. Ojha, On semi-symmetric non-metric connection, *Filomat*, 2012, **26**(2): 269-275

[3] S. S. Chern, W. H. Chern and K. S. Lam, Lectures on Differential Geometry, W. C., 2000

[4] U. C. De and D. Kamila, On a type of semi-symmetric non-metric connection on a Riemannian manifold, *J. Indian Inst. Sci.*, 1995, **75**, 707-710

[5] U. C. De and B. C. Biswas, On a type of semi-symmetric non-metric connection on a Riemannian manifold, *Istanbul Univ. Mat. Derg.*, 1996/1997, **55/56**, 237-238

[6] K. A. Dunu, A Geometrical model for scalar-tensor theories of gravitation, *Tensor, N. S.*, 1975, **29**, 214-216

[7] S. B. Edgar, Curvature copies and uniqueness of Bianchi-type equation; the symmetric connection case, *Class. Quantum, Grav.*, 1993, **10**, 2545-2555

[8] F. Fu, X. Yang and P. Zhao, Geometrical and physical characteristics of a class of conformal mappings, *Journal of Geometry and Physics*, **62**(6):1467-1479

[9] A. Hayden, Subspaces of a space with torsion, *Proc. London Math. Soc.*, 1932, **34**, 27-50

[10] Y. Han, T. Ho and P. Zhao, Some invariant of quarter-symmetric metric connections under the projective transformation, *Filomat*, 2013, **27**, 679-691

[11] Y. Han., T. Y. Ho and P. Zhao, A Schur's lemma based on a semi-symmetric non-metric connection, *International Journal of Geometry*, 2016, **5**(1):47-53

[12] Y. Han, F. Fu and P. Zhao, A class of non-holonomic projective connections on sub-Riemannian manifolds, *Filomat*, 2017, **31**(7): 1295-1303

[13] Y. Han, T. Y. Ho and F. Fu, A classfication of conformal-Weyl manifolds in view of non-metric connections, *Facta Universitatis(WIS), ser. Math. Inform.*, 2016, **31**(2):513-527

[14] T. Y. Ho, On the projective semi-symmetric connection and conformal semi-symmetric connection on the Riemannian manifold, *Journal of Kim Il Sung university(Natural Science)*, 2013, 2: 3-10

[15] T. Imai, Notes on semi-symmetric metric connection, *Tensor. N. S.*, 1972, **24**,9-16

[16] H. V. Le, Statistical manifold are statistical models, *J. Geom.*, 2005, **24**, 83-93

[17] J. Li, G. He and P. Zhao, On submanifolds in a Riemannian manifold with a semi-symmetric non-metric connection, *Symmetry*, 2017, **9**, 112

[18] E. S. Stepanova, Dual symmetric statistical manifolds, *Journal of Mathematical Sciences*, 2007, **147**(1):6507-6509

[19] I. Suhendro, A new semi-symmetric unified field theory of the classical fields of gravity and electromagnetism, *Progress in Physics*, 2007, **4**, 47-62

[20] W. Tang, T. Y. Ho, K. I. Ri, F. Fu and P. Zhao, On a generalized quarter symmetric metric recurrent connection, *Filomat*, 2018, **32**(1): 207-215

[21] K. Yano, On a semi-symmetric metric connection, *Rev. Roum. Math. Pure Appl.*, 1970, **15**, 1579-1586

[22] P. Zhao, Some Properties of projective semi-symmetric connection, *International Mathematical Forum*, 2008, **3**(7): 341-347

[23] P. Zhao and H. Song, An invariant of the projective semi-symmetric connection, *Chin. Quart., J. of Math.*, 2001, **16**(4): 49-54

[24] P. Zhao, H. Song and X. Yang, Some invariant properties ofsemi-symmetric metric recurrent connections and curvature tensor expressions, *Chin. Quart, J. of Math.*, 2004, **19**(4): 355-361