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On Generalized Implicit Operator Equilibrium Problems

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Abstract. In this paper, we introduce and study a class of generalized implicit operator equilibrium problems (In short, GIOEP) and derive some propositions for this class of problems. We also prove some new existence results for the solution of this problem by using Fan KKM theorems in Hausdorff topological vector spaces. The results presented in this paper generalize and unify the corresponding results of several authors which are extensions of previously known results.

1. Introduction

Equilibrium problem is being intensively studied, beginning with Blum and Oettli [1] where they proposed it as a generalization of optimization and variational inequality problem. In 2005, Kazmi and Raouf[5] studied a class of operator equilibrium problems and established some existence results for the solution of this problem.
Let Y be a Hausdorff topological vector space. Then we define the ordering relationships on Y with respect to cone P in Y as follows : For $A, B \subseteq Y$,

$$B - A \subseteq P \;\Leftrightarrow\; A \leq B \Leftrightarrow\; a \leq b, \forall a \in A, b \in B,$$

$$B - A \nsubseteq P \Leftrightarrow\; A \nleq B \Leftrightarrow\; a \nleq b, \forall a \in A, b \in B.$$

If the $intP \neq \phi$, then the weak ordering in Y is defined as follows:

$$B - A \subseteq intP \;\Leftrightarrow\; A < B \;\Leftrightarrow\; a < b, \forall a \in A, b \in B,$$

$$B - A \nsubseteq intP \;\Leftrightarrow\; A \nless B \;\Leftrightarrow\; a \nless b, \forall a \in A, b \in B.$$

Now we will work under the following setting:

Let X and Y be Hausdorff topological vector spaces and let $L(X, Y)$ be the space of all continuous linear operators from X to Y. Let $K \subseteq L(X, Y)$ be a non-empty convex set. Let $C : K \to 2^Y$ be a set-valued mapping such that for each $f \in K$, $C(f)$ is a closed and convex cone in Y with non-empty interior, 2^Y denotes the set of all non-empty subsets of Y.

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In this paper, we consider the following generalized implicit operator equilibrium problem (In short, GIOEP): Find $f^* \in K$ such that

$$F(h(f^*), g) \nsubseteq -intC(f^*), \quad \forall g \in K \tag{1.1}$$

where $F : K \times K \to 2^Y$ be a set- valued map and $h : K \to K$ be a map from K into K itself.

Some special cases of GIOEP

(i) If $F : K \times K \to 2^Y$, $h : K \to K$ is the identity map on K, and $C : K \to 2^Y$ be a set valued map such that $C(f)$ is open, $\forall f \in K$, then (1.1) reduces to the problem of finding $f^* \in K$ such that

$$F(f^*, g) \nsubseteq -C(f^*), \quad \forall g \in K \tag{1.2}$$

which is called the generalized operator equilibrium problem studied by Raouf and Kim [10].

(ii) If F is single valued map, then (1.2) reduces to the problem of finding $f^* \in K$ such that

$$F(f^*, g) \notin -C(f^*), \quad \forall g \in K,$$

which is called the operator equilibrium problem studied by Kazmi and Raouf [5].

(iii) If F is single valued map and $F(f^*, g) = \langle \eta(f^*, g), T(f^*) \rangle$, $\forall g \in K$, where $T : K \to X$ and $\eta : K \times K \to K$, then (1.2) reduces to the problem of finding $f^* \in K$ such that

$$\langle \eta(f^*, g), T(f^*) \rangle \notin -C(f^*), \quad \forall g \in K,$$

which is called the operator variational-like inequality problem.

(iv) If F is single valued map and $F(f^*, g) = \langle f^* - g, T(f^*) \rangle$, $\forall g \in K$, where $T : K \to X$, then (1.2) reduces to the operator variational inequality problem considered by Domokos and Kolumban [2].

(v) If F is single valued map and $F(f^*, g) = \phi(f^*) - \phi(g)$, where $\phi : K \to Y$, then (1.1) reduces to the problem of finding $f^* \in K$ such that

$$\phi(f^*) - \phi(g) \notin -C(f^*), \quad \forall g \in K,$$

which appears to be new. We call it operator minimization problem.

(vi) If $K \subseteq X$, then (1.2) reduces to generalized vector equilibrium problem studied by Konnov and Yao [6].

(vii) If F is single valued map and If $K \subseteq X$, then (1.2) reduces to vector equilibrium problem studied by Kazmi [4].

(viii) If F is single valued map and If $K \subseteq X$, $Y = \mathbb{R}$, then (1.2) reduces to equilibrium problem of finding $f \in K$ such that $F(f, g) \geq 0$, $\forall g \in K$, considered and studied by Blum and Oettli [1].

The main objective of this paper is to study the existence of solution of implicit operator equilibrium problem in Hausdorff topological vector spaces. In section- 2, we recall some necessary definitions and results which are needed in the latter section. We prove some new existence results for the solution of implicit operator equilibrium problems in section-3. Our results extend and unify corresponding results of Konnov and Yao [6], Raouf and Kim [10], Ram [8], Ram and Khanna [9], Kazmi [4], Kazmi and Raouf [5] and Li, Huang and Kim [7].

2. Preliminaries

Now we give some definitions and preliminary results needed in the latter sections.

Definition 2.1. *A set-valued map $T : K \to 2^Y$ is called a KKM-Map if for every finite subset $\{x_1, x_2, ..., x_n\}$ of K, $co\,\{x_1, x_2, ...x_n\} \subseteq \bigcup_{i=1}^{n} T(x_i)$, where co denotes the convex hull.*

Definition 2.2. *Let $F : K \times K \to 2^Y$ be a set valued map and $h : K \to K$*

(i) $F(f, g)$ is a Q-function with respect to g, if for any given $f \in K$,

$$F(f, tg_1 + (1-t)g_2) \subseteq tF(f, g_1) + (1-t)F(f, g_2) + Q, \ \ \forall g_1, g_2 \in K \text{ and } t \in [0, 1],$$

where Q is a closed and convex cone of Y such that $intQ \neq \phi$.

(ii) h is a affine mapping, if for any $g_1, g_2 \in K$ and $t \in \mathbb{R}$,

$$h(tg_1 + (1-t)g_2) = t\,h(g_1) + (1-t)h(g_2)$$

Remark Let $F : K \times K \to 2^Y$ be a set valued map. $F(f, g)$ is a Q-function with respect to g, if for any given $f \in K$, $F(f, \sum_{i=1}^{n} t_i g_i) \subseteq \sum_{i=1}^{n} t_i F(f, g_i) + Q$, $\forall\, g_i \in K$ and $t_i \in [0, 1]$ $(i = 1, 2, ..., n)$ with $\sum_{i=1}^{n} t_i = 1$.

Definition 2.3. *A set-valued map $T : X \to 2^Y$ is called upper-semicontinuous (for short, u.s.c) at $x_0 \in X$ if for any net $\{x_\lambda\}$ in X such that $x_\lambda \to x_0$ and for any net $\{y_\lambda\}$ in Y with $y_\lambda \in T(x_\lambda)$ such that $y_\lambda \to y_0$ in Y, we have $y_0 \in T(x_0)$. T is called upper semicontinuous on X if it is upper semicontinuous at each point of X.*

Definition 2.4. *Let $F : K \times K \to 2^Y$ be a set-valued. Then we say that $F(f, g)$ is hemi-continuous with respect to g, if for any given $f \in K$,*

$$lim_{t \to 0^+} F(f, tg_1 + (1-t)g_2) = F(f, g_2), \ \forall g_1, g_2 \in K.$$

To prove the existence results for the solutions of problem (1.1), we shall use the following lemmas:

Lemma 2.5. *[3] Let K be a non-empty convex subset of a Hausdorff topological vector space X. Let $T : K \to 2^X$ be a KKM-map, such that for any $y \in K$, $T(y)$ is closed and $T(y^*)$ is contained in a compact set $B \subseteq X$ for some $y^* \in K$. Then there exist $x^* \in B$ such that $x^* \in T(y)$, for all $y \in K$. That is, $\bigcap_{y \in K} T(y) \neq \phi$.*

Lemma 2.6. Let (Y, P) be an ordered topological vector space with a closed and convex cone P. Then for any $A, B, C \subseteq Y$, we have

(i) $A - B \subseteq -intP$ and $A \nsubseteq -intP \implies B \nsubseteq -intP$.

(ii) $A + B \subseteq -P$ and $A + C \nsubseteq -intP \implies C - B \nsubseteq -intP$.

(iii) $A + C - B \nsubseteq -intP$ and $-B \subseteq -P \implies A + C \nsubseteq -intP$.

(iv) $A + B \nsubseteq -intP$ and $B - C \subseteq -P \implies A + C \nsubseteq -intP$.

Proof. (i) Suppose $A - B \subseteq -intP$ and $A \nsubseteq -intP$. We have to show that $B \nsubseteq -intP$. Since $A - B \subseteq -intP \implies -B \subseteq -intP - A \implies B \subseteq intP + A \implies B \subseteq intP + Y \setminus -intP \implies B \subseteq Y \setminus -intP \implies B \nsubseteq -intP$.

(ii) Suppose $A + B \subseteq -P$ and $A + C \nsubseteq -intP$. We have to show that $C - B \nsubseteq -intP$. Suppose, if possible $C - B \subseteq -intP$, then $A + C = A + B + C - B \subseteq -P + \{-intP\} \subseteq -intP$, which is a contradiction.

(iii) Suppose, if possible $A + C \subseteq -intP$, then $A + C - B \subseteq -intP - P \subseteq -intP$, which is a contradiction.

(iv) Suppose, if possible $A + C \subseteq -intP$, then $A + B = A + C + B - C \subseteq -intP - P \subseteq -intP$, which is a contradiction.

3. Existence results

In this section, we prove some new existence results for the solutions of generalized implicit operator equilibrium problem (1.1) .

Theorem 3.1. *Let $K \subseteq L(X, Y)$ be a non-empty convex set, $h : K \to K$ a mapping. Let $F : K \times K \to 2^Y$ be a set-valued mapping. Suppose that the following assumption holds:*

(1) h is continuous,

(2) $F(f, g)$ is continuous w.r.t f,

(3) the set-valued map $W : K \to 2^Y$ defined by $W(f) = Y \setminus \{-intC(f)\}$, $\forall f \in K$, is upper semicontinuous on K,

(4) there exists a set-valued map $G : K \times K \to 2^Y$ such that

(i) $G(h(f), f) \nsubseteq -intC(f)$, $\forall f \in K$,

(ii) $G(h(f), g) - F(h(f), g) \subseteq -intC(f)$, $\forall f, g \in K$,

(iii) $\{g \in K : G(f, g) \subseteq -intC(f)\}$ is convex , $\forall f \in K$.

(5) Furthermore, suppose that there exists a non empty compact and convex subset B of K such that for each $f \in K \setminus B$ there exists $g \in B$ such that $F(h(f), g) \subseteq -intC(f)$.

Then there exists $f^* \in K$ such that $F(h(f^*), g) \nsubseteq -intC(f^*)$, $\forall g \in K$.

Proof. For each $g \in K$, define the set-valued map $S : K \to 2^Y$ as

$$S(g) = \{f \in B : F(h(f), g) \nsubseteq -intC(f)\}.$$

We first prove that $S(g)$ is closed, $\forall g \in K$. For this, let $\{f_\alpha\}$ be a net in $S(g)$ such that $f_\alpha \to f$. Then $f \in B$ (as B is compact). It follows from $f_\alpha \in S(g)$ that

$$F(h(f_\alpha), g) \nsubseteq -intC(f_\alpha)$$

$$\implies F(h(f_\alpha), g) \subseteq W(f_\alpha) = Y \setminus \{-intC(f_\alpha)\}.$$

Again, since $F(f, g)$ is continuous w.r.t f and h is also continuous, we have

$$F(h(f_\alpha), g) \to F(h(f), g).$$

Therefore by the upper semi-continuity of W, we have

$$F(h(f), g) \subseteq W(f).$$

$$\text{This implies } F(h(f), g) \nsubseteq -intC(f) \Rightarrow f \in S(g).$$

Hence $S(g)$ is closed, $\forall g \in K$.

Now we will show that

$$\bigcap_{g \in K} S(g) \neq \phi.$$

Since B is compact, it is sufficient to show that the family $\{S(g)\}_{g \in K}$ has the finite intersection property. For this, let $\{g_1, g_2, \cdots, g_n\}$ be a finite subset of K. Set $D = conv[B \cup \{g_1, g_2, \cdots, g_n\}]$. Clearly, D is compact and convex subset of K. Next, for each $g \in K$, we define two set-valued mapping, $T_1, T_2 : K \to 2^D$ as follows:

$$T_1(g) = \{f \in D : F(h(f), g) \nsubseteq -intC(f)\}$$

and

$$T_2(g) = \{f \in D : G(h(f), g) \nsubseteq -intC(f)\}.$$

By assumption (i), (ii) of (4), we have

$$G(h(g), g) \nsubseteq -intC(g)$$

and

$$G(h(g), g) - F(h(g), g) \subseteq -intC(g).$$

This implies by Lemma 2.6(i), $F(h(g), g) \nsubseteq -intC(g)$ and so $T_1(g) \neq \phi$.
Since $T_1(g)$ is a closed subset of a compact set D. Therefore $T_1(g)$ is compact. Now we will show that T_2 is a KKM-Map. Suppose there exists a finite subset $\{f_1, f_2, \cdots, f_n\}$ of D and $\lambda_i \geq 0$, $i = 1, 2, \cdots, n$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that

$$\bar{f} = \sum_{i=1}^{n} \lambda_i f_i \nsubseteq \bigcup_{j=1}^{n} T_2(f_j),$$

then $G(h(\bar{f}), f_j) \subseteq -intC(\bar{f})$, $j = 1, 2, \cdots, n$.
From assumption (4)(iii), we have

$$G(h(\bar{f}), \bar{f}) \subseteq -intC(\bar{f}),$$

which is a contradiction to (4)(i). Hence T_2 is a KKM-mapping.
From assumption (4)(ii) and Lemma 2.6(i), we have

$$T_2(g) \subseteq T_1(g), \forall g \in K.$$

Infact, $f \in T_2(g) \implies G(h(f), g) \nsubseteq -intC(f)$, and by assumption (4)(ii), we have,

$$G(h(f), g) - F(h(f), g) \subseteq -intC(f) \implies F(h(f), g) \nsubseteq -intC(f) \implies f \in T_1(g).$$

So, T_1 is also a KKM -mapping.

From Lemma 2.5, there exists $f^* \in D$ such that $f^* \in T_1(g)$, $\forall g \in K$.

This implies there exists $f^* \in D$ such that

$$F(h(f^*), g) \nsubseteq -intC(f^*), \ \forall g \in K.$$

Therefore by assumption (5), we get $f^* \in B$ and moreover, $f^* \in S(g_i)$, $i = 1, 2, \cdots, n$. Hence $\{S(g)\}_{g \in K}$ has the finite intersection property. □

Theorem 3.2. *Let $K \subseteq L(X, Y)$ be a non-empty convex set and $h : K \to K$ be a mapping. Let $F_i : K \times K \to 2^Y (i = 1, 2)$ be two set-valued mappings. Let $Q = \bigcap_{f \in K} \{-C(f)\}$ such that $intQ \neq \phi$. Suppose that the following assumption holds:*

(1) $C(h(f)) \subseteq C(f)$, $\forall f \in K$,

(2) h is affine and continuous,

(3) (i) $F_1(h(f), f) = \{0\}$, $\forall f \in K$, (ii) $F_1(f, g) + F_2(g, f) \subseteq \{-C(f)\} \cap \{-C(g)\}$, $\forall f, g \in K$, (iii) $F_1(h(f), g) - F_1(f, h(g)) \subseteq \{-C(f)\} \cap \{-C(g)\}$, $\forall f, g \in K$, (iv) $F_1(f, g)$ is hemicontinuous with respect to f and continuous with respect to g, (v) $F_1(f, g)$ is Q-function with respect to g,

(4) (i) $F_2(h(f), f) = \{0\}$, $\forall f \in K$, (ii) $F_2(f, g)$ is continuous with respect to f, (iii) $F_2(f, g)$ is Q-function with respect to g,

(5) the set-valued map $W : K \to 2^Y$, defined by $W(f) = Y \setminus \{-intC(f)\}$, $\forall f \in K$, is upper semicontinuous on K.

(6) *Furthermore, suppose that there exists a non-empty, compact and convex subset B of K, such that for each $f \in K \setminus B$, there exists $g \in B$ such that*

$$F_1(h(f), g) + F_2(h(f), g) \subseteq -intC(f).$$

Then there exists $f^* \in K$ such that $F_1(h(f^*), g) + F_2(h(f^*), g) \nsubseteq -intC(f^*), \ \forall g \in K.$

For the proof of above theorem, we need the following two propositions for which the hypotheses remain the same as in Theorem 3.2

Proposition 3.3. *$\exists \ f^* \in B$ such that $F_2(h(f^*), g) - F_1(g, h(f^*)) \nsubseteq -intC(f^*), \ \forall g \in K.$*

Proof. Consider the set $S(g) = \{f \in B : F_2(h(f), g) - F_1(g, h(f)) \nsubseteq -intC(f)\}, \ \forall g \in K$. Then for any $g \in K$, $S(g)$ is closed. Infact, let f_α be a net in $S(g)$ such that $f_\alpha \to f$. Then $f \in B$ (since B is compact) and

$$F_2(h(f_\alpha), g) - F_1(g, h(f_\alpha)) \nsubseteq -intC(f_\alpha),$$

i.e

$$F_2(h(f_\alpha), g) - F_1(g, h(f_\alpha)) \subseteq Y \setminus \{-intC(f_\alpha)\} = W(f_\alpha).$$

Since h is continuous, $F_1(f, g)$ is continuous with respect to g and $F_2(f, g)$ is continuous with respect to f, we have

$$F_2(h(f_\alpha), g) - F_1(g, h(f_\alpha)) \to F_2(h(f), g) - F_1(g, h(f)).$$

The upper semicontinuity of set valued mapping W implies that $F_2(h(f), g) - F_1(g, h(f)) \subseteq W(f)$ and so $F_2(h(f), g) - F_1(g, h(f)) \nsubseteq -intC(f)$. Thus $f \in S(g)$ and so $S(g)$ is closed.
Now we prove that S is a KKM-map. Suppose that there exists a finite subset $\{f_1, f_2, \cdots, f_n\}$ of B and $\lambda_i \geq 0, \ i = 1, 2, \cdots, n$, with $\sum_{i=1}^n \lambda_i = 1$, such that

$$\bar{f} = \sum_{i=1}^{n} \lambda_i f_i \nsubseteq \bigcup_{j=1}^{n} S(f_j).$$

Then

$$F_2(h(\bar{f}), f_j) - F_1(f_j, h(\bar{f})) \subseteq -intC(\bar{f}), j = 1, 2, \cdots, n. \tag{3.1}$$

It follows from assumptions(1) and (3)(ii) that

$$F_1(f_j, h(\bar{f})) + F_1(h(\bar{f}), f_j) \subseteq \left\{-C(f_j)\right\} \cap \left\{-C(h(\bar{f}))\right\} \subseteq -C(\bar{f}). \tag{3.2}$$

By adding (3.1)and (3.2), we have

$$F_2(h(\bar{f}), f_j) + F_1(h(\bar{f}), f_j) \subseteq -intC(\bar{f}) - C(\bar{f}) \subseteq -intC(\bar{f}), \ j = 1, 2, \cdots, n.$$

Since $C(\bar{f})$ is the convex cone, we have

$$\sum_{j=1}^{n} F_2(h(\bar{f}), f_j) + \sum_{j=1}^{n} F_1(h(\bar{f}), f_j) \subseteq -intC(\bar{f}). \tag{3.3}$$

Since $F_1(f, g)$ and $F_2(f, g)$ are Q-function with respect to g, then $F_1(f, g) + F_2(f, g)$ is also Q-function with respect to g and hence

$$F_1(h(\bar{f}), \bar{f}) - \sum_{j=1}^{n} F_1(h(\bar{f}), f_j) + F_2(h(\bar{f}), \bar{f}) - \sum_{j=1}^{n} F_2(h(\bar{f}), f_j) \subseteq Q \subseteq -C(\bar{f}). \tag{3.4}$$

From (3.3) and (3.4), we have

$$F_1(h(\bar{f}),\bar{f}) + F_2(h(\bar{f}),\bar{f}) \subseteq -intC(\bar{f}) - C(\bar{f}) \subseteq -intC(\bar{f}),$$

a contradiction with

$$F_1(h(\bar{f}),\bar{f}) = F_2(h(\bar{f}),\bar{f}) = 0.$$

Thus S is a KKM-map. Since $S(g)$ is contained in a compact set B, by Lemma 2.5, there exists $f^* \in B$ such that

$$F_2(h(f^*),g) - F_1(g,h(f^*)) \nsubseteq -intC(f^*), \quad \forall g \in K.$$

□

Proposition 3.4. *The following statements are equivalent:*

(1) $\exists\, f^* \in B : F_2(h(f^*),g) - F_2(g,h(f^*)) \nsubseteq -intC(f^*), \forall g \in K,$

(2) $\exists\, f^* \in B : F_1(h(f^*),g) + F_2(h(f^*),g) \nsubseteq -intC(f^*), \quad \forall g \in K.$

Proof. (2) $\implies$ (1): Let (2) holds. Then there exists $f^* \in B$ such that

$$F_1(h(f^*),g) + F_2(h(f^*),g)) \nsubseteq -intC(f^*), \forall g \in K$$

From assumptions (1) and (3)(ii), we have

$$F_1(h(f^*),g) + F_1(g,h(f^*)) \subseteq \{-C(g)\} \cap \{-C(h(f^*)\} \subseteq -C(f^*)$$

By Lemma (2.6)(ii), there exists $f^* \in B$ such that

$$F_2(h(f^*),g) - F_1(g,h(f^*)) \nsubseteq -intC(f^*), \forall g \in K.$$

(1) $\Longrightarrow$ (2): Let (1) holds. Then there exists $f^* \in B$ such that

$$F_2(h(f^*),g) - F_1(g,h(f^*)) \nsubseteq -intC(f^*), \forall g \in K.$$

$$\text{Let } \; f_t = tg + (1-t)f^* \subseteq K, 0 < t \leq 1.$$

Since $C(f^*)$ is the convex cone, then

$$tF_1(h(f_t),g) - (1-t)F_1(f_t,h(f^*)) - tF_1(h(f_t),g)) + (1-t)F_2(h(f^*),f_t) \nsubseteq -intC(f^*). \tag{3.5}$$

Since $F_1(f,g)$ is Q-function with respect to g and $F_1(h(f_t),f_t)) = \{0\}$, we have

$$-tF_1(h(f_t),g) - (1-t)F_1(h(f_t),f^*) \subseteq Q = \bigcap_{g\in K}\{-C(g)\} \subseteq -C(f^*). \tag{3.6}$$

By assumption(3)(iii), we have

$$F_1(h(f_t),f^*) - F_1(f_t,h(f^*)) \subseteq \{-C(f^*)\} \cap \{-C(f_t)\} \subseteq -C(f^*).$$

Since $C(f^*)$ is the convex cone, it follows that

$$(1-t)F_1(h(f_t),f^*) - (1-t)F_1(f_t,h(f^*)) \subseteq -C(f^*) \tag{3.7}$$

By adding (3.6) and (3.7),we have

$$-tF_1(h(f_t),g) - (1-t)F_1(f_t,h(f^*)) \subseteq -C(f^*) - C(f^*) \subseteq -C(f^*). \tag{3.8}$$

Then (3.5),(3.8), and Lemma 2.6(iii) imply that

$$tF_1(h(f_t), g) + (1-t)F_2(h(f^*), f_t) \nsubseteq -intC(f^*). \tag{3.9}$$

Since $F_2(f,g)$ is Q-function with respect to g and $F_2(h(f^*), f^*) = \{0\}$, we have

$$F_2(h(f^*), f_t) - tF_2(h(f^*), g) \subseteq Q = \cap_{g\in K}\{-C(g)\} \subseteq -C(f^*).$$

Since $C(f^*)$ is the convex cone, we have

$$(1-t)F_2(h(f^*), f_t) - t(1-t)F_2(h(f^*), g) \subseteq -C(f^*). \tag{3.10}$$

Now (3.9), (3.10) and Lemma (2.6)(iv) imply that

$$tF_1(h(f_t), g) + t(1-t)F_2(h(f^*), g) \nsubseteq -intC(f^*).$$

Dividing by t, we have

$$F_1(h(f_t), g) + (1-t)F_2(h(f^*), g) \nsubseteq -intC(f^*).$$

$$\text{that is} \qquad F_1(h(f_t), g) + (1-t)F_2(h(f^*), g) \subseteq W(f^*).$$

Letting $t \to 0$ and hence $f_t \to f^*$. Since $W(f^*)$ is closed, h is affine and $F_1(f,g)$ is hemicontinuous with respect to f, we have

$$F_1(h(f^*), g) + F_2(h(f^*), g) \subseteq W(f^*)$$

$$\text{and thus } F_1(h(f^*), g) + F_2(h(f^*), g) \nsubseteq -intC(f^*).$$

□

Proof of theorem Let $\{g_1, g_2, \cdots, g_n\}$ be a finite subset of K and $D = conv[B \cup \{g_1, g_2, \cdots, g_n\}]$. Then D is a compact and convex subset of K. Then by Proposition 3.3, there exists $f^* \in D$ such that

$$F_2(h(f^*), g) - F_1(g, h(f^*)) \nsubseteq -intC(f^*),\ \forall g \in K.$$

In particular,

$$F_2(h(f^*), g_i) - F_1(g_i, h(f^*) \nsubseteq -intC(f^*),\ \ i = 1, 2, \cdots, n.$$

Hence every finite subfamily of the family of closed sets

$$T(g) = \{f \in D : F_2(h(f), g) - F_1(g, h(f)) \nsubseteq -intC(f)\}, \forall g \in K.$$

has non empty intersection. Since D is compact, we have

$$\bigcap_{g\in K} T(g) \neq \phi.$$

From Proposition 3.4, we have

$$\bigcap_{g\in K} S(g) \neq \phi.$$

Thus there exists $f^* \in D$ such that

$$F_1(h(f^*), g) + F_2(h(f^*), g) \nsubseteq -intC(f^*), \forall g \in K.$$

From assumption (6), we have $f^* \in B$ such that

$$F_1(h(f^*), g) + F_2(h(f^*), g) \nsubseteq -intC(f^*), \forall g \in K.$$

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