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On Approximately Dual Frames for Hilbert C^*-Modules

Ljiljana Arambašić[a]

[a]*Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia*

Abstract. In this paper we discuss approximately dual frames for a frame or an outer frame of a Hilbert C^*-module. We show that every frame or an outer frame, up to a scalar multiple, is approximately dual to itself. This enables us to get a canonical dual frame of a given frame $(x_n)_n$ as a limit of approximately dual frames defined by $(x_n)_n$.

1. Introduction

A *(right) Hilbert C^*-module over a C^*-algebra* A (or a *(right) Hilbert A-module*) is a linear space X which is a right A-module equipped with an A-valued inner-product $\langle \cdot , \cdot \rangle : X \times X \to \mathsf{A}$ such that

(1) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ for $x, y, z \in X$, $\alpha, \beta \in \mathbb{C}$,
(2) $\langle x, ya\rangle = \langle x, y\rangle a$ for $x, y \in X$, $a \in \mathsf{A}$,
(3) $\langle x, y\rangle^* = \langle y, x\rangle$ for $x, y \in X$,
(4) $\langle x, x\rangle \geq 0$ for $x \in X$; if $\langle x, x\rangle = 0$ then $x = 0$,

and such that X is a Banach space with respect to the norm defined by $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$, $x \in X$.

Obviously, Hilbert C^*-modules generalize Hilbert spaces, as well as C^*-algebras (which can be regarded as Hilbert C^*-modules over itself with the inner product $\langle a, b\rangle = a^*b$). Also, $X = \mathsf{A}^N = \mathsf{A} \oplus \ldots \oplus \mathsf{A}$, $N \in \mathbb{N}$, (N copies of A) with the inner product $\langle (a_1, \ldots, a_N), (b_1, \ldots, b_N)\rangle = \sum_{n=1}^N a_n^* b_n$ is a Hilbert A-module. In the modular frame theory one of the most important Hilbert C^*-modules is $\ell^2(\mathsf{A})$, the *generalized Hilbert space over* A, which is defined as the set

$$\ell^2(\mathsf{A}) = \{(a_n)_n : a_n \in \mathsf{A}, n \in \mathbb{N}, \sum_{n=1}^{\infty} a_n^* a_n \text{ converges in norm of } \mathsf{A}\}$$

with the modular action defined by $(a_n)_n a = (a_n a)_n$ and the inner product $\langle (a_n)_n, (b_n)_n\rangle = \sum_{n=1}^{\infty} a_n^* b_n$.

We say that a Hilbert A-module X is *countably generated* if there exists a sequence $(x_n)_n$ in X such that the closed linear span of the set $\{x_n a : n \in \mathbb{N}, a \in \mathsf{A}\}$ is equal to X. X is *algebraically finitely generated* Hilbert A-module if there exists a finite sequence $(x_n)_{n=1}^N$ such that $X = \{\sum_{n=1}^N x_n a_n : a_n \in \mathsf{A}\}$. We assume that the class of countably generated Hilbert C^*-modules includes all algebraically finitely generated Hilbert C^*-modules.

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Email address: arambas@math.hr (Ljiljana Arambašić)

If X and Y are Hilbert A-modules we denote by $\mathbb{B}(X, Y)$ the Banach space of all adjointable operators from X to Y. For $x \in X$ and $y \in Y$ let $\theta_{y,x} \in \mathbb{B}(X, Y)$ be the map defined by $\theta_{y,x}(z) = y\langle x, z\rangle$. The closure of the linear span of all $\theta_{y,x}$'s is denoted by $\mathbb{K}(X, Y)$; the elements of $\mathbb{K}(X, Y)$ are usually called generalized compact operators. In the case $Y = X$ we write $\mathbb{K}(X)$ and $\mathbb{B}(X)$. For basic facts on Hilbert C^*-modules we refer the reader to [10–12, 14].

Let $M(\mathsf{A})$ be the multiplier algebra of a C^*-algebra A. It is known that, for a Hilbert A-module X, there exists a Hilbert $M(\mathsf{A})$-module $M(X)$ containing X as the ideal submodule associated with the ideal A in $M(\mathsf{A})$; i.e., $X = M(X)\mathsf{A}$. It holds

$$X = \{x \in M(X) : \langle x, v\rangle \in \mathsf{A},\ \forall v \in M(X)\}. \tag{1}$$

The extended module $M(X)$ is called the multiplier module of X.

In particular, if we regard a C^*-algebra A as a Hilbert C^*-module over itself then $M(X)$ coincides with $M(\mathsf{A})$. Further, if X is a Hilbert C^*-module over a unital C^*-algebra, or if X is algebraically finitely generated Hilbert A-module, then $M(X)$ coincides with X ([3], page 547 and Proposition 2.4). In general, X can be naturally identified as $\mathbb{K}(\mathsf{A}, X)$ and $M(X)$ as $\mathbb{B}(\mathsf{A}, X)$ (see [3, 13]).

If X and Y are Hilbert A-modules, then each operator $T \in \mathbb{B}(X, Y)$ has a unique extension $T_M \in \mathbb{B}(M(X), M(Y))$. The map $T \mapsto T_M$ is a bijection of $\mathbb{B}(X, Y)$ and $\mathbb{B}(M(X), M(Y))$ such that $\|T_M\| = \|T\|$ and $(T_M)^* = (T^*)_M$ for all T in $\mathbb{B}(X, Y)$. To simplify notation we shall use the same letter T for the extended map T_M.

Let us now introduce basic modular frame definitions.

Modular frames were introduced by M. Frank and D. Larson. In their papers [5–7] they managed to show that, even in this more general situation, many of the most important results from the Hilbert space frame theory still hold. Recently, the concept of outer frames for Hilbert C^*-modules has been introduced in [2]. It is shown there that outer frames share many properties of frames and that it is natural to include them in the study of modular frames. Actually, the only difference between frames and outer frames is that some of the vectors of outer frames belong to the multiplier module $M(X)$ and not to X.

Definition 1.1. *Let X be a Hilbert A-module and $(x_n)_n$ a sequence in $M(X)$ such that there exist positive constants A and B so that*

$$A\langle x, x\rangle \le \sum_{n=1}^{\infty} \langle x, x_n\rangle\langle x_n, x\rangle \le B\langle x, x\rangle, \quad \forall x \in X, \tag{2}$$

where the sum in the middle converges in norm. If $x_n \in X$ for all $n \in \mathrm{N}$, then we say that $(x_n)_n$ is a frame *for X, and if at least one x_n is in $M(X) \setminus X$, then we say that $(x_n)_n$ is an* outer frame *for X.*

The constants A and B are called frame bounds*.*

If we can choose $A = B = 1$, then an (outer) frame $(x_n)_n$ is called an (outer) Parseval frame *for X.*

Sequences satisfying the second inequality in (2) are called Bessel sequences *if all of its elements are in X, and* outer Bessel sequences *if at least one x_n is in $M(X) \setminus X$.*

Given a Bessel sequence or an outer Bessel sequence $(x_n)_n$ for a Hilbert A-module X, we define the *analysis operator* of $(x_n)_n$ by $U : X \to \ell^2(\mathsf{A})$, $Ux = (\langle x_n, x\rangle)_n$. Then U is an adjointable map and its adjoint operator U^*, called the *synthesis operator*, is given by $U^*((a_n)_n) = \sum_{n=1}^{\infty} x_n a_n$. (If $(x_n)_{n=1}^N$ is a finite Bessel sequence or outer Bessel sequence then we understand $U : X \to \mathsf{A}^N$ and $U^*((a_n)_{n=1}^N) = \sum_{n=1}^N x_n a_n$.)

Furthermore, if $(x_n)_n$ is a frame or an outer frame for X, then (2) is equivalent to $A \cdot I \le U^*U \le B \cdot I$, where I is the identity operator in $\mathbb{B}(X)$. It follows that U is bounded from below and U^*U is an invertible operator in $\mathbb{B}(X)$, so we have the following *reconstruction formula*

$$x = \sum_{n=1}^{\infty} x_n\langle (U^*U)^{-1}x_n, x\rangle = \sum_{n=1}^{\infty} (U^*U)^{-1}x_n\langle x_n, x\rangle, \quad \forall x \in X. \tag{3}$$

In particular, $(x_n)_n$ is an (outer) Parseval frame for X if and only if $U^*U = I$, that is, if and only if $x = \sum_{n=1}^{\infty} x_n\langle x_n, x\rangle$ for all $x \in X$.

Obviously, it follows from (3) that every Hilbert C^*-module possessing a frame is countably generated. It is also known that every countably generated Hilbert C^*-module possesses a frame. There are Hilbert C^*-modules (which, obviously, cannot be countably generated) that possess outer frames but not frames (e.g. take a non-unital σ-unital C^*-algebra A and conclude similarly as in [2], p. 409). For more results about frames in Hilbert C^*-modules, besides mentioned papers [2, 5–7], see e.g. [1, 8, 9].

In this paper we introduce the notion of approximately dual frames for general Hilbert C^*-modules; the discussion includes both frames and outer frames. We show that frames which are close enough to a dual frame of a given frame $(x_n)_n$ are necessarily approximately dual frames of $(x_n)_n$. Further, if a frame or an outer frame has a unique dual, then each of its approximately dual frames also has a unique dual. We also show that every frame or outer frame is, up to a scalar multiple, approximately dual to itself and its canonical dual can be obtained as a limit of approximately dual frames defined in terms of the original frame or outer frame.

2. Approximately dual frames in Hilbert C^*-modules

Let us start with the definition of dual frames in a Hilbert C^*-module.

Definition 2.1. *Let $(x_n)_n$ be a frame or an outer frame for a Hilbert A-module X. We say that a frame or an outer frame $(y_n)_n$ is* dual *to $(x_n)_n$ if*

$$x = \sum_{n=1}^{\infty} y_n\langle x_n, x\rangle, \qquad \forall x \in X. \tag{4}$$

Suppose $(y_n)_n$ is a dual frame for $(x_n)_n$. Let U and V in $\mathbb{B}(X, \ell^2(\mathsf{A}))$ be analysis operators for $(x_n)_n$ and $(y_n)_n$, respectively. Then (4) can be written as $V^*U = I$. It follows from here that $U^*V = I$, so we have

$$x = \sum_{n=1}^{\infty} x_n\langle y_n, x\rangle, \qquad \forall x \in X. \tag{5}$$

This means that $(x_n)_n$ is also dual to $(y_n)_n$, that is, duality is a symmetric relation. Therefore we say that $(x_n)_n$ and $(y_n)_n$ are dual to each other, or shortly, that $(x_n)_n$ and $(y_n)_n$ are dual frames.

Remark 2.2. *We assumed in Definition 2.1 that $(x_n)_n$ and $(y_n)_n$ are frames or outer frames. However, it was enough to assume that $(x_n)_n$ and $(y_n)_n$ are Bessel sequences or outer Bessel sequences. Indeed, since their analysis operators satisfy the relation $V^*U = I$ (that is (4)), it follows that V^* is necessarily surjective, so by Theorem 3.19 from [2], $(y_n)_n$ is a frame or an outer frame for X. The same conclusion holds for $(x_n)_n$.*

Every frame or outer frame has at least one dual. Namely, by (3), the sequences $(x_n)_n$ and $((U^*U)^{-1}x_n)_n$ are dual frames; thereby, they are either both frames or both outer frames. However, a frame and an outer frame can be dual to each other as well ([2, Example 4.4]).

The importance of dual frames comes from the reconstruction property (4). However, if $(x_n)_n$ and $(y_n)_n$ are frames or outer frames such that V^*U is invertible (not necessarily I), where U and V are associated analysis operators, then if follows from

$$V^*Ux = \sum_{n=1}^{\infty} y_n\langle x_n, x\rangle, \qquad \forall x \in X \tag{6}$$

that

$$x = \sum_{n=1}^{\infty} (V^*U)^{-1} y_n\langle x_n, x\rangle, \qquad \forall x \in X, \tag{7}$$

so we still have a perfect reconstruction of each $x \in X$.

Let us now introduce a concept of approximately dual frames in Hilbert C^*-modules which extends approximately dual frames for Hilbert spaces introduced in [4].

Definition 2.3. *Let $(x_n)_n$ be a frame or an outer frame for a Hilbert A-module X with U in $\mathbb{B}(X, \ell^2(\mathsf{A}))$ the associated analysis operator. We say that a frame or an outer frame $(y_n)_n$ is* approximately dual to $(x_n)_n$ *if* $\|I - V^*U\| < 1$.

Observe that, just as in the dual frame situation, approximate duality is a symmetric relation, since $\|I - V^*U\| < 1$ if and only if $\|I - U^*V\| < 1$.

Let us begin with a simple example as an illustration that there are many approximately dual frames.

Example 2.4. *Suppose $(x_n)_n$ and $(y_n)_n$ are dual frames for a Hilbert C^*-module X with analysis operators U and V, respectively. Let $x_0, y_0 \in M(X)$ be such that $\|x_0\|\|y_0\| < 1$. Then $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ are frames or outer frames with analysis operators U_1 and V_1 such that*

$$V_1^*U_1x = y_0\langle x_0, x\rangle + \sum_{n=1}^{\infty} y_n\langle x_n, x\rangle = y_0\langle x_0, x\rangle + x, \quad \forall x \in X,$$

so we have

$$\|(V_1^*U_1 - I)x\| = \|y_0\langle x_0, x\rangle\| \leq \|y_0\|\|x_0\|\|x\|, \quad \forall x \in X.$$

Therefore, $\|V_1^*U_1 - I\| \leq \|y_0\|\|x_0\| < 1$, which means that $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ are approximately dual frames in X.

The following lemma says that, as with dual frames, it is enough to assume that $(x_n)_n$ and $(y_n)_n$ are Bessel sequences or outer Bessel sequences. Also, we shall see that every pair of approximately dual frames defines two pairs of dual frames.

Lemma 2.5. *Let $(x_n)_n$ and $(y_n)_n$ be Bessel sequences or outer Bessel sequences for a Hilbert C^*-module X with the analysis operators U and V, respectively. If $\|I - V^*U\| < 1$ then $(x_n)_n$ and $(y_n)_n$ are frames or outer frames approximately dual to each other. Moreover, $((U^*V)^{-1}x_n)_n$ and $(y_n)_n$, as well as $(x_n)_n$ and $((V^*U)^{-1}y_n)_n$ are dual frames.*

Proof. Since $\|I - V^*U\| < 1$, V^*U in $\mathbb{B}(X)$ is invertible. Further, it follows from (7) and Remark 2.2 that $(x_n)_n$ and $((V^*U)^{-1}y_n)_n$ are dual frames. We similarly get

$$x = \sum_{n=1}^{\infty} (U^*V)^{-1}x_n\langle y_n, x\rangle, \quad \forall x \in X,$$

so $((U^*V)^{-1}x_n)_n$ and $(y_n)_n$ are also dual frames. Then, obviously, $(x_n)_n$ and $(y_n)_n$ are approximately dual frames. □

Let $(x_n)_n$ be a given frame or an outer frame for a Hilbert C^*-module X. In the following proposition we prove that around any dual frame of $(x_n)_n$ there is a neighborhood (in the sense of the ball around its analysis operator) whose elements define approximately dual frames of $(x_n)_n$. This result cannot be proved in general if the outer frames for X are out of the picture.

Proposition 2.6. *Let $(x_n)_n$ be a frame or an outer frame for a Hilbert A-module X with frame bounds A and B, and $(y_n)_n$ its dual frame. Let U and V be their analysis operators, respectively. If $W \in \mathbb{B}(X, \ell^2(\mathsf{A}))$ is such that $\|W - V\| < \frac{1}{\sqrt{B}}$, then W is the analysis operator of a frame or an outer frame for X which is approximately dual to $(x_n)_n$.*

Proof. Since $(x_n)_n$ and $(y_n)_n$ are dual to each other, $V^*U = I$. If $W \in \mathbb{B}(X, \ell^2(\mathsf{A}))$ is such that $\|W - V\| < \frac{1}{\sqrt{B}}$, then

$$\|I - W^*U\| = \|V^*U - W^*U\| \leq \|V^* - W^*\|\|U\| = \|V - W\|\|U\| < \frac{\|U\|}{\sqrt{B}} \leq 1,$$

since $\|U\|$ is the optimal upper bound for $(x_n)_n$ and therefore $\|U\| \leq \sqrt{B}$. It follows from here that W^*U is an invertible operator, in particular, surjective, so W^* is surjective as well. By [2, Theorem 3.19], W^* is the synthesis operator of the frame or outer frame for X. Since $\|I - W^*U\| < 1$, $(z_n)_n$ is an approximately dual frame for $(x_n)_n$. □

The following proposition deals with a frame or an outer frame which has a unique dual frame.

Proposition 2.7. *Let $(x_n)_n$ and $(y_n)_n$ be approximately dual frames for a full Hilbert A-module X. If $(x_n)_n$ has a unique dual, then $(y_n)_n$ also has a unique dual.*

Proof. Let U and V be the associated analysis operators for $(x_n)_n$ and $(y_n)_n$, respectively. If $(x_n)_n$ has a unique dual frame, then by [2, Theorem 4.15], U is invertible in $\mathbb{B}(X, \ell^2(\mathsf{A}))$. Since $(x_n)_n$ and $(y_n)_n$ are approximately dual frames, $U^*V \in \mathbb{B}(X)$ is invertible. Let $W = (U^*V)^{-1}U^* \in \mathbb{B}(\ell^2(\mathsf{A}), X)$. Then $WV = (U^*V)^{-1}U^*V = I_X$ and

$$VW = V(U^*V)^{-1}U^* = (U^*)^{-1}U^*V(U^*V)^{-1}U^* = (U^*)^{-1}U^* = I_{\ell^2(\mathsf{A})}.$$

Therefore, $W \in \mathbb{B}(X, \ell^2(\mathsf{A}))$ is invertible and, again by [2, Theorem 4.15], $(y_n)_n$ has a unique dual. □

Just as in the Hilbert space situation, the dual frame $((V^*U)^{-1}y_n)_n$ of $(x_n)_n$ (mentioned in Lemma 2.5) can be obtained as a limit of approximately dual frames of $(x_n)_n$. What follows is basically the content of Proposition 3.2 from [4], but written in terms of modular frames.

Proposition 2.8. *Let $(x_n)_n$ and $(y_n)_n$ be approximately dual frames for a Hilbert A-module X with analysis operators U and V, respectively. For every $N \in \mathbb{N}$ the sequence $(z_n^N)_n$ defined as*

$$z_n^N = \sum_{k=0}^{N} (I - V^*U)^k y_n, \quad n \in \mathbb{N}$$

is an approximately dual frame of $(x_n)_n$, and the dual $((V^*U)^{-1}y_n)_n$ of $(x_n)_n$ can be obtained as the limit

$$(V^*U)^{-1}y_n = \lim_{N\to\infty} z_n^N, \quad n \in \mathbb{N}. \tag{8}$$

Proof. Since $\|I - V^*U\| < 1$ we have

$$(V^*U)^{-1} = (I - (I - V^*U))^{-1} = \sum_{k=0}^{\infty} (I - V^*U)^k,$$

so

$$(V^*U)^{-1}y_n = \lim_{N\to\infty} \sum_{k=0}^{N} (I - V^*U)^k y_n, \quad n \in \mathbb{N}.$$

This gives (8) and we only need to see that, for each N, $(x_n)_n$ and $(z_n^N)_n$ are approximately dual frames.

Obviously, $(z_n^N)_n$ is a Bessel or an outer Bessel sequence in X, as an image of a frame or an outer frame $(y_n)_n$ under adjointable operator. Its synthesis operator is $Z_N^* = \sum_{k=0}^{N} (I - V^*U)^k V^* \in \mathbb{B}(\ell^2(\mathsf{A}), X)$. Then we have

$$\begin{aligned} \|I - Z_N^*U\| &= \|I - \sum_{k=0}^{N} (I - V^*U)^k (I - (I - V^*U))\| \\ &= \|(I - V^*U)^{N+1}\| \leq \|I - V^*U\|^{N+1} < 1 \end{aligned}$$

so $(z_n^N)_n$ is an approximately dual frame of (x_n). □

Observe that, in the preceding proposition, $\lim_{N\to\infty} \|I - Z_N^* U\| = 0$, so for every $x \in X$ we have

$$\|x - \sum_{n=1}^{\infty} z_n^N \langle x_n, x\rangle\| \leq \|I - V^*U\|^{N+1}\|x\| \to 0 \text{ as } N \to \infty.$$

The previous statement gives us a dual frame $((V^*U)^{-1}y_n)_n$ of $(x_n)_n$ as a limit of approximately dual frames. Here we start with a pair of frames or outer frames $(x_n)_n$ and $(y_n)_n$ which are approximately dual to each other.

A natural question to ask ourselves is whether we can get the canonical dual frame $((U^*U)^{-1}x_n)_n$ of $(x_n)_n$ in this way, starting with $(x_n)_n$. To answer this, we should see when a frame or an outer frame is approximately dual to itself. Recall that a frame or an outer frame is dual to itself if and only if it is a Parseval frame.

Proposition 2.9. *Let $(x_n)_n$ be a frame or an outer frame for a Hilbert A-module X with the optimal frame bounds A and B. Then $(x_n)_n$ is approximately dual to itself if and only if $B < 2$.*

Proof. Let $U \in \mathbb{B}(X, \ell^2(\mathsf{A}))$ be the associated analysis operator. Then $A \cdot I \leq U^*U \leq B \cdot I$, so the spectrum $\sigma(U^*U)$ of U^*U (in the C^*-algebra $\mathbb{B}(X)$) is contained in the closed interval $[A, B]$. Since A and B are optimal frame bounds, A, B belong to $\sigma(U^*U)$. Then $[A-1, B-1]$ is the smallest interval that contains $\sigma(U^*U - I)$. Since $U^*U - I$ is selfadjoint it holds

$$\|U^*U - I\| = \max\{|A-1|, |B-1|\}.$$

Suppose $B < 2$. Then $-1 < A - 1 < B - 1 < 1$, so $\|U^*U - I\| < 1$, which means that $(x_n)_n$ is approximately dual to itself.

Suppose $(x_n)_n$ is approximately dual to itself, that is $\|U^*U - I\| < 1$. Then $|B-1| < 1$ wherefrom $B < 2$. □

In particular, if $(x_n)_n$ is a frame or an outer frame with the optimal upper bound B, then for any $\lambda \in \mathbb{C}$ such that $|\lambda| < \sqrt{\frac{2}{B}}$, the frame $(\lambda x_n)_n$ is approximately dual to itself. In other words, every frame or outer frame is, up to a scalar multiple, approximately dual to itself.

Now the following corollary follows directly from Propositions 2.9 and 2.8.

Corollary 2.10. *Let $(x_n)_n$ be a frame or an outer frame for a Hilbert A-module X such that $B < 2$. Let $U \in \mathbb{B}(X, \ell^2(\mathsf{A}))$ be its analysis operator. For every $N \in \mathbb{N}$ the sequence defined by*

$$z_n^N = \sum_{k=0}^{N} (I - U^*U)^k x_n, \quad n \in \mathbb{N}$$

is an approximately dual frame for $(x_n)_n$. For the canonical dual frame $((U^*U)^{-1}x_n)_n$ of $(x_n)_n$ it holds

$$(U^*U)^{-1}x_n = \lim_{N\to\infty} z_n^N, \quad n \in \mathbb{N}.$$

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