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Systems of Implicit Fractional Fuzzy Differential Equations with Nonlocal Conditions

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Abstract. In this paper, two types of fixed point theorems are employed to study the solvability of nonlocal problem for implicit fuzzy fractional differential systems under Caputo gH-fractional differentiability in the framework of generalized metric spaces. First of all, we extend Krasnoselskii's fixed point theorem to the vector version in the generalized metric space of fuzzy numbers. Under the Lipschitz conditions, we use Perov's fixed point theorem to prove the global existence of the unique mild fuzzy solution in both types (i) and (ii). When the nonlinearity terms are not Lipschitz, we combine Perov's fixed point theorem with vector version of Krasnoselskii's fixed point theorem to prove the existence of mild fuzzy solutions. Based on the advantage of vector-valued metrics and convergent matrix, we attain some properties of mild fuzzy solutions such as the boundedness, the attractivity and the Ulam - Hyers stability. Finally, a computational example is presented to demonstrate the effectivity of our main results.

1. Introduction

In many real world problems, there is often a need to interpret and solve the problems operating in the environment inherent uncertainties and vagueness. When engineers want to handle these disadvantages, they may use either stochastic and statistical models or fuzzy models, but stochastic and statistical uncertainty occur due to the natural randomness in the process. It is generally expressed by a probability density or frequency distribution function. For the estimation of the distribution, it requires sufficient information about the variables and parameters involved in it. On the other hand, fuzzy set theory refers to the uncertainty when we may have lack of knowledge or incomplete information about the variables and parameters. In general, science and engineering systems are governed by ordinary and partial differential equations [10, 13], but the type of differential equation (DEs) depends upon the applications, domains, complicated environments, the effect of coupling, and so on. As such, the complicacy needs to be handled by recently developed differential equations contained uncertainty or fuzziness [15, 17, 27].

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Since Agarwal et al. [1] introduced the concept of solutions for fuzzy fractional differential equations, this subject has become an important area of research due to its wide range of applications in various disciplines, namely physics, chemistry, biology, economics, chaotic theory and in engineering systems such as fluid mechanics, viscoelasticity, civil, mechanical, aerospace, and chemical and so forth, see [4, 5, 15, 19, 34]. Hence, this topic has been paid more and more attention from many scientists and mathematicians (see [2, 16, 18, 28, 35, 36] for therein). In the flow of development, scientists have proposed many techniques to solve the analysis as well as numerical solutions of fractional fuzzy differential equations, see [6–8, 20, 25, 26, 29]. In the differential equations subject, much attentions have been given to different types of problems with nonlocal conditions. These conditions were used to describe of motion phenomena with better effect than the classical conditions, see [14, 21, 25, 30].

Motivation by aforesaid, in this paper we study the global existence and some properties of solutions to the following nonlocal problem for implicit fuzzy fractional differential systems

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^{q}_{j}x(t) = g_1(t, x(t), y(t)) + h_1(t, {}^{C}_{gH}\mathcal{D}^{q}_{j}x(t), {}^{C}_{gH}\mathcal{D}^{q}_{j}y(t)) \\ {}^{C}_{gH}\mathcal{D}^{q}_{j}y(t) = g_2(t, x(t), y(t)) + h_2(t, {}^{C}_{gH}\mathcal{D}^{q}_{j}x(t), {}^{C}_{gH}\mathcal{D}^{q}_{j}y(t)) \end{cases} \quad t \in J_\infty = [0, \infty) \tag{1}$$

subject to nonlocal conditions

$$\begin{cases} x(0) + \sum\limits_{k\in J_1} a_k x(t_k) = \sum\limits_{k\in J_2} a_k x(t_k) \\ y(0) + \sum\limits_{k\in Q_1} \tilde{a}_k y(t_k) = \sum\limits_{k\in Q_2} \tilde{a}_k y(t_k) \\ J_1 \cup J_2 = Q_1 \cup Q_2 = \{1, 2, ..., m\} \\ J_1 \cap J_2 = Q_1 \cap Q_2 = \emptyset \end{cases} \tag{2}$$

where ${}^{C}_{gH}\mathcal{D}^{q}_{j}$ $(j = 1, 2)$ are the Caputo gH-differentiability of order $q \in [0, 1]$ defined in Definition 3.2 in [23], $g_i, h_i \in C(J_\infty \times E_c \times E_c, E_c)$ are fuzzy-valued continuous mappings.

The main contributions in this paper are three folds:

1. Our considered model is more general since it is a combination of three types of differential equations: implicit DEs, fractional DEs and set-valued DEs. Boundary conditions are divided to a vast of form, not restricted to periodic behavior or local conditions.

2. A standard technique for solving nonlocal problems bases on the transformation of the problem into a fixed point problem with suitable integral type operator. Then, a fixed point theorem will guarantee the existence of solutions. In our model, by applying suitable setting, nonlocal problem (1) − (2) is transformed into a fixed point problem of an operator T, which can be written as a sum of two operators G and H. Next, we consider fixed point problem in proportion to different hypotheses of forcing functions.

   2.a In the first case with the assumption that g_i, h_i all satisfy global Lipschitz conditions, thank to some auxiliary lemmas and approximation method, operator T is brought into Lipschitz-like form with a respective convergent to zero matrix. Then, by applying Perov's fixed point theorem, the unique global existence of the problem is attained.

   2.b However, for weaker hypotheses of forcing functions, that only requires Lipschitz condition of one part of nonlinearities and another part is considered under linearity assumptions, this becomes a complex problem since topological fixed point theorems, which essentially base on compactness, can not be applied for this case. This situation can be relaxed by using Krasnoselskii's fixed point theorem to analyze operator T into a sum of a generalized contraction and a completely continuous mapping. Krasnoselskii's theorem (see [31]), which is considered as a combination of generalized contraction principle and Schauder's fixed point theorem in Banach space, has become an important result for nonlinear analysis with a large number of applications. In our paper, we extend Krasnoselskii's fixed point theorem to fuzzy-valued functions metric space without linearity in structure and then we apply to study the solvability of nonlocal problem (1) – (2).

3. Some quantities properties of mild fuzzy solutions of nonlocal problem (1) – (2) are attained such as bounded fuzzy solutions, decay fuzzy solutions, attractivity set. These help us describe the asymptotic behavior of fuzzy solutions on infinite time. Moreover, this paper initiates Ulam - Hyers stability notions for implicit fractional fuzzy differential system with nonlocal conditions, which have not been studied before.

The paper is organized as follows: In Section 2, an extension of Krasnoselskii's fixed point theorem is stated and proved. In addition, we review some notions of fuzzy numbers and fuzzy metric space. After revisiting some fundamentals of Caputo gH-derivatives of fuzzy-valued functions, Section 3 focuses on stating the nonlocal problem for implicit fuzzy fractional differential system, constructing the hypotheses for the problem and presenting our main results about the solvability and some qualitative properties of solutions. For more clarity, in Section 4, an application example is given to demonstrate these results. Finally, conclusion with some helps of Appendix are in two last Sections 5 and 6.

2. Krasnoselskii's fixed point theorem in generalized metric space

2.1. *Generalized metric space of fuzzy numbers*

We will introduce some notions and necessary preliminaries used throughout the paper. For more details, see in previous works [21–25].

Let $(X, \lesssim_X)$ be an ordered set and X^2 be the 2−time Cartesian product of X. Consider $x = (x_1, x_2), y = (y_1, y_2) \in X^2$, denote $\lesssim_{X^2}$ by an order relation in X^2. Here, $x \lesssim_{X^2} y$ if and only if $x_1 \lesssim_X y_1$ and $x_2 \lesssim_X y_2$. The inverse order relation $\gtrsim_{X^2}$ is defined similarly. For $r = (r_1, r_2)$, $s = (s_1, s_2) \in X^2$, we define $\max\{r, s\} := \{\max\{r_1, s_1\}, \max\{r_2, s_2\}\}$.

We call X is a generalized metric space (in the sense of Perov) if there exists a vector-valued mapping $d : X \times X \to \mathbb{R}^2_+$ such that

(i) $d(u, v) \gtrsim_{\mathbb{R}^2} 0_{\mathbb{R}^2}$ for all $u,\ v \in X$ and $d(u, v) = 0_{\mathbb{R}^2} \Rightarrow u = v$, where $0_{\mathbb{R}^2} = (0, 0)$;

(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;

(iii) $d(u, w) \lesssim_{\mathbb{R}^2} d(u, v) + d(v, w)$ for all $u, v, w \in X$.

Let E be the space of fuzzy numbers, which are certain functions $u : \mathbb{R} \to [0, 1]$, satisfying normal, fuzzy convex, upper semi-continuous and compact supported. The α−level sets of fuzzy number u are defined by

$$[u]^\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\} & \text{if } 0 < \alpha \leq 1 \\ cl(supp\, u) & \text{if } \alpha = 0. \end{cases}$$

Fuzzy number u has a nice property that its α−level sets can be represented in intervals. Namely, parametric form $[u]^\alpha = [u^-_\alpha, u^+_\alpha]$, where u^-_α, u^+_α denote the left-hand endpoint and the right-hand endpoint, respectively. Denote

$$d_\infty(u, v) = \sup_{0\leq\alpha\leq1} d_H\left(\left[u\right]^\alpha, \left[v\right]^\alpha\right),\ u, v \in E$$

by the supremum metric in E. It is obvious that (E, d_∞) is a complete metric space (see [11]).

The sum and scalar multiplication of fuzzy numbers in E are defined via their level sets. If there exists $w \in E$ such that $u = v + w$, we call $w = u \ominus v$ the Hukuhara difference of u and v. For $u, v \in E$, the generalized Hukuhara difference (gH-difference for short) of u and v, denoted by $u \ominus_{gH} v$, is defined by the element $w \in E$ such that

$$u \ominus_{gH} v \Leftrightarrow u = v + w \text{ or } v = u + (-1)w.$$

Denote E_c by the space of fuzzy numbers $u \in E$ such that mapping $\alpha \mapsto [u]^\alpha$ is continuous with respect to Hausdorff metric on $[0,1]$. In addition, for $J_\infty = [0,\infty)$, we denote $C(J_\infty, E_c)$ by the set of all continuous functions $f : J_\infty \to E_c$ with metric

$$D(u,\overline{u}) = \max\{D_\tau(u,\overline{u}); \tilde{D}_\tau(u,\overline{u})\},\ \tau \in (0,1),$$

where $D_\tau(u,\overline{u}) = \sup\limits_{[0,\tau]} d_\infty(u(t),\overline{u}(t))$ and $\tilde{D}_\tau(u,\overline{u}) = \sup\limits_{[\tau,\infty)}\{d_\infty(u(t),\overline{u}(t))e^{-\theta(t-\eta)}\}$, η and θ are given positive real numbers satisfying $0 < \eta < \tau < 1,\ \theta > 0$.

Define vector-valued metric on $C(J_\infty, E_c) \times C(J_\infty, E_c)$

$$\rho(w,\tilde{w}) = \begin{bmatrix} D(x,\tilde{x}) \\ D(y,\tilde{y}) \end{bmatrix},\ w = (x,y),\ \tilde{w} = (\tilde{x},\tilde{y}) \in C(J_\infty, E_c) \times C(J_\infty, E_c).$$

Similar to Lemma 2.3 in [25], $(C(J_\infty, E_c) \times C(J_\infty, E_c), \rho)$ is a generalized semi-linear Banach space having cancelation property, i.e., $\rho(cu + w, cv + w) = |c|\rho(u,v)$ for all $u,v,w \in C(J_\infty, E_c) \times C(J_\infty, E_c)$.

2.2. *Krasnoselskii's fixed point Theorem in fuzzy-valued functions metric space*

A square matrix M with non-negative elements is said to be convergent to zero if $M^k \to 0$ as $k \to \infty$. The property of converging to zero of matrix M is equivalent to each of the following conditions (see Lemma 2 in [32])

(i) The eigenvalues of M are located inside the unit disc of the complex plane;

(ii) $I - M$ is nonsingular and $(I - M)^{-1}$ has non-negative elements.

Lemma 2.1. *Suppose that $f : C(J_\infty, E_c) \times C(J_\infty, E_c) \to C(J_\infty, E_c) \times C(J_\infty, E_c)$ is a generalized contraction, i.e there exists a convergent to zero matrix M such that*

$$\rho(f(u), f(v)) \le M\rho(u,v),\ \forall u,v \in C(J_\infty, E_c) \times C(J_\infty, E_c).$$

Then for each $w \in C(J_\infty, E_c) \times C(J_\infty, E_c)$, equation $z = f(z) + w$ has unique solution. Moreover, mapping $\varphi_f : C(J_\infty, E_c) \times C(J_\infty, E_c) \to C(J_\infty, E_c) \times C(J_\infty, E_c)$, defined by $\varphi_f(w) = z$, is continuous.

Proof. For each $w \in C(J_\infty, E_c) \times C(J_\infty, E_c)$, consider mapping

$$h : C(J_\infty, E_c) \times C(J_\infty, E_c) \to C(J_\infty, E_c) \times C(J_\infty, E_c)$$

defined by $h(\nu) = w + f(\nu)$ for all $\nu \in C(J_\infty, E_c) \times C(J_\infty, E_c)$. The inequality

$$\rho(h(\nu), h(\overline{\nu})) \le \rho(f(\nu), f(\overline{\nu})) \le M\rho(\nu,\overline{\nu})$$

implies that h is a generalized contraction. From Perov's theorem (Theorem 1, [32]), h has a unique fixed point z^*, i.e for each $w \in C(J_\infty, E_c) \times C(J_\infty, E_c)$, there always exists a unique point $z^* \in C(J_\infty, E_c) \times C(J_\infty, E_c)$ such that $w + f(z^*) = z^*$, that follows equation $z = f(z) + w$ has unique solution. Thus, mapping φ_f, given by $\varphi_f(w) = z$, is well-defined.

For $w, w' \in C(J_\infty, E_c) \times C(J_\infty, E_c)$ and $\nu = \varphi_f(w)$, $\nu' = \varphi_f(w')$, we have

$$\begin{aligned}\rho(\varphi_f(w), \varphi_f(w')) &= \rho(w + f(\nu), w' + f(\nu')) \\ &\le \rho(w,w') + \rho(f(\nu), f(\nu')) \\ &\le \rho(w,w') + M\rho(\nu,\nu').\end{aligned}$$

Thus, $(I - M)\rho(\varphi_f(w), \varphi_f(w')) \le \rho(w,w')$. Since M is a convergent to zero matrix, we have

$$\rho(h(\nu), h(\overline{\nu})) \le \rho(f(\nu), f(\overline{\nu})) \le M\rho(\nu,\overline{\nu})$$

Consequently, for all $\epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$, we can choose $\delta = (I - M)\epsilon$. Thus for all $w,\ w' \in C(J_\infty, E_c) \times C(J_\infty, E_c)$ satisfying $\rho(w,w') < \delta$, we have $\rho(\varphi_f(w), \varphi_f(w')) < \epsilon$. It implies that φ_f is continuous. $\square$

Traditional Krasnoselskii's fixed point theorem operates in Banach space. This base space has linearity property. In the next step, we will extend Krasnoselskii's fixed point theorem to the semi-linear Banach space $C(J_\infty, E_c) \times C(J_\infty, E_c)$, that is lack of linearity, as follows.

Theorem 2.2 (Krasnoselskii's fixed point theorem in semi-linear Banach space)**.** *Let B be a nonempty closed, bounded, convex subset of $C(J_\infty, E_c) \times C(J_\infty, E_c)$ and operator $T : B \to C(J_\infty, E_c) \times C(J_\infty, E_c)$ satisfying*

(i) $T = G + H$ with G is a completely continuous operator and H is a generalized contraction with a convergent to zero matrix M.

(ii) $G(B) + H(B) \subset B$.

Then T has at least one fixed point in B.

Proof. Firstly, for arbitrary $v \in B$, we consider an operator $h_v : B \to B$ defined by $h_v(w) = H(w) + G(v)$. From (ii), h_v is well-defined. In another hand, for $w, w' \in B$

$$\rho(h_v(w), h_v(w')) \leq \rho(H(w), H(w')) \leq M\rho(w, w').$$

Thus h_v is a generalized contraction. By using Perov's theorem, we imply that there exists a unique $\overline{w}_v \in B$ such that $h_v(\overline{w}_v) = \overline{w}_v$, or equivalently $\overline{w}_v = H(\overline{w}_v) + G(v)$.

Consider an operator $c : B \to B$ such that $c(v) = \overline{w}_v$. Then $c(.)$ satisfies equation $c(v) = H(c(v)) + G(v)$ for all $v \in B$. From Lemma 2.1, the mapping $\varphi_H : C(J_\infty, E_c) \times C(J_\infty, E_c) \to C(J_\infty, E_c) \times C(J_\infty, E_c)$ given by $\varphi_H(w) = z$, with $v = H(z) + w$, is well-defined and continuous. We can rewrite $c(B) = \varphi_H(G(B))$. Since $G(B)$ is relatively compact, $c(B)$ is relatively compact, too. By using Theorem 3.4 in [3], there exists $v \in B$ such that $c(v) = v$, i.e $H(v) + G(v) = v$. The proof is complete. □

3. Main results

3.1. Caputo gH-derivatives of fuzzy-valued functions

Let $f : J_\infty \longrightarrow E$, $t_0 \in J_\infty$ and h be a number such that $t_0 + h \in J_\infty$. If there exists an element $Df(t_0) \in E$ such that

$$Df(t_0) = \lim_{h \to 0} \frac{1}{h}\Big[f(t_0 + h) \ominus_{gH} f(t_0)\Big]$$

then f is called generalized Hukuhara differentiable (gH-differentiable) at t_0. Denote $C^1(J_\infty, E)$ by the space of all fuzzy-valued continuously gH-differentiable functions defined on J_∞.

Definition 3.1 ([12], Definition 26)**.** *Assume that $f : J_\infty \to E$ is gH-differentiable at $t_0 \in J_\infty$ and $[f(t)]^\alpha = [f_\alpha^-(t), f_\alpha^+(t)]$ for all $\alpha \in [0, 1]$, $t \in J_\infty$. We say that*

(i) f is (i)-gH-differentiable at t_0 if $[Df(t_0)]^\alpha = [(f_\alpha^-)'(t_0), (f_\alpha^+)'(t_0)]$ for all $\alpha \in [0, 1]$. Denote by $D_1 f(t_0)$ the (i)-gH-derivative of f at t_0.

(ii) f is (ii)-gH-differentiable at t_0 if $[Df(t_0)]^\alpha = [(f_\alpha^+)'(t_0), (f_\alpha^-)'(t_0)]$ for all $\alpha \in [0, 1]$. Denote by $D_2 f(t_0)$ the (ii)-gH-derivative of f at t_0.

We recall the left-side mixed Riemann-Liouville fractional integral of order $q \in (0, 1]$ for real valued function $f : J_\infty \to \mathbb{R}$ by

$$^{RL}I_{0^+}^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s) ds, \ t \in J_\infty. \tag{3}$$

Definition 3.2. *Let $q \in (0,1]$ and $u \in C^1(J_\infty, E)$, $[u(t)]^\alpha = [u^-_\alpha(t), u^+_\alpha(t)]$ for all $t \in J_\infty$ and $\alpha \in [0,1]$. The mixed Riemann - Liouville fractional integral of order q for fuzzy-valued function u, denoted by*

$$ {}^{RL}_{F}\mathcal{I}^q_{0^+} u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s) ds, \tag{4}$$

is defined by level sets as follows

$$[{}^{RL}_{F}\mathcal{I}^q_{0^+} u(t)]^\alpha = [{}^{RL}I^q_{0^+} u^-_\alpha(t), {}^{RL}I^q_{0^+} u^+_\alpha(t)], \ \ t \in J_\infty, \ \alpha \in [0,1],$$

provided that ${}^{RL}I^q_{0^+} u^+_\alpha(t), {}^{RL}I^q_{0^+} u^-_\alpha(t)$ are defined by (3).

Definition 3.3. *The Caputo gH-derivative of order $q \in [0,1)$ of $u \in C^1(J_\infty, E)$ in type j $(j = 1,2)$ is defined by*

$${}^{C}_{gH}\mathcal{D}^q_j u(t) = {}^{RL}_{F}\mathcal{I}^{1-q}_{0^+} D_j u(t), \ t \in J_\infty,$$

provided that the expression on the right-hand side is defined.

Denote $\mathscr{C}^q_j(J_\infty, E_c) = \{u : J_\infty \to E_c \,|\, {}^{C}_{gH}\mathcal{D}^q_j u(\cdot)$ exists and it is continuous$\}$.

3.2. The nonlocal problem for implicit fuzzy fractional differential systems under Caputo gH derivatives

In this paper, we consider the nonlocal problem for implicit fuzzy fractional differential system (1) - (2) where $x, y \in \mathscr{C}^q_j(J_\infty, E_c)$, a_k, $\tilde{a}_k$ are positive numbers satisfying inequalities

$$0 < \sum_{k\in J_2} a_k - \sum_{k \in J_1} a_k < 1; \ 0 < \sum_{k \in Q_2} \tilde{a}_k - \sum_{k\in Q_1} \tilde{a}_k < 1. \tag{5}$$

Denote $a = (1 + \sum\limits_{k\in J_1} a_k - \sum\limits_{k\in J_2} a_k)^{-1}$, $\tilde{a} = (1 + \sum\limits_{k\in Q_1} \tilde{a}_k - \sum\limits_{k \in Q_2} \tilde{a}_k)^{-1}$. By changing of variables

$$u(t) = {}^{C}_{gH}\mathcal{D}^q_j x(t), \ v(t) = {}^{C}_{gH}\mathcal{D}^q_j y(t), \tag{6}$$

we have following lemma.

Lemma 3.4. *By setting*

$$\begin{aligned}
A_1(u) &= a\Big[\sum_{k\in J_2} a_k {}^{RL}_{F}\mathcal{I}^q_{0^+} u(t_k) \ominus \sum_{k \in J_1} a_k {}^{RL}_{F}\mathcal{I}^q_{0^+} u(t_k)\Big] \\
A_2(u) &= -a\Big[\sum_{k\in J_1} a_k {}^{RL}_{F}\mathcal{I}^q_{0^+} u(t_k) \ominus \sum_{k \in J_2} a_k {}^{RL}_{F}\mathcal{I}^q_{0^+} u(t_k)\Big] \\
\tilde{A}_1(v) &= \tilde{a}\Big[\sum_{k\in Q_2} \tilde{a}_k {}^{RL}_{F}\mathcal{I}^q_{0^+} v(t_k) \ominus \sum_{k \in Q_1} \tilde{a}_k {}^{RL}_{F}\mathcal{I}^q_{0^+} v(t_k)\Big] \\
\tilde{A}_2(v) &= -\tilde{a}\Big[\sum_{k\in Q_1} \tilde{a}_k {}^{RL}_{F}\mathcal{I}^q_{0^+} v(t_k) \ominus \sum_{k \in Q_2} \tilde{a}_k {}^{RL}_{F}\mathcal{I}^q_{0^+} v(t_k)\Big],
\end{aligned}$$

we have following assertions

1. *Assume that $x, y \in \mathscr{C}^q_1(J_\infty, E_c)$ satisfy nonlocal conditions (2). Then we can present $x(t)$, $y(t)$ in the following forms*

$$\begin{cases} x(t) := \mathcal{F}_1[u](t) = A_1(u) + {}^{RL}_{F}\mathcal{I}^q_{0^+} u(t) \\ y(t) := \tilde{\mathcal{F}}_1[v](t) = \tilde{A}_1(v) + {}^{RL}_{F}\mathcal{I}^q_{0^+} v(t) \end{cases} \quad t \in J_\infty. \tag{7}$$

2. *Assume that $x, y \in \mathscr{C}_2^q(J_\infty, E_c)$ satisfy nonlocal conditions* (2). *Then we can present $x(t)$, $y(t)$ in the following forms*

$$\begin{cases} x(t) := \mathcal{F}_2[u](t) = A_2(u) \ominus (-1)_F^{RL}\mathcal{I}_{0^+}^q u(t) \\ y(t) := \tilde{\mathcal{F}}_2[v](t) = \tilde{A}_2(v) \ominus (-1)_F^{RL}\mathcal{I}_{0^+}^q v(t) \end{cases} \quad t \in J_\infty. \tag{8}$$

Proof. We will prove the first equation of (7) and (8), the second equation is proved similarly. Indeed, by taking integral both side of ${}_{gH}^{C}\mathcal{D}_j^q x(t) = u(t)$, we have

$$\begin{aligned} &{}_F^{RL}\mathcal{I}_{0^+}^q\Big({}_F^{RL}\mathcal{I}_{0^+}^{1-q} D_j x(t)\Big) = {}_F^{RL}\mathcal{I}_{0^+}^q u(t) \\ \Rightarrow &\int_0^t D_j x(s)ds = {}_F^{RL}\mathcal{I}_{0^+}^q u(t) \\ \Rightarrow & x(t) \ominus_{gH} x(0) = {}_F^{RL}\mathcal{I}_{0^+}^q u(t) \end{aligned}$$

Case 1: If $x, y \in \mathscr{C}_1^q(J_\infty, E_c)$ then we have

$$x(t) = x(0) + {}_F^{RL}\mathcal{I}_{0^+}^q u(t). \tag{9}$$

Thus $x(t_k) = x(0) + {}_F^{RL}\mathcal{I}_{0^+}^q u(t_k)$. Substitute this equation into (2), we obtain

$$x(0) + \sum_{k\in J_1} a_k\Big[x(0) + {}_F^{RL}\mathcal{I}_{0^+}^q u(t_k)\Big] = \sum_{k\in J_2} a_k\Big[x(0) + {}_F^{RL}\mathcal{I}_{0^+}^q u(t_k)\Big]. \tag{10}$$

From definition of Hukuhara difference, inequalities (5) and equality (10), one can see that the differences $\sum_{k\in J_2} a_k {}_F^{RL}\mathcal{I}_{0^+}^q u(t_k) \ominus \sum_{k\in J_1} a_k {}_F^{RL}\mathcal{I}_{0^+}^q u(t_k)$ and $\Big[1 + \sum_{k\in J_1} a_k\Big]x(0) \ominus \sum_{k\in J_2} a_k x(0)$ exist and $\Big[1 + \sum_{k\in J_1} a_k - \sum_{k\in J_2} a_k\Big]x(0) = \sum_{k\in J_2} a_k {}_F^{RL}\mathcal{I}_{0^+}^q u(t_k) \ominus \sum_{k\in J_1} a_k {}_F^{RL}\mathcal{I}_{0^+}^q u(t_k)$. Thus, we obtain

$$x(0) = a\Big[\sum_{k\in J_2} a_k {}_F^{RL}\mathcal{I}_{0^+}^q u(t_k) \ominus \sum_{k\in J_1} a_k {}_F^{RL}\mathcal{I}_{0^+}^q u(t_k)\Big].$$

Then, substituting $x(0)$ into equation (9), the first equation of (7) is attained.

Case 2: If $x, y \in \mathscr{C}_2^q(J_\infty, E_c)$ then we have

$$x(t) = x(0) \ominus (-1)_F^{RL}\mathcal{I}_{0^+}^q u(t). \tag{11}$$

Substitute $x(t_k) = x(0) \ominus (-1)_F^{RL}\mathcal{I}_{0^+}^q u(t_k)$ into (2), we obtain

$$x(0) + \sum_{k\in J_1} a_k\Big[x(0) \ominus (-1)_F^{RL}\mathcal{I}_{0^+}^q u(t_k)\Big] = \sum_{k\in J_2} a_k\Big[x(0) \ominus (-1)_F^{RL}\mathcal{I}_{0^+}^q u(t_k)\Big].$$

By using analogous arguments as in **Case 1** associated with the property $(-1)(u \ominus v) = (-1)u \ominus (-1)v$ (see Lemma 2.3 in [22]), we obtain

$$\Big[1 + \sum_{k\in J_1} a_k - \sum_{k\in J_2} a_k\Big]x(0) = (-1)\Big[\sum_{k\in J_1} a_k {}_F^{RL}\mathcal{I}_{0^+}^q u(t_k) \ominus \sum_{k\in J_2} a_k {}_F^{RL}\mathcal{I}_{0^+}^q u(t_k)\Big].$$

It implies that $x(0) = -a\Big[\sum_{k\in J_1} a_k {}_F^{RL}\mathcal{I}_{0^+}^q u(t_k) \ominus \sum_{k\in J_2} a_k {}_F^{RL}\mathcal{I}_{0^+}^q u(t_k)\Big] = A_2(u)$. Therefore, by substituting the expression of $x(0)$ into (11), we obtain the first equation of (8). □

Now we can rewrite equations (1) in following forms

$$\begin{cases} u(t) = g_1(t, \mathcal{F}_1[u](t), \tilde{\mathcal{F}}_1[v](t)) + h_1(t, u(t), v(t)) \\ v(t) = g_2(t, \mathcal{F}_1[u](t), \tilde{\mathcal{F}}_1[v](t)) + h_2(t, u(t), v(t)) \end{cases} \quad \text{for } j = 1 \tag{12}$$

and

$$\begin{cases} u(t) = g_1(t, \mathcal{F}_2[u](t), \tilde{\mathcal{F}}_2[v](t)) + h_1(t, u(t), v(t)) \\ v(t) = g_2(t, \mathcal{F}_2[u](t), \tilde{\mathcal{F}}_2[v](t)) + h_2(t, u(t), v(t)) \end{cases} \quad \text{for } j = 2. \tag{13}$$

Remark 3.5. *When $x,\ y \in \mathscr{C}^q_j(J_\infty, E_c)$, $j = 1, 2$, then u, v belong to $C(J_\infty, E_c)$. Moreover, for each fuzzy solution $(u, v) \in C(J_\infty, E_c) \times C(J_\infty, E_c)$ of system (12) (or (13)), from formula (7) (or (8)), we can determine $x, y \in C(J_\infty, E_c)$ through u, v. We have following definition*

Definition 3.6. *A pair of functions $(x, y) \in C(J_\infty, E_c) \times C(J_\infty, E_c)$ is called*

(i) A mild fuzzy solution in type (i) *of nonlocal problem (1) - (2) if it satisfies the integral system (7) and (u, v) satisfies the system (12);*

(ii) A mild fuzzy solution in type (ii) *of nonlocal problem (1) - (2) if it satisfies the integral system (8) and (u, v) satisfies the system (13).*

3.3. Hypotheses

Following hypotheses will be used throughout this paper.

Hypothesis 1 (h_i satisfy global Lipschitz conditions). *Assume that there exist non-negative real numbers $\overline{b}_i, \overline{c}_i$ $(i = 1, 2)$ such that*

$$d_\infty(h_i(t, u, v), h_i(t, \overline{u}, \overline{v})) \le \overline{b}_i d_\infty(u, \overline{u}) + \overline{c}_i d_\infty(v, \overline{v}), \tag{14}$$

for all $(u, v), (\overline{u}, \overline{v}) \in E_c \times E_c$ and $t \in J_\infty$.

Hypothesis 2 (g_i satisfy global Lipschitz conditions). *Assume that there exist non-negative real numbers σ_{i1}, σ_{i2} $(i = 1, 2)$ such that*

$$d_\infty(g_i(t, \varphi, \psi), g_i(t, \overline{\varphi}, \overline{\psi})) \le \sigma_{i1} d_\infty(\varphi, \overline{\varphi}) + \sigma_{i2} d_\infty(\psi, \overline{\psi}), \tag{15}$$

for $(\varphi, \psi), (\overline{\varphi}, \overline{\psi}) \in E_c \times E_c$ and $t \in J_\infty$.

Hypothesis 3 (the growth of g_i are at most linear). *Assume that the growth of $g_i(t, \varphi, \psi)$ $(i = 1, 2)$ with respect to φ and ψ is at most linear on each of the two subintervals $[0, t_m]$ and $[t_m, \infty)$, that is there exists non-negative real numbers $b_i, c_i, e_i, B_i, C_i, E_i$ such that*

$$d_\infty(g_i(t, \varphi, \psi), \hat{0}) \le \begin{cases} b_i d_\infty(\varphi, \hat{0}) + c_i d_\infty(\psi, \hat{0}) + e_i & t \in [0, t_m]; \\ \left(B_i d_\infty(\varphi, \hat{0}) + C_i d_\infty(\psi, \hat{0}) + E_i\right) e^{-2\theta(t-\eta)} & t \in [t_m, \infty), \end{cases} \tag{16}$$

for all $(\varphi, \psi), (\overline{\varphi}, \overline{\psi}) \in E_c \times E_c$ and $\hat{0}$ is zero fuzzy number.

Denote $\hat{C}(J_\infty, E_c)$ by the space of all $(u, v) \in C(J_\infty, E_c) \times C(J_\infty, E_c)$ such that for all $t \in J_\infty$, the following Hukuhara differences exist

$$\begin{cases} X(u)(t) := A_2(u) \ominus (-1)^{RL}_F \mathcal{I}^q_{0^+} u(t) \\ Y(v)(t) := \tilde{A}_2(v) \ominus (-1)^{RL}_F \mathcal{I}^q_{0^+} v(t) \end{cases} \tag{17}$$

where A_2, $\tilde{A}_2$ are defined in Lemma 3.4.

Hypothesis 4. *Assume that $\hat{C}(J_\infty, E_c) \neq \emptyset$ and if $(u, v) \in \hat{C}(J_\infty, E_c)$ then there exist Hukuhara differences*

$$\begin{cases} A_2(X(u)) \ominus (-1)^{RL}_{F}\mathcal{I}^q_{0^+}X(u)(t) \\ \tilde{A}_2(Y(v)) \ominus (-1)^{RL}_{F}\mathcal{I}^q_{0^+}Y(v)(t) \end{cases} \quad \text{for all } t \in J_\infty. \tag{18}$$

Remark 3.7. *For simplicity in presentation, these notations are introduced*

$$\beta_1 = \frac{t_m^q}{\Gamma(q+1)}\Big(1 + a\sum_{k=1}^{m} a_k\Big) \text{ and } \beta_2 = \frac{t_m^q}{\Gamma(q+1)}\Big(1 + \tilde{a}\sum_{k=1}^{m} \tilde{a}_k\Big)$$

$$M_0 = \begin{bmatrix} b_1\beta_1 & c_1\beta_2 \\ b_2\beta_1 & c_2\beta_2 \end{bmatrix}, \quad M_1 = \begin{bmatrix} \bar{b}_1 & \bar{c}_1 \\ \bar{b}_2 & \bar{c}_2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} \sigma_{11}\beta_1 & \sigma_{12}\beta_2 \\ \sigma_{21}\beta_1 & \sigma_{22}\beta_2 \end{bmatrix} \text{ and } M_\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$$\overline{M}_k = \begin{bmatrix} aa_k & 0 \\ 0 & \tilde{a}\tilde{a}_k \end{bmatrix} \ (k = \overline{1, m}), \quad {}^{RL}\mathcal{I}^q_{0^+}\begin{pmatrix} \xi(t) \\ \nu(t) \end{pmatrix} = \frac{1}{\Gamma(q)}\begin{bmatrix} \int_0^t (t-s)^{q-1}\xi(s)ds \\ \int_0^t (t-s)^{q-1}\nu(s)ds \end{bmatrix}$$

On the other hand, if $x(u(t), v(t))$, $y(u(t), v(t))$ are defined by (7) or (8) then they can be rewritten in following compact form

$$G_1(u, v)(t) = g_1(t, x(u(t), v(t)), y(u(t), v(t))) \text{ and } H_1(u, v)(t) = h_1(t, u(t), v(t))$$
$$G_2(u, v)(t) = g_2(t, x(u(t), v(t)), y(u(t), v(t))) \text{ and } H_2(u, v)(t) = h_2(t, u(t), v(t))$$

Therefore, system (12) and (13) can be rewritten as follows

$$\begin{cases} u(t) = G_1(u, v)(t) + H_1(u, v)(t) \\ v(t) = G_2(u, v)(t) + H_2(u, v)(t). \end{cases} \tag{19}$$

We will transfer system (19) into fixed point problem by consider operators

$$T_1(u, v)(t) = G_1(u, v)(t) + H_1(u, v)(t), \ T_2(u, v)(t) = G_2(u, v)(t) + H_2(u, v)(t). \tag{20}$$

For $j = 1$, we consider operator

$$T_{(\mathrm{i})}(u, v)(t) := (T_1(u, v)(t), T_2(u, v)(t)),$$

where $x(u(t), v(t))$, $y(u(t), v(t))$ are given in (7).
For $j = 2$, we consider operator

$$T_{(\mathrm{ii})}(u, v)(t) := (T_1(u, v)(t), T_2(u, v)(t)),$$

where $x(u(t), v(t))$, $y(u(t), v(t))$ are given in (8).

3.4. Global existence of mild fuzzy solutions

Theorem 3.8. *Assume that following assumptions hold*

1. *g_i $(i = 1, 2)$ are continuous functions satisfying hypothesis (**H2**);*
2. *h_i $(i = 1, 2)$ satisfy hypothesis (**H1**) and $h_i(t, \hat{0}, \hat{0}) = \hat{0}$, where $\hat{0}$ is zero fuzzy number;*
3. *The spectral radius of the matrix $M_1 + M_2$ is less than one.*

Then the problem (1) − (2) *has a unique mild fuzzy solution in type* (i) *in $C(J_\infty, E_c) \times C(J_\infty, E_c)$. In addition, if hypothesis (**H4**) is satisfied, it ensures the global unique existence of mild fuzzy solution in type* (ii) *of the problem.*

Proof. For all $t \in J_\infty$ and $(u,v),(\overline{u},\overline{v}) \in C(J_\infty, E_c)\times C(J_\infty,E_c)$ defined by (19), we have

$$\begin{bmatrix} d_\infty(T_1(u,v)(t), T_1(\overline{u},\overline{v})(t)) \\ d_\infty(T_2(u,v)(t), T_2(\overline{u},\overline{v})(t)) \end{bmatrix} \lesssim_{\mathbb{R}^2} \begin{bmatrix} d_\infty(G_1(u,v)(t), G_1(\overline{u},\overline{v})(t)) \\ d_\infty(G_2(u,v)(t), G_2(\overline{u},\overline{v})(t)) \end{bmatrix} + \begin{bmatrix} d_\infty(H_1(u,v)(t), H_1(\overline{u},\overline{v})(t)) \\ d_\infty(H_2(u,v)(t), H_2(\overline{u},\overline{v})(t)) \end{bmatrix}$$

When $j = 1$, by applying Lemma 6.2 and Lemma 6.7, one receives

$$\begin{aligned} \begin{bmatrix} D(T_1(u,v), T_1(\overline{u},\overline{v})) \\ D(T_1(u,v), T_1(\overline{u},\overline{v})) \end{bmatrix} &\lesssim_{\mathbb{R}^2} \begin{bmatrix} \sigma_{11}\beta_1 & \sigma_{12}\beta_2 \\ \sigma_{21}\beta_1 & \sigma_{22}\beta_2 \end{bmatrix} \begin{bmatrix} D(u,\overline{u}) \\ D(v,\overline{v}) \end{bmatrix} + \begin{bmatrix} \overline{b}_1 & \overline{c}_1 \\ \overline{b}_1 & \overline{c}_2 \end{bmatrix} \begin{bmatrix} D(u,\overline{u}) \\ D(v,\overline{v}) \end{bmatrix} \\ &= \begin{bmatrix} \sigma_{11}\beta_1 + \overline{b}_1 & \sigma_{12}\beta_2 + \overline{c}_1 \\ \sigma_{21}\beta_1 + \overline{b}_2 & \sigma_{22}\beta_2 + \overline{c}_2 \end{bmatrix} \begin{bmatrix} D(u,\overline{u}) \\ D(v,\overline{v}) \end{bmatrix}. \end{aligned} \tag{21}$$

Since the assumption matrix $M_1 + M_2$ has spectral radius less than 1, it implies that $T_{(i)}$ is a generalized contraction. Therefore, by applying Perov's theorem, the unique existence of fixed point of operator $T_{(i)}$ is shown.

When $j = 2$, $(x(t), y(t))$ is defined by (8). From Hypothesis **(H4)**, we have $T_{(ii)}$ is well-defined. Thanks to Lemma 2.2 in [22] and analogous arguments in Lemma 6.2 and Lemma 6.7 (see **Appendix**), we receive again the generalized contractive property of operator $T_{(ii)}$. This follows the unique global existence of mild fuzzy solution in type (ii) of the problem. The proof is complete. □

Remark 3.9. *The proof of Theorem 3.8 is based on Perov's fixed point theorem with Lipschitz property of mapping h_i and g_i, $(i = 1,2)$. If these conditions are released, in concretely if the hypothesis **(H2)** is not satisfied, mappings g_i $(i = 1,2)$ are only bounded at most linear, then the use of Perov's theorem to prove the global existence of the problem becomes useless. In this case, by using our extended result of Krasnoselskii theorem in Section 2, we receive a result on the global existence of mild fuzzy solutions as follows.*

Theorem 3.10. *Assume that*

1. *g_i $(i = 1,2)$ are compact functions satisfying hypothesis (**H3**);*
2. *h_i $(i = 1,2)$ are functions which satisfy hypothesis (**H1**) and $h_i(t,\hat{0},\hat{0}) = \hat{0}$;*
3. *The following inequality is fulfilled*

$$b_1\beta_1 + \overline{b}_1 + c_2\beta_2 + \overline{c}_2 < \min\{2, 1 + (b_1\beta_1 + \overline{b}_1)(c_2\beta_2 + \overline{c}_2) - (b_2\beta_1 + \overline{b}_2)(c_1\beta_2 + \overline{c}_1)\}.$$

Then the problem (1)-(2) has at least one mild fuzzy solution in type (i). *Moreover, if hypothesis (**H4**) also holds for all $t \in J_\infty$, this problem has at least one mild fuzzy solution in type* (ii).

Proof. Denote $G(u,v) = \begin{bmatrix} G_1(u,v) & G_2(u,v) \end{bmatrix}^T$, $(u,v) \in C(J_\infty,E_c)\times C(J_\infty,E_c)$. Firstly, when $j = 1$, we consider the global existence of mild fuzzy solution in type (i). The proof will be given via three steps.
Step 1. G is completely continuous.

Let B be a bounded subset of $C(J_\infty,E_c)\times C(J_\infty,E_c)$, i.e there exist R_1, $R_2 > 0$ such that $B \subseteq \{(u,v) \in C(J_\infty,E_c) : D(u,\hat{0}) \le R_1;\ D(v,\hat{0}) \le R_2\}$. We will apply Theorem 4.1 in [33] to prove that $G(B)$ is relatively compact in $C(J_\infty,E_c)\times C(J_\infty,E_c)$, which is equivalent to

- $G(B) = (G_1(B), G_2(B))$ is equicontinuous on B,
- $G(B)(t)$ is relatively compact in $E_c \times E_c$ for $t \in J_\infty$.

Since g_i $(i = 1,2)$ are continuous on $J_\infty \times E_c \times E_c$ for all $\epsilon > 0$, there exists $\delta_0 > 0$ such that $|t - t'| + d_\infty(x(t), x(t')) + d_\infty(y(t), y(t')) < \delta_0$ then

$$d_\infty(g_1(t, x(t), y(t)), g_1(t', x(t'), y(t'))) < \epsilon. \tag{22}$$

For all $(u,v)\in B$, let t, $t'\in J_\infty$ $(t<t')$ such that $|t-t'|<\delta$, then

$$\begin{aligned} d_\infty(x(t),x(t')) &= d_\infty(A_1(u)+{}^{RL}_{F}\mathcal{I}^q_{0^+}u(t), A_1(u)+{}^{RL}_{F}\mathcal{I}^q_{0^+}u(t')) \le d_\infty({}^{RL}_{F}\mathcal{I}^q_{0^+}u(t), {}^{RL}_{F}\mathcal{I}^q_{0^+}u(t')) \\ &\le \frac{1}{\Gamma(q)}\Big(d_\infty\Big(\int_0^t (t-s)^{q-1}u(s)ds, \int_0^t (t'-s)^{q-1}u(s)ds\Big) + d_\infty\Big(\int_t^{t'}(t'-s)^{q-1}u(s)ds, \hat{0}\Big)\Big) \\ &\le \frac{1}{\Gamma(q)}d_\infty\Big(\int_0^t [(t'-s)^{q-1}-(t-s)^{q-1}]u(s)ds, \hat{0}\Big) + \int_t^{t'}(t'-s)^{q-1}d_\infty(u(s),\hat{0})ds. \end{aligned} \tag{23}$$

Case 1: For t, $t'\in[0,t_m]$, (23) becomes

$$\begin{aligned} \Gamma(q)d_\infty(x(t),x(t')) &\le D_{t_m}(u,\hat{0})\Big[\int_0^t [(t'-s)^{q-1}-(t-s)^{q-1}]ds + \int_t^{t'}(t'-s)^{q-1}ds\Big] \\ &\le \frac{R_1}{q}[t'^q - t^q - (t'-t)^q] + \frac{R_1}{q}(t'-t)^q = \frac{R_1}{q}(t'^q - t^q). \end{aligned}$$

Case 2: For t, $t'\in[t_m,\infty)$, (23) becomes

$$\begin{aligned} \Gamma(q)d_\infty(x(t),x(t')) &\le \tilde{D}_{t_m}(u,\hat{0})\Big(\int_0^t [(t'-s)^{q-1}-(t-s)^{q-1}]e^{\theta(s-\eta)}ds + \int_t^{t'}(t'-s)^{q-1}e^{\theta(s-\eta)}ds\Big) \\ &\le R_1\Big(\int_0^{t'}(t'-s)^{q-1}e^{\theta(s-\eta)}ds - \int_0^t (t-s)^{q-1}e^{\theta(s-\eta)}ds\Big) \\ &\le R_1 e^{-\theta\eta}\Big[t'^q E_{1,1+q}(\theta t') - t^q E_{1,1+q}(\theta t)\Big], \end{aligned}$$

where $E_{1,1+q}(.)$ is the Mittag-Leffler function.
Case 3: For case $t\in[0,t_m]$ and $t'\in[t_m,\infty)$,

$$\begin{aligned} \Gamma(q)d_\infty(x(t),x(t')) &\le \int_0^t [(t'-s)^{q-1}-(t-s)^{q-1}]d_\infty(u(s),\hat{0})ds + \int_t^{t_m}(t'-s)^{q-1}d_\infty(u(s),\hat{0})ds \\ &\quad + \int_{t_m}^{t'}(t'-s)^{q-1}d_\infty(u(s),\hat{0})e^{-\theta(s-\eta)}e^{\theta(s-\eta)}ds \\ &\le \frac{D_{t_m}(u,\hat{0})}{q}[t'^q - t^q - (t'-t_m)^q] + \tilde{D}_{t_m}(u,\hat{0})\int_{t_m}^{t'}(t'-s)^{q-1}e^{\theta(s-\eta)}ds \\ &\le \frac{R_1}{q}[t'^q - t^q - (t'-t_m)^q] + R_1 e^{\theta(t_m-\eta)}(t'-t_m)^q E_{1,q+1}(\theta t' - \theta t_m) \\ &\le R_1\Big(\frac{t'^q - t^q}{q} + e^{\theta(t_m-\eta)}(t'-t)^q E_{1,q+1}(\theta t' - \theta t)\Big). \end{aligned}$$

It follows from case 1 to 3 that if t, t' are closed, then $x(t)$, $x(t')$ are also closed. It means that there exists $\delta_1>0$ such that if $|t-t'|<\delta_1$ then $d_\infty(x(t),x(t'))<\frac{\delta_0}{3}$. Similarly, there also exists $\delta_2>0$ such that if $|t-t'|<\delta_2$ then $d_\infty(y(t),y(t'))<\frac{\delta_0}{3}$.

Put $\delta=\min\{\delta_1,\delta_2,\frac{\delta_0}{3}\}$. Thanks to (22) for all $\epsilon>0$, there exists $\delta>0$ such that for all $(x,y)\in B$ and t, $t'\in J_\infty$ satisfying $|t-t'|<\delta$ then

$$d_\infty(g_i(t,x(t),y(t)), g_i(t',x(t'),y(t')))<\epsilon.$$

Therefore, $G_i(B)$ $(i=1,2)$ are equicontinuous on B.

According to Lemma 4.1 in [33], $G(B)(t)$ is relatively compact in $E_c\times E_c$ if $G(B)(t)$ is compact-supported subset of $E_c\times E_c$ and level-equicontinuous for all $t\in J_\infty$. Since differences $\sum\limits_{k\in J_2} a_k{}^{RL}_{F}\mathcal{I}^q_{0^+}u(t_k)\ominus\sum\limits_{k\in J_1} a_k{}^{RL}_{F}\mathcal{I}^q_{0^+}u(t_k)$,

$\sum\limits_{k\in Q_2} \tilde{a}_k {}_F^{RL}\mathcal{I}_{0^+}^q v(t_k) \ominus \sum\limits_{k\in Q_1} \tilde{a}_k {}_F^{RL}\mathcal{I}_{0^+}^q v(t_k)$ exist, there are compact sets K_1, $K_2 \subset \mathbb{R}$ such that $\Big[\sum\limits_{k\in J_2} a_k {}_F^{RL}\mathcal{I}_{0^+}^q u(t_k) \ominus \sum\limits_{k\in J_1} a_k {}_F^{RL}\mathcal{I}_{0^+}^q u(t_k)\Big]^0 \subset K_1$ and $\Big[\sum\limits_{k\in Q_2} \tilde{a}_k {}_F^{RL}\mathcal{I}_{0^+}^q v(t_k) \ominus \sum\limits_{k\in Q_1} \tilde{a}_k {}_F^{RL}\mathcal{I}_{0^+}^q v(t_k)\Big]^0 \subset K_2$. In addition, for each $t \in J_\infty$, there exist compact sets K_{t3}, K_{t4} satisfying $[{}_F^{RL}\mathcal{I}_{0^+}^q u(t)]^0 \subset K_{t3}$, $[{}_F^{RL}\mathcal{I}_{0^+}^q v(t)]^0 \subset K_{t4}$. Thus, we have

$$[x(t)]^0 = \frac{a}{\Gamma(q)}\Big[\sum_{k\in J_2} a_k {}_F^{RL}\mathcal{I}_{0^+}^q u(t_k) \ominus \sum_{k\in J_1} a_k {}_F^{RL}\mathcal{I}_{0^+}^q u(t_k)\Big]^0 + [{}_F^{RL}\mathcal{I}_{0^+}^q u(t)]^0 \subset \frac{a}{\Gamma(q)}K_1 + K_{t3}.$$

$$[y(t)]^0 = \frac{\tilde{a}}{\Gamma(q)}\Big[\sum_{k\in Q_2} \tilde{a}_k {}_F^{RL}\mathcal{I}_{0^+}^q v(t_k) \ominus \sum_{k\in Q_1} \tilde{a}_k {}_F^{RL}\mathcal{I}_{0^+}^q v(t_k)\Big]^0 + [{}_F^{RL}\mathcal{I}_{0^+}^q v(t)]^0 \subset \frac{\tilde{a}}{\Gamma(q)}K_2 + K_{t4}.$$

This follows $\Big[G_i(u,v)(t)\Big]^0 \subset g_i\Big(J_\infty, \lambda_1 K_1 + \lambda_3 K_{t3}, \lambda_2 K_2 + \lambda_4 K_{t4}\Big)$ $(i = 1,2)$, which are compact sets. Hence, $G_i(B)(t)$, $i = 1,2$, are compact-supported subsets of E_c. The second condition can be inferred from hypotheses g_i $(i = 1,2)$ are compact functions.

Step 2. Let $(u,v), (\overline{u},\overline{v}) \in C(J_\infty, E_c) \times C(J_\infty, E_c)$. According to Lemma 6.2, we have

$$D(H_1(u,v), H_1(\overline{u},\overline{v})) \le \overline{b}_1 D(u,\overline{u}) + \overline{c}_1 D(v,\overline{v});$$
$$D(H_2(u,v), H_2(\overline{u},\overline{v})) \le \overline{b}_2 D(u,\overline{u}) + \overline{c}_2 D(v,\overline{v}).$$

It can be rewritten in following form

$$\begin{bmatrix} D(H_1(u,v), H_1(\overline{u},\overline{v})) \\ D(H_2(u,v), H_2(\overline{u},\overline{v})) \end{bmatrix} \lesssim_{\mathbb{R}^2} M_1 \begin{bmatrix} D(u,\overline{u}) \\ D(v,\overline{v}) \end{bmatrix}. \tag{24}$$

From assumption $(M_0 + M_1)^k \to 0$ as $k \to \infty$, we deduce that $M_1^k \to 0$ as $k \to \infty$. Hence H is generalized contractive in Perov's sense.

Step 3. In this step, we will prove that there exists a nonempty, bounded, closed and convex subset B of $C(J_\infty, E_c) \times C(J_\infty, E_c)$ such that $G(\mathrm{B}) + H(\mathrm{B}) \subset \mathrm{B}$. From Lemma 6.6, on each of two intervals $[0, t_m]$ and $[t_m, \infty)$, we have

$$D_{t_m}(G_1(u,v), \hat{0}) \le b_1\beta_1 D_{t_m}(u,\hat{0}) + c_1\beta_2 D_{t_m}(v,\hat{0}) + e_1, \tag{25}$$

$$\begin{aligned} \tilde{D}_{t_m}(G_1(u,v), \hat{0}) &\le \Big(B_1\beta_1 D_{t_m}(u,\hat{0}) + C_1\beta_2 D_{t_m}(v,\hat{0}) + E_1\Big)e^{-3\theta(t-\eta)} \\ &\quad + \Big(B_1\tilde{Q}(\theta,q)\tilde{D}_{t_m}(u,\hat{0}) + C_1\tilde{Q}(\theta,q)\tilde{D}_{t_m}(v,\hat{0})\Big)e^{-2\theta(t-\eta)} \\ &\le \Big(B_1\beta_1 D_{t_m}(u,\hat{0}) + C_1\beta_2 D_{t_m}(v,\hat{0}) + E_1\Big)e^{-3\theta(t_m-\eta)} \\ &\quad + \Big(B_1\tilde{Q}(\theta,q)\tilde{D}_{t_m}(u,\hat{0}) + C_1\tilde{Q}(\theta,q)\tilde{D}_{t_m}(v,\hat{0})\Big)e^{-2\theta(t_m-\eta)} \end{aligned} \tag{26}$$

where $\tilde{Q}(\theta,q) = \dfrac{Q(\theta,q)}{\Gamma(q+1)}$, $Q(\theta,q)$ is defined in Remark 6.8.

Now by taking advantage from the special choice of metric $\tilde{D}_{t_m}$, concretely from the choice of $\eta < t_m$, we can choose $\theta > 0$ large enough such that

$$B_1 e^{-\theta(t_m-\eta)} \le b_1; \qquad C_1 e^{-\theta(t_m-\eta)} \le c_1; \qquad E_1 e^{-\theta(t-\eta)} \le e_1. \tag{27}$$

By combining (26) with (27), we get

$$\begin{aligned} \tilde{D}_{t_m}(G_1(u,v), \hat{0}) &\le b_1\beta_1 D_{t_m}(u,\hat{0}) + c_1\beta_2 D_{t_m}(v,\hat{0}) + e_1 \\ &\quad + \frac{B_1\tilde{Q}(\theta,q)\tilde{D}_{t_m}(u,\hat{0})}{e^{2\theta(t_m-\eta)}} + \frac{C_1\tilde{Q}(\theta,q)\tilde{D}_{t_m}(v,\hat{0})}{e^{2\theta(t_m-\eta)}}. \end{aligned} \tag{28}$$

From (25) and (28), it implies that

$$D(G_1(u,v),\hat{0}) \le \left(b_1\beta_1 + \frac{B_1\tilde{Q}(\theta,q)}{e^{2\theta(t_m-\eta)}}\right)D(u,\hat{0}) + \left(c_1\beta_2 + \frac{C_1\tilde{Q}(\theta,q)}{e^{2\theta(t_m-\eta)}}\right)D(v,\hat{0}) + e_1. \tag{29}$$

By similar estimation, we also get

$$D(G_2(u,v),\hat{0}) \le \left(b_2\beta_1 + \frac{B_2\tilde{Q}(\theta,q)}{e^{2\theta(t_m-\eta)}}\right)D(u,\hat{0}) + \left(c_2\beta_2 + \frac{C_2\tilde{Q}(\theta,q)}{e^{2\theta(t_m-\eta)}}\right)D(v,\hat{0}) + e_2. \tag{30}$$

The inequalities (29), (30) can be put under the vectorial form

$$\begin{bmatrix} D(G_1(u,v),\hat{0}) \\ D(G_2(u,v),\hat{0}) \end{bmatrix} \lesssim_{\mathbb{R}^2} M_\theta \begin{bmatrix} D(u,\hat{0}) \\ D(v,\hat{0}) \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{31}$$

where

$$M_\theta = M_0 + M_Q = \begin{bmatrix} b_1\beta_1 & c_1\beta_2 \\ b_2\beta_1 & c_2\beta_2 \end{bmatrix} + \begin{bmatrix} \frac{B_1\tilde{Q}(\theta,q)}{e^{2\theta(t_m-\eta)}} & \frac{C_1\tilde{Q}(\theta,q)}{e^{2\theta(t_m-\eta)}} \\ \frac{B_2\tilde{Q}(\theta,q)}{e^{2\theta(t_m-\eta)}} & \frac{C_2\tilde{Q}(\theta,q)}{e^{2\theta(t_m-\eta)}} \end{bmatrix}. \tag{32}$$

On another hand, from (24) we deduce that

$$\begin{bmatrix} D(H_1(u,v),\hat{0}) \\ D(H_2(u,v),\hat{0}) \end{bmatrix} \lesssim_{\mathbb{R}^2} \begin{bmatrix} D(H_1(u,v),H_1(\hat{0},\hat{0})) \\ D(H_2(u,v),H_2(\hat{0},\hat{0})) \end{bmatrix} + \begin{bmatrix} D(H_1(\hat{0},\hat{0}),\hat{0}) \\ D(H_2(\hat{0},\hat{0}),\hat{0}) \end{bmatrix} \lesssim_{\mathbb{R}^2} M_1 \begin{bmatrix} D(u,\hat{0}) \\ D(v,\hat{0}) \end{bmatrix} \tag{33}$$

for each $(u,v) \in C(J_\infty, E_c) \times C(J_\infty, E_c)$.

Put $\mathrm{B} = \{(u,v) \in C(J_\infty,E_c) \times C(J_\infty,E_c) \mid D(u,\hat{0}) \le R_1;\ D(v,\hat{0}) \le R_2\}$, with $R_1 > 0$, $R_2 > 0$. According to estimations (31) and (33), the condition $G(\mathrm{B}) + H(\mathrm{B}) \subset \mathrm{B}$ is guaranteed by inequality $\left(M_\theta + M_1\right)R + E \le R$, $R = [R_1\ R_2]^T$, which is equivalent to

$$E \le \left(I - M_\theta - M_1\right)R. \tag{34}$$

Since $M_\theta + M_1 = M_0 + M_Q + M_1$, $\rho(M_0 + M_1) < 1$ and the entries of M_Q are small as desired for large enough $\theta > 0$, we can choose θ such that $\rho(M_\theta + M_1) < 1$. According to property of matrix with spectral radius less than 1, the inequality (34) is equivalent to $R \ge \left(I - M_\theta - M_1\right)^{-1}E$. This leads to there exist $R_1, R_2 > 0$ for which the inwardness condition $G(\mathrm{B}) + H(\mathrm{B}) \subset \mathrm{B}$ is satisfied. Applying Krasnoselskii's fixed point theorem in Section 2, we receive the existence of at least one fixed point of operator $T_{(\mathrm{i})}$ in B, i.e nonlocal problem (1) - (2) has at least one (i) - mild fuzzy solution.

In addition, based on analogous arguments used in the proof of Lemma 5.2 in [22], we can check that $(\hat{C}(J_\infty,E_c),\rho)$ is a generalized semi-linear Banach space. Thus, if hypothesis (**H4**) holds, the operator $T_{(\mathrm{ii})} : \hat{C}(J_\infty,E_c) \times \hat{C}(J_\infty,E_c) \to \hat{C}(J_\infty,E_c) \times \hat{C}(J_\infty,E_c)$ is well-defined, where X, $Y : \hat{C}(J_\infty,E_c) \to \hat{C}(J_\infty,E_c)$ are defined in (17). By applying analogous argument, the existence of (ii) - mild fuzzy solution of nonlocal problem is also attained. □

3.5. The behavior of fuzzy solutions

In this subsection, assume that all hypotheses of Theorem 3.10 are fulfilled, that guarantees the global existence of at least one mild fuzzy solution in type (i) (or type (ii)).

Let $0_{\mathbb{R}^2} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and ρ_0 be a vector-valued metric defined by

$$\rho_0(w(t),\overline{w}(t)) = \begin{bmatrix} d_\infty(x(t),\overline{x}(t)) \\ d_\infty(y(t),\overline{y}(t)) \end{bmatrix},$$

where $w(t) = \begin{bmatrix} x(t) & y(t) \end{bmatrix}^T, \overline{w}(t) = \begin{bmatrix} \overline{x}(t) & \overline{y}(t) \end{bmatrix}^T \in E_c \times E_c$. Our main aim is about the boundedness, attractivity and stability of mild fuzzy solutions in the sense of type (i) and (ii) on $J_\infty = [0, \infty)$.

Indeed, let us assume that $z(t) = \begin{bmatrix} u(t) & v(t) \end{bmatrix}^T, \overline{z}(t) = \begin{bmatrix} \overline{u}(t) & \overline{v}(t) \end{bmatrix}^T$ satisfying system (12) (or (13)). Then, we have the following results

Theorem 3.11. *If z and $\overline{z}$ are fuzzy solutions of the system (12) (or (13)) then the following limit holds*

$$\lim_{t\to\infty} \rho_0(z(t), \overline{z}(t)) = 0_{\mathbb{R}^2}.$$

Proof. Assume that $z(t) = \begin{bmatrix} u(t) & v(t) \end{bmatrix}^T, \overline{z}(t) = \begin{bmatrix} \overline{u}(t) & \overline{v}(t) \end{bmatrix}^T$ are fuzzy solutions of system (12) (or (13)). Then, for each $t \in [t_m, \infty)$, from the formula (19) and Hypothesis **(H1)**, we have

$$\begin{bmatrix} d_\infty(u(t), \overline{u}(t)) \\ d_\infty(v(t), \overline{v}(t)) \end{bmatrix} \lesssim_{\mathbb{R}^2} (I - M_1)^{-1} \begin{bmatrix} d_\infty(G_1(u, v)(t), G_1(\overline{u}, \overline{v})(t)) \\ d_\infty(G_2(u, v)(t), G_2(\overline{u}, \overline{v})(t)) \end{bmatrix}. \tag{35}$$

In addition, since $x(u(t), v(t))$ and $y(u(t), v(t))$ are defined by (7) (or (8)) and g_i $(i = 1, 2)$ satisfy linearity assumptions (**H3**), the following inequality holds

$$\begin{aligned} d_\infty(G_1(u, v)(t), G_1(\overline{u}, \overline{v})(t)) &\le d_\infty(G_1(u, v)(t), \hat{0}) + d_\infty(G_1(\overline{u}, \overline{v})(t), \hat{0}) \\ &\le \Big(B_1(d_\infty(x(t), \hat{0}) + d_\infty(\overline{x}(t), \hat{0})) + C_1(d_\infty(y(t), \hat{0}) + d_\infty(\overline{y}(t), \hat{0})) + 2E_1\Big) e^{-2\theta(t-\eta)} \\ &\le 2B_1R_1 \left(\frac{\beta_1 t^q}{t_m^q e^{2\theta(t-\eta)}} + \frac{\tilde{Q}(\theta, q)}{e^{\theta(t-\eta)}} \right) + 2C_1R_2 \left(\frac{\tilde{\beta}_1 t^q}{e^{2\theta(t-\eta)} t_m^q} + \frac{\tilde{Q}(\theta, q)}{e^{\theta(t-\eta)}} \right) + \frac{2E_1}{e^{2\theta(t-\eta)}}. \end{aligned}$$

Since $e^{-2\theta(t-\eta)}$ can be as small as desired as t tends to $+\infty$, it follows that

$$\lim_{t\to\infty} d_\infty(G_1(u, v)(t), G_1(\overline{u}, \overline{v})(t)) = 0.$$

Similarly, we also obtain $\lim_{t\to\infty} d_\infty(G_2(u, v)(t), G_2(\overline{u}, \overline{v})(t)) = 0$. Thus, $\lim_{t\to\infty} \rho_0(z(t), \overline{z}(t)) = 0_{\mathbb{R}^2}$. The proof is complete. $\square$

For simplicity, we can denote zero fuzzy-valued function $\tilde{0} : \mathbb{R} \to E,\ t \mapsto \tilde{0}(t) = \hat{0}$, where $\hat{0}$ is zero fuzzy number and denote $\Theta = (\hat{0}, \hat{0}) \in E_c \times E_c$. Then, we have

Theorem 3.12. *The fuzzy solutions of system (12) (or (13)) converge to Θ. Moreover, the behaviors of mild fuzzy solutions in type* (i) *(or* (ii)*) of the nonlocal problem (1) - (2) are asymptotic to a set $\mathcal{E}_0$ as $t \to +\infty$.*

Proof. Assume that $z(t) = \begin{bmatrix} u(t) & v(t) \end{bmatrix}^T$ is a fuzzy solution of the system (12) (or (13)), i.e., $u(t)$, $v(t)$ satisfy system (19), we have

$$\begin{bmatrix} d_\infty(u(t), \hat{0}) \\ d_\infty(v(t), \hat{0}) \end{bmatrix} \lesssim_{\mathbb{R}^2} \begin{bmatrix} d_\infty(G_1(u, v)(t), \hat{0}) \\ d_\infty(G_2(u, v)(t), \hat{0}) \end{bmatrix} + \begin{bmatrix} d_\infty(H_1(u, v)(t), \hat{0}) \\ d_\infty(H_2(u, v)(t), \hat{0}) \end{bmatrix}$$

For each $t \in [t_m, \infty)$, from (27), (60) we can choose θ large enough such that

$$\begin{aligned} d_\infty(G_1(u, v)(t), \hat{0}) &\le \Big(B_1\beta_1 D_{t_m}(u, \hat{0}) + C_1\beta_2 D_{t_m}(v, \hat{0}) + E_1\Big) e^{-2\theta(t-\eta)} \\ &\quad + \Big(B_1\tilde{Q}(\theta, q)\tilde{D}_{t_m}(u, \hat{0}) + C_1\tilde{Q}(\theta, q)\tilde{D}_{t_m}(v, \hat{0})\Big) e^{-\theta(t-\eta)} \\ &\le b_1\beta_1 d_\infty(u(t), \hat{0}) + c_1\beta_1 d_\infty(v(t), \hat{0}) + E_1 e^{-2\theta(t-\eta)} \\ &\quad + \frac{B_1 Q(\theta, q)}{\Gamma(q+1)e^{\theta(t-\eta)}} d_\infty(u(t), \hat{0}) + \frac{C_1 Q(\theta, q)}{\Gamma(q+1)e^{\theta(t-\eta)}} d_\infty(v(t), \hat{0}) \\ &\le \left(b_1\beta_1 + \frac{B_1 Q(\theta, q)}{\Gamma(q+1)e^{\theta(t_m-\eta)}}\right) d_\infty(u(t), \hat{0}) \\ &\quad + \left(c_1\beta_1 + \frac{C_1 Q(\theta, q)}{\Gamma(q+1)e^{\theta(t_m-\eta)}}\right) d_\infty(v(t), \hat{0}) + E_1 e^{-2\theta(t-\eta)}. \end{aligned}$$

By using similar arguments and applying Corollary 6.3, the following estimation holds for all $t \geq t_m$

$$\begin{aligned}
\begin{bmatrix} d_\infty(u(t),\hat{0}) \\ d_\infty(v(t),\hat{0}) \end{bmatrix} &\lesssim_{\mathbb{R}^2} \begin{bmatrix} b_1\beta_1 + \frac{B_1 Q(\theta,q)}{\Gamma(q+1)e^{\theta(t-\eta)}} & c_1\beta_2 + \frac{C_1 Q(\theta,q)}{\Gamma(q+1)e^{\theta(t-\eta)}} \\ b_2\beta_1 + \frac{B_2 Q(\theta,q)}{\Gamma(q+1)e^{\theta(t-\eta)}} & c_2\beta_2 + \frac{C_2 Q(\theta,q)}{\Gamma(q+1)e^{\theta(t-\eta)}} \end{bmatrix} \begin{bmatrix} d_\infty(u(t),\hat{0}) \\ d_\infty(v(t),\hat{0}) \end{bmatrix} \\
&\quad + \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} \begin{bmatrix} d_\infty(u(t),\hat{0}) \\ d_\infty(v(t),\hat{0}) \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} e^{-2\theta(t-\eta)} \\
&\lesssim_{\mathbb{R}^2} \begin{bmatrix} b_1\beta_1 + \frac{B_1 Q(\theta,q)}{\Gamma(q+1)e^{\theta(t_m-\eta)}} & c_1\beta_2 + \frac{C_1 Q(\theta,q)}{\Gamma(q+1)e^{\theta(t_m-\eta)}} \\ b_2\beta_1 + \frac{B_2 Q(\theta,q)}{\Gamma(q+1)e^{\theta(t_m-\eta)}} & c_2\beta_2 + \frac{C_2 Q(\theta,q)}{\Gamma(q+1)e^{\theta(t_m-\eta)}} \end{bmatrix} \begin{bmatrix} d_\infty(u(t),\hat{0}) \\ d_\infty(v(t),\hat{0}) \end{bmatrix} \\
&\quad + \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} \begin{bmatrix} d_\infty(u(t),\hat{0}) \\ d_\infty(v(t),\hat{0}) \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} e^{-2\theta(t-\eta)} \\
&= \left(\tilde{M}_\theta + M_1\right) \begin{bmatrix} d_\infty(u(t),\hat{0}) \\ d_\infty(v(t),\hat{0}) \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} e^{-2\theta(t-\eta)}
\end{aligned}$$

where $\tilde{M}_\theta = \begin{bmatrix} b_1\beta_1 + \frac{B_1 \tilde{Q}(\theta,q)}{e^{\theta(t_m-\eta)}} & c_1\beta_2 + \frac{C_1 \tilde{Q}(\theta,q)}{e^{\theta(t_m-\eta)}} \\ b_2\beta_1 + \frac{B_2 \tilde{Q}(\theta,q)}{e^{\theta(t_m-\eta)}} & c_2\beta_2 + \frac{C_2 \tilde{Q}(\theta,q)}{e^{\theta(t_m-\eta)}} \end{bmatrix}$, $\tilde{Q}(\theta,q) = \dfrac{Q(\theta,q)}{\Gamma(q+1)}$ and $M_1 = \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix}$.

This inequality is equivalent to

$$\begin{bmatrix} d_\infty(u(t),\hat{0}) \\ d_\infty(v(t),\hat{0}) \end{bmatrix} \lesssim_{\mathbb{R}^2} \left(I - M_1 - \tilde{M}_\theta\right)^{-1} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} e^{-2\theta(t-\eta)} := \tilde{M}_{1,\theta}\, e^{-2\theta(t-\eta)}. \tag{36}$$

Since $0 < \eta < t_m$ and $\theta > 0$ can be chosen big enough, $e^{-2\theta(t_m-\eta)}$ can be as small as we desired as t tends to $+\infty$, that implies $\lim_{t\to\infty} \rho_0(z(t),\Theta) = 0_{\mathbb{R}^2}$.

On the other hand, since $u(t)$, $v(t)$ are continuous in closed interval $[0,t_m]$, there exist positive real numbers P_1, P_2 such that

$$d_\infty(u(t),\hat{0}) \leq P_1, \quad d_\infty(v(t),\hat{0}) \leq P_2 \quad \text{for all } t \in [0,t_m]. \tag{37}$$

Let $w(t) = \begin{bmatrix} x(t) & y(t) \end{bmatrix}^T$ be a mild fuzzy solution in type (i) (or (ii)) of the nonlocal problem (1) - (2). Then, from Lemma 3.4, we have

$$\begin{aligned}
\begin{bmatrix} d_\infty(x(t),\hat{0}) \\ d_\infty(y(t),\hat{0}) \end{bmatrix} &\lesssim_{\mathbb{R}^2} \begin{bmatrix} a\sum\limits_{k=1}^{m} a_k d_\infty\left({}^{RL}_{F}\mathcal{I}^q_{0_+} u(t_k),\hat{0}\right) \\ \tilde{a}\sum\limits_{k=1}^{m} \tilde{a}_k d_\infty\left({}^{RL}_{F}\mathcal{I}^q_{0_+} v(t_k),\hat{0}\right) \end{bmatrix} + \begin{bmatrix} d_\infty\left({}^{RL}_{F}\mathcal{I}^q_{0_+} u(t),\hat{0}\right) \\ d_\infty\left({}^{RL}_{F}\mathcal{I}^q_{0_+} v(t),\hat{0}\right) \end{bmatrix} \\
&\lesssim_{\mathbb{R}^2} \begin{bmatrix} a\sum\limits_{k=1}^{m} \dfrac{a_k}{\Gamma(q)} \displaystyle\int_0^{t_k} (t_k-s)^{q-1} d_\infty(u(s),\hat{0})ds \\ \tilde{a}\sum\limits_{k=1}^{m} \dfrac{\tilde{a}_k}{\Gamma(q)} \displaystyle\int_0^{t_k} (t_k-s)^{q-1} d_\infty(v(s),\hat{0})ds \end{bmatrix} + \frac{1}{\Gamma(q)} \begin{bmatrix} \displaystyle\int_0^{t} (t-s)^{q-1} d_\infty(u(s),\hat{0})ds \\ \displaystyle\int_0^{t} (t-s)^{q-1} d_\infty(v(s),\hat{0})ds \end{bmatrix} \\
&= \begin{bmatrix} a\sum\limits_{k=1}^{m} \dfrac{a_k t_k^q}{\Gamma(q+1)} P_1 \\ \tilde{a}\sum\limits_{k=1}^{m} \dfrac{\tilde{a}_k t_k^q}{\Gamma(q+1)} P_2 \end{bmatrix} + \frac{1}{\Gamma(q)} \begin{bmatrix} \theta_1 \displaystyle\int_0^{t} (t-s)^{q-1} e^{-2\theta(s-\eta)} ds \\ \theta_2 \displaystyle\int_0^{t} (t-s)^{q-1} e^{-2\theta(s-\eta)} ds \end{bmatrix} \\
&\leq \sum_{k=1}^{m} \frac{t_m^q}{\Gamma(q+1)} \begin{bmatrix} P_1 a a_k \\ P_2 \tilde{a}\tilde{a}_k \end{bmatrix} + \frac{1}{\Gamma(q)} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \int_0^t (t-s)^{q-1} e^{-2\theta(s-\eta)} ds
\end{aligned}$$

where P_1, P_2 are real numbers defined in (37) and $\tilde{M}_{1,\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ defined in (36).

In Lemma 6.10, if $\lambda = 2\theta$ is a big enough number such that $2\theta t_m > 1$ then we immediately obtain

$$\begin{aligned}
\begin{bmatrix} d_\infty(x(t),\hat{0}) \\ d_\infty(y(t),\hat{0}) \end{bmatrix} &\lesssim_{\mathbb{R}^2} \sum_{k=1}^{m} \frac{t_m^q}{\Gamma(q+1)} \overline{M}_k \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \frac{\tilde{M}_{1,\theta}}{\Gamma(q)} \left[\frac{e^{-2\theta(t-\eta)}}{2\theta} + e^{-2\theta(t-\eta-1)} \left(\frac{1}{q} - \frac{1}{2\theta} \right) \right] \\
&\lesssim_{\mathbb{R}^2} \sum_{k=1}^{m} \frac{t_m^q}{\Gamma(q+1)} \overline{M}_k \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \frac{\tilde{M}_{1,\theta}}{\Gamma(q)} \left[\frac{e^{2\theta\eta}}{2\theta} + e^{-2\theta(t-\eta-1)} \left(\frac{1}{q} - \frac{1}{2\theta} \right) \right] \\
&\lesssim_{\mathbb{R}^2} \left(\sum_{k=1}^{m} \frac{t_m^q}{\Gamma(q+1)} \overline{M}_k \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \frac{e^{2\theta\eta}}{2\theta} \frac{\tilde{M}_{1,\theta}}{\Gamma(q)} \right) + \left(\frac{1}{q} - \frac{1}{2\theta} \right) \frac{\tilde{M}_{1,\theta}}{\Gamma(q)} e^{-2\theta(t-\eta-1)}.
\end{aligned} \tag{38}$$

Because the first term in the right hand side of (38) is a constant vector while the second term tends to zero vector when $t \to \infty$, we imply that there exists a set $\mathcal{E}_0 \subset E \times E$ such that mild fuzzy solution $w(t) = \begin{bmatrix} x(t) & y(t) \end{bmatrix}^T$ of nonlocal problem (1) - (2) is asymptotic to $\mathcal{E}_0$ as $t \to \infty$. The proof is complete. □

Theorem 3.13. *All mild fuzzy solutions of nonlocal problem (1) - (2) are bounded.*

Proof. When $j = 1$, suppose that $(x, y) \in C(J_\infty, E_c) \times C(J_\infty, E_c)$ is a mild fuzzy solution in type (i) of nonlocal problem (1) - (2). Then,

$$\begin{aligned}
\begin{bmatrix} d_\infty(x(t),\hat{0}) \\ d_\infty(y(t),\hat{0}) \end{bmatrix} &\lesssim_{\mathbb{R}^2} \begin{bmatrix} a \sum\limits_{k=1}^{m} a_k d_\infty \left({}_F^{RL}\mathcal{I}_{0_+}^q u(t_k), \hat{0} \right) \\ \tilde{a} \sum\limits_{k=1}^{m} \tilde{a}_k d_\infty \left({}_F^{RL}\mathcal{I}_{0_+}^q v(t_k), \hat{0} \right) \end{bmatrix} + \begin{bmatrix} d_\infty \left({}_F^{RL}\mathcal{I}_{0_+}^q u(t), \hat{0} \right) \\ d_\infty \left({}_F^{RL}\mathcal{I}_{0_+}^q v(t), \hat{0} \right) \end{bmatrix} \\
&\lesssim_{\mathbb{R}^2} \sum_{k=1}^{m} \overline{M}_k {}^{RL}\mathcal{I}_{0_+}^q \begin{pmatrix} d_\infty(u(t_k),\hat{0}) \\ d_\infty(v(t_k),\hat{0}) \end{pmatrix} + {}^{RL}\mathcal{I}_{0_+}^q \begin{pmatrix} d_\infty(u(t),\hat{0}) \\ d_\infty(v(t),\hat{0}) \end{pmatrix},
\end{aligned} \tag{39}$$

where $\overline{M}_k$ ($k = \overline{1, m}$) are defined in Remark 3.7. Then, inequality (39) is equivalent to

$$\begin{aligned}
\begin{bmatrix} d_\infty(x(t),\hat{0}) \\ d_\infty(y(t),\hat{0}) \end{bmatrix} &\lesssim_{\mathbb{R}^2} \sum_{k=1}^{m} \overline{M}_k {}^{RL}\mathcal{I}_{0_+}^q \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} + {}^{RL}\mathcal{I}_{0_+}^q \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \\
&\lesssim_{\mathbb{R}^2} \left(\sum_{k=1}^{m} \overline{M}_k \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \right) \frac{t_m^q}{\Gamma(q+1)} := \mathcal{S}_1.
\end{aligned}$$

where P_1, P_2 are real numbers defined in (37). It follows

$$\begin{bmatrix} D_{t_m}(x,\hat{0}) \\ D_{t_m}(y,\hat{0}) \end{bmatrix} \lesssim_{\mathbb{R}^2} \mathcal{S}_1.$$

For $t \in [t_m, \infty)$, from the proof of Theorem 3.12, we have

$$\begin{bmatrix} d_\infty(u(t),\hat{0}) \\ d_\infty(v(t),\hat{0}) \end{bmatrix} \lesssim_{\mathbb{R}^2} \overline{M} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} e^{-2\theta(t_m-\eta)} := \mathcal{J}_1.$$

Thus, inequality (39) becomes

$$\begin{bmatrix} d_\infty(x(t),\hat{0}) \\ d_\infty(y(t),\hat{0}) \end{bmatrix} \lesssim_{\mathbb{R}^2} \sum_{k=1}^{m} \overline{M}_k \frac{t_k^q}{\Gamma(q+1)} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \frac{t^q}{\Gamma(q+1)} \mathcal{J}_1.$$

Dividing both sides by $e^{\theta(t-\eta)}$ and taking supremum w.r.t $t \in [t_m, \infty)$, we obtain

$$\begin{bmatrix} \tilde{D}_{t_m}(x,\hat{0}) \\ \tilde{D}_{t_m}(y,\hat{0}) \end{bmatrix} \lesssim_{\mathbb{R}^2} \sum_{k=1}^{m} \overline{M}_k \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \frac{t_m^q}{e^{\theta(t_m-\eta)}\Gamma(q+1)} + \frac{q^q}{\theta e^q e^{-\theta\eta}\Gamma(q+1)} \mathcal{J}_1 := \mathcal{S}_2.$$

Therefore, we have $\rho_0((x,y),(\hat{0},\hat{0})) \lesssim_{\mathbb{R}_2} \max\{\mathcal{S}_1,\mathcal{S}_2\}$, which implies the boundedness of mild fuzzy solutions in type (i) of nonlocal problem (1) - (2).

When $j=2$, $(x(t),y(t))$ is defined by (8). Thanks to Lemma 2.2 in [22], the inequality (39) still holds. Therefore, by repeating arguments of the case when $j=1$, we also receive the boundedness of mild fuzzy solution in type (ii). Hence, the proof is complete. □

3.6. Ulam-Hyers stability of nonlocal problem

In this subsection, we will develop the Ulam - Hyers stable for differential systems in [9] to the problem (1) – (2).

Theorem 3.14. *Assume that following hypotheses hold*

1. g_i $(i=1,2)$ *are continuous functions satisfying hypothesis (**H2**);*
2. h_i $(i=1,2)$ *satisfy hypothesis (**H1**) and* $h_i(t,\hat{0},\hat{0})=\hat{0}$, *where* $\hat{0}$ *is zero fuzzy number;*
3. *The spectral radius of the matrix* M_1+M_2 *is less than* 1.

Then nonlocal problem (1) – (2) *is Ulam - Hyers stable in type* (i), *i.e there exists a matrix* $\mathcal{M}\in Mat_{2\times 2}(\mathbb{R}_+)$ *such that for each* ϵ_1, $\epsilon_2>0$ *and* $(\tilde{x},\tilde{y})\in\mathscr{C}_1^q(J_\infty,E_c)\times\mathscr{C}_1^q(J_\infty,E_c)$, *for which*

$$\begin{cases} d_\infty\left({}^{C}_{gH}\mathcal{D}^q_j\tilde{x}(t), g_1(t,\tilde{x}(t),\tilde{y}(t))+h_1(t,{}^{C}_{gH}\mathcal{D}^q_j\tilde{x}(t),{}^{C}_{gH}\mathcal{D}^q_j\tilde{y}(t))\right)\le\epsilon_1\\ d_\infty\left({}^{C}_{gH}\mathcal{D}^q_j\tilde{y}(t), g_2(t,\tilde{x}(t),\tilde{y}(t))+h_2(t,{}^{C}_{gH}\mathcal{D}^q_j\tilde{x}(t),{}^{C}_{gH}\mathcal{D}^q_j\tilde{y}(t))\right)\le\epsilon_2\\ x(0)+\sum\limits_{k\in J_1}a_kx(t_k)=\sum\limits_{k\in J_2}a_kx(t_k)\\ y(0)+\sum\limits_{k\in Q_1}\tilde{a}_ky(t_k)=\sum\limits_{k\in Q_2}\tilde{a}_ky(t_k)\end{cases}\tag{40}$$

there exists a mild fuzzy solution in type (i) (x,y) *of nonlocal problem (1)-(2) satisfying*

$$\begin{bmatrix}D(x,\tilde{x})\\ D(y,\tilde{y})\end{bmatrix}\lesssim_{\mathbb{R}^2}\mathcal{M}\begin{bmatrix}\epsilon_1\\ \epsilon_2\end{bmatrix}.$$

In addition, if hypotheses (17) - (18) *are fulfilled then the Ulam - Hyers stable in type* (ii) *of nonlocal problem (1)-(2) are also attained.*

Proof. For simplicity in presentation, denote $M_1^*=(I-M_1)^{-1}$ and $M_{1,\sigma}^*=(I-M_1)^{-1}M_\sigma$.

We recall that $(\tilde{x},\tilde{y})\in\mathscr{C}_j^q(J_\infty,E_c)\times\mathscr{C}_j^q(J_\infty,E_c)$ $(j=1,\,2)$ is a mild fuzzy solution of differential inequations system (40) (see Definition 4.4 in [24]) if for each $t\in J_\infty$, there exist $\Phi_1(t)$, $\Phi_2(t)\in C(J_\infty,E_c)$ such that

(i) $\begin{bmatrix}d_\infty(\Phi_1(t),\hat{0})\\ d_\infty(\Phi_2(t),\hat{0})\end{bmatrix}\lesssim_{\mathbb{R}^2}\begin{bmatrix}\epsilon_1\\ \epsilon_2\end{bmatrix}$

(ii) and

$$\begin{cases}{}^{C}_{gH}\mathcal{D}^q_j\tilde{x}(t)=g_1(t,\tilde{x}(t),\tilde{y}(t))+h_1(t,{}^{C}_{gH}\mathcal{D}^q_j\tilde{x}(t),{}^{C}_{gH}\mathcal{D}^q_j\tilde{y}(t))+\Phi_1(t)\\ {}^{C}_{gH}\mathcal{D}^q_j\tilde{y}(t)=g_2(t,\tilde{x}(t),\tilde{y}(t))+h_2(t,{}^{C}_{gH}\mathcal{D}^q_j\tilde{x}(t),{}^{C}_{gH}\mathcal{D}^q_j\tilde{y}(t))+\Phi_2(t)\end{cases}\quad j=1,\,2.\tag{41}$$

For $j=1$, applying transformation $\tilde{u}(t)={}^{C}_{gH}\mathcal{D}^q_1\tilde{x}(t)$, $\tilde{v}(t)={}^{C}_{gH}\mathcal{D}^q_1\tilde{y}(t)$ and using analogous arguments in the proof of Lemma 3.4, we obtain that if the pair $(\tilde{x}(t),\tilde{y}(t))$ satisfies nonlocal conditions (2), then it is a solution of the following fuzzy integral system

$$\begin{cases}\tilde{x}(t)=A_1(\tilde{u})+{}^{RL}_{F}\mathcal{I}^q_{0^+}\tilde{u}(t)=\mathcal{F}_1[\tilde{u}](t)\\ \tilde{y}(t)=\tilde{A}_1(\tilde{v})+{}^{RL}_{F}\mathcal{I}^q_{0^+}\tilde{v}(t)=\tilde{\mathcal{F}}_1[\tilde{v}](t)\end{cases}\quad\text{for }\ t\in J_\infty.\tag{42}$$

Thus, we can transform (41) into the following form

$$\begin{cases} \tilde{u}(t) = g_1(t, \mathcal{F}_1[\tilde{u}](t), \tilde{\mathcal{F}}_1[\tilde{v}](t)) + h_1(t, \tilde{u}(t), \tilde{v}(t)) + \Phi_1(t) \\ \tilde{v}(t) = g_2(t, \mathcal{F}_1[\tilde{u}](t), \tilde{\mathcal{F}}_1[\tilde{v}](t)) + h_2(t, \tilde{u}(t), \tilde{v}(t)) + \Phi_2(t) \end{cases} \tag{43}$$

In addition, when $(\tilde{u}, \tilde{v}) \in C(J_\infty, E_c) \times C(J_\infty, E_c)$ is a fuzzy solution of system (43), we can find out $(\tilde{x}, \tilde{y}) \in C(J_\infty, E_c) \times C(J_\infty, E_c)$ from the formula (42). It leads to the definition of mild fuzzy solution in type (i) of differential inequations system (40).

Indeed, assume that $(\tilde{x}, \tilde{y}) \in C(J_\infty, E_c) \times C(J_\infty, E_c)$ is a mild fuzzy solution in type (i) of differential inequations system (40). Since $\tilde{u}(t)$, $\tilde{v}(t)$ satisfy system (43) and g_i, h_i $(i = 1, 2)$ satisfy the global Lipschitz assumptions (**H1**), (**H2**), one gets

$$\begin{bmatrix} d_\infty(u(t), \tilde{u}(t)) \\ d_\infty(v(t), \tilde{v}(t)) \end{bmatrix} \lesssim_{\mathbb{R}^2} M_\sigma \begin{bmatrix} d_\infty(x(t), \tilde{x}(t)) \\ d_\infty(y(t), \tilde{y}(t)) \end{bmatrix} + M_1 \begin{bmatrix} d_\infty(u(t), \tilde{u}(t)) \\ d_\infty(v(t), \tilde{v}(t)) \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}.$$

It follows

$$\begin{bmatrix} d_\infty(u(t), \tilde{u}(t)) \\ d_\infty(v(t), \tilde{v}(t)) \end{bmatrix} \lesssim_{\mathbb{R}^2} M^*_{1,\sigma} \begin{bmatrix} d_\infty(x(t), \tilde{x}(t)) \\ d_\infty(y(t), \tilde{y}(t)) \end{bmatrix} + M^*_1 \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}.$$

If (x, y) and $(\tilde{x}, \tilde{y})$ are mild fuzzy solutions in type (i) of nonlocal problems (1) - (2) and (40) - (2) then we have

$$\begin{aligned} \begin{bmatrix} d_\infty(x(t), \tilde{x}(t)) \\ d_\infty(y(t), \tilde{y}(t)) \end{bmatrix} &\lesssim_{\mathbb{R}^2} \begin{bmatrix} d_\infty(x(t), \mathcal{F}_1[\tilde{u}]) \\ d_\infty(y(t), \tilde{\mathcal{F}}_1[\tilde{v}]) \end{bmatrix} + \begin{bmatrix} d\,(\tilde{x}(t), \mathcal{F}_1[\tilde{u}]) \\ d_\infty(\tilde{y}(t), \tilde{\mathcal{F}}_1[\tilde{v}]) \end{bmatrix} \\ &\lesssim_{\mathbb{R}^2} \begin{bmatrix} d_\infty(\mathcal{F}_1[u], \mathcal{F}_1[\tilde{u}]) \\ d_\infty(\tilde{\mathcal{F}}_1[v], \tilde{\mathcal{F}}_1[\tilde{v}]) \end{bmatrix} \\ &\lesssim_{\mathbb{R}^2} \begin{bmatrix} d_\infty(A_1(u), A_1(\tilde{u})) \\ d_\infty(\tilde{A}_1(v), \tilde{A}_1(\tilde{v})) \end{bmatrix} + \begin{bmatrix} d_\infty({}^{RL}_{F}\mathcal{I}^q_{0^+} u(t), {}^{RL}_{F}\mathcal{I}^q_{0^+} \tilde{u}(t)) \\ d_\infty({}^{RL}_{F}\mathcal{I}^q_{0^+} v(t), {}^{RL}_{F}\mathcal{I}^q_{0^+} \tilde{v}(t)) \end{bmatrix} \\ &\lesssim_{\mathbb{R}^2} \sum_{k=1}^m \overline{M}_k {}^{RL}\mathcal{I}^q_{0^+} \begin{bmatrix} d_\infty(u(t_k), \tilde{u}(t_k)) \\ d_\infty(v(t_k), \tilde{v}(t_k)) \end{bmatrix} + {}^{RL}\mathcal{I}^q_{0^+} \begin{bmatrix} d_\infty(u(t), \tilde{u}(t)) \\ d_\infty(v(t), \tilde{v}(t)) \end{bmatrix} \end{aligned} \tag{44}$$

Case 1: For $t \in [0, t_m]$, the estimation (44) becomes

$$\begin{aligned} \begin{bmatrix} d_\infty(x(t), \tilde{x}(t)) \\ d_\infty(y(t), \tilde{y}(t)) \end{bmatrix} &\lesssim_{\mathbb{R}^2} \sum_{k=1}^m \overline{M}_k {}^{RL}\mathcal{I}^q_{0^+} \left(M^*_{1,\sigma} \begin{bmatrix} d_\infty(x(t_k), \tilde{x}(t_k)) \\ d_\infty(y(t_k), \tilde{y}(t_k)) \end{bmatrix} + M^*_1 \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} \right) \\ &\quad + {}^{RL}\mathcal{I}^q_{0^+} \left(M^*_{1,\sigma} \begin{bmatrix} d_\infty(x(t), \tilde{x}(t)) \\ d_\infty(y(t), \tilde{y}(t)) \end{bmatrix} + M^*_1 \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} \right) \\ &\lesssim_{\mathbb{R}^2} \sum_{k=1}^m \overline{M}_k \left(M^*_1 \begin{bmatrix} \frac{\epsilon_1 t_m^q}{\Gamma(q+1)} \\ \frac{\epsilon_2 t_m^q}{\Gamma(q+1)} \end{bmatrix} + M^*_{1,\sigma} {}^{RL}\mathcal{I}^q_{0^+} \begin{pmatrix} D_{t_m}(x, \tilde{x}) \\ D_{t_m}(y, \tilde{y}) \end{pmatrix} \right) \\ &\quad + M^*_1 \begin{bmatrix} \frac{\epsilon_1 t_m^q}{\Gamma(q+1)} \\ \frac{\epsilon_2 t_m^q}{\Gamma(q+1)} \end{bmatrix} + M^*_{1,\sigma} {}^{RL}\mathcal{I}^q_{0^+} \begin{pmatrix} D_{t_m}(x, \tilde{x}) \\ D_{t_m}(y, \tilde{y}) \end{pmatrix} \end{aligned}$$

Taking supremum the left side when $t \in [0, t_m]$, we obtain

$$\begin{bmatrix} D_{t_m}(x, \tilde{x}) \\ D_{t_m}(y, \tilde{y}) \end{bmatrix} \lesssim_{\mathbb{R}^2} \left(M^*_{1,\sigma} + \sum_{k=1}^m \overline{M}_k M^*_{1,\sigma} \right) \begin{bmatrix} \frac{D_{t_m}(x,\tilde{x}) t_m^q}{\Gamma(q+1)} \\ \frac{D_{t_m}(y,\tilde{y}) t_m^q}{\Gamma(q+1)} \end{bmatrix} + \left(M^*_1 + \sum_{k=1}^m \overline{M}_k M^*_1 \right) \begin{bmatrix} \frac{\epsilon_1 t_m^q}{\Gamma(q+1)} \\ \frac{\epsilon_2 t_m^q}{\Gamma(q+1)} \end{bmatrix}.$$

Put

$$L_1 = \frac{t_m^q}{\Gamma(q+1)}\left(M_1^* + \sum_{k=1}^{m}\overline{M}_k M_1^*\right);\ N_1 = \left(I - \frac{t_m^q}{\Gamma(q+1)}M_{1,\sigma}^* - \frac{t_m^q}{\Gamma(q+1)}\sum_{k=1}^{m}\overline{M}_k M_{1,\sigma}^*\right)^{-1}.$$

Then, we obtain $\begin{bmatrix} D_{t_m}(x,\tilde{x}) \\ D_{t_m}(y,\tilde{y}) \end{bmatrix} \lesssim_{\mathbb{R}^2} N_1 L_1 \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$.

Case 2. For $t \in [t_m, +\infty)$, by applying analogous estimations as **Case 1**, (44) becomes

$$\begin{aligned}\begin{bmatrix} d_\infty(x(t),\tilde{x}(t)) \\ d_\infty(y(t),\tilde{y}(t)) \end{bmatrix} \lesssim_{\mathbb{R}^2} & \sum_{k=1}^{m}\overline{M}_k\left(M_1^*\begin{bmatrix} \frac{\epsilon_1 t_m^q}{\Gamma(q+1)} \\ \frac{\epsilon_2 t_m^q}{\Gamma(q+1)} \end{bmatrix} + M_{1,\sigma}^*\begin{bmatrix} \frac{D_{t_m}(x,\tilde{x})t_m^q}{\Gamma(q+1)} \\ \frac{D_{t_m}(y,\tilde{y})t_m^q}{\Gamma(q+1)} \end{bmatrix}\right) \\ & + M_1^*\begin{bmatrix} \frac{\epsilon_1 t^q}{\Gamma(q+1)} \\ \frac{\epsilon_2 t^q}{\Gamma(q+1)} \end{bmatrix} + M_{1,\sigma}^*\,{}^{RL}\mathcal{I}_{0^+}^q\left(\begin{bmatrix} d_\infty(x(t),\tilde{x}(t)) \\ d_\infty(y(t),\tilde{y}(t)) \end{bmatrix} e^{-\theta(t-\eta)}e^{\theta(t-\eta)}\right).\end{aligned}$$

In addition, for $\theta \geq q$, we have $\max\limits_{[t_m,\infty)} \frac{t^q}{e^{\theta t}} = \frac{q^q}{\theta e^q}$. Thus, dividing both sides by $e^{\theta(t-\eta)}$, we get

$$\begin{aligned}\begin{bmatrix} d_\infty(x(t),\tilde{x}(t)) \\ d_\infty(y(t),\tilde{y}(t)) \end{bmatrix} e^{-\theta(t-\eta)} \lesssim_{\mathbb{R}^2} & \left(\sum_{k=1}^{m}\overline{M}_k M_1^*\frac{t_m^q e^{-\theta(t-\eta)}}{\Gamma(q+1)} + M_1^*\frac{e^{\theta\eta-q}q^q}{\theta\Gamma(q+1)}\right)\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} \\ & + \left(M_{1,\sigma}^* + \sum_{k=1}^{m}\overline{M}_k M_{1,\sigma}^*\right)\begin{bmatrix} \frac{D_{t_m}(x,\tilde{x})t_m^q}{\Gamma(q+1)} \\ \frac{D_{t_m}(y,\tilde{y})t_m^q}{\Gamma(q+1)} \end{bmatrix} e^{-\theta(t-\eta)} + M_{1,\sigma}^*\begin{bmatrix} \tilde{Q}(\theta,q)\tilde{D}_{t_m}(x,\tilde{x}) \\ \tilde{Q}(\theta,q)\tilde{D}_{t_m}(y,\tilde{y}) \end{bmatrix}.\end{aligned}$$

Then, we have

$$\begin{aligned}\begin{bmatrix} \tilde{D}_{t_m}(x,\tilde{x}) \\ \tilde{D}_{t_m}(y,\tilde{y}) \end{bmatrix} \lesssim_{\mathbb{R}^2} & \left(\sum_{k=1}^{m}\overline{M}_k M_1^*\frac{t_m^q e^{-\theta(t_m-\eta)}}{\Gamma(q+1)} + M_1^*\frac{e^{\theta\eta-q}q^q}{\theta\Gamma(q+1)}\right)\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} + \tilde{Q}(\theta,q)M_{1,\sigma}^*\begin{bmatrix} \tilde{D}_{t_m}(x,\tilde{x}) \\ \tilde{D}_{t_m}(y,\tilde{y}) \end{bmatrix} \\ & + \frac{t_m^q e^{-\theta(t_m-\eta)}}{\Gamma(q+1)}\left(M_{1,\sigma}^* + \sum_{k=1}^{m}\overline{M}_k M_{1,\sigma}^*\right)L_1 N_1\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}.\end{aligned}$$

Put

$$L_2 = \sum_{k=1}^{m}\overline{M}_k M_1^*\frac{t_m^q e^{-\theta(t_m-\eta)}}{\Gamma(q+1)} + M_1^*\frac{e^{\theta\eta-q}q^q}{\theta\Gamma(q+1)} + \frac{t_m^q e^{-\theta(t_m-\eta)}}{\Gamma(q+1)}\left(M_{1,\sigma}^* + \sum_{k=1}^{m}\overline{M}_k M_{1,\sigma}^*\right)L_1 N_1.$$

$$N_2 = \left(I - \tilde{Q}(\theta,q)M_{1,\sigma}^*\right)^{-1}.$$

Then it follows

$$\begin{bmatrix} \tilde{D}_{t_m}(x,\tilde{x}) \\ \tilde{D}_{t_m}(y,\tilde{y}) \end{bmatrix} \lesssim_{\mathbb{R}^2} N_2 L_2 \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}.$$

Let $\mathcal{M} = \max\{N_1L_1, N_2L_2\}$. Then, $\begin{bmatrix} D(x,\tilde{x}) \\ D(y,\tilde{y}) \end{bmatrix} \lesssim_{\mathbb{R}^2} \mathcal{M}\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$, which implies that nonlocal problem (1) - (2) is Ulam - Hyers stable in type (i).

Moreover, if all hypotheses **(H1)**, **(H2)** and **(H4)** are satisfied, then by the setting $\tilde{u}(t) = {}_{gH}^{C}\mathcal{D}_2^q\tilde{x}(t)$, $\tilde{v}(t) = {}_{gH}^{C}\mathcal{D}_2^q\tilde{y}(t)$ and by analogous arguments in the proof of Lemma 3.4, we have $(\tilde{x}(t),\tilde{y}(t))$, satisfying nonlocal conditions (2), is a fuzzy solution of following integral system

$$\begin{cases} \tilde{x}(t) = A_2(\tilde{u}) \ominus (-1)^{RL}_{F}\mathcal{I}^q_{0^+}\tilde{u}(t) = \mathcal{F}_2[\tilde{u}](t) \\ \tilde{y}(t) = \tilde{A}_2(\tilde{v}) \ominus (-1)^{RL}_{F}\mathcal{I}^q_{0^+}\tilde{v}(t) = \tilde{\mathcal{F}}_2[\tilde{v}](t) \end{cases} \quad \text{for } t \in J_\infty. \tag{45}$$

Then, system (41) can be transformed into following form

$$\begin{cases} \tilde{u}(t) = g_1(t, \mathcal{F}_2[\tilde{u}](t), \tilde{\mathcal{F}}_2[\tilde{v}](t)) + h_1(t, \tilde{u}(t), \tilde{v}(t)) + \Phi_1(t) \\ \tilde{v}(t) = g_2(t, \mathcal{F}_2[\tilde{u}](t), \tilde{\mathcal{F}}_2[\tilde{v}](t)) + h_2(t, \tilde{u}(t), \tilde{v}(t)) + \Phi_2(t) \end{cases} \tag{46}$$

Therefore, to find $(\tilde{x}, \tilde{y})$, we only need to find out fuzzy solution $(\tilde{u}, \tilde{v})$ of system (46). As a corollary, we have $(\tilde{x}, \tilde{y}) \in C(J_\infty, E_c) \times C(J_\infty, E_c)$, that implies the definition of mild fuzzy solution in type (ii) of differential inequations system (40). Then, by repeating analogous arguments as in Case $j = 1$, Ulam - Hyers stability in type (ii) of nonlocal problem (1) - (2) is attained. □

4. Example

To illustrate obtained results, consider following implicit fuzzy fractional differential system

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^{\frac{1}{2}}_{j}x(t) = \dfrac{\sqrt{\pi}}{4\sqrt{4t+1}}K \\ {}^{C}_{gH}\mathcal{D}^{\frac{1}{2}}_{j}y(t) = \dfrac{\sqrt{t}}{\pi\sqrt{4t+1}}{}^{C}_{gH}\mathcal{D}^{\frac{1}{2}}_{j}x(t) \end{cases} \quad j = 1, 2, \tag{47}$$

$t \in J_\infty = [0, \infty)$, with nonlocal conditions

$$\begin{cases} x(0) + x\left(\frac{1}{12}\right) = \frac{\sqrt{\pi+1}}{2}x\left(\frac{1}{4\pi}\right) \\ y(0) + (2+\sqrt{3})\,y\left(\frac{1}{12}\right) = \frac{\pi+1+\sqrt{\pi^2+\pi}}{2}y\left(\frac{1}{4\pi}\right) \end{cases} \tag{48}$$

where K is a triangular fuzzy number with its level sets $[K]^\alpha = [\alpha, 2-\alpha]$, $\alpha \in [0, 1]$.

By the transformation ${}^{C}_{gH}\mathcal{D}^{\frac{1}{2}}_{j}x(t) = u(t)$, ${}^{C}_{gH}\mathcal{D}^{\frac{1}{2}}_{j}y(t) = v(t)$, where $x, y \in \mathscr{C}^q_j(J_\infty, E_c)$ combined with nonlocal conditions (48), we can represent $x(t)$, $y(t)$ in the form (49) w.r.t $j = 1$ and the form (50) w.r.t $j = 2$

$$\begin{cases} x(t) = a\left[\dfrac{\sqrt{\pi+1}}{2\sqrt{\pi}}\displaystyle\int_0^{\frac{1}{4\pi}}\dfrac{u(s)}{\sqrt{\frac{1}{4\pi}-s}}ds \ominus \dfrac{1}{\sqrt{\pi}}\displaystyle\int_0^{\frac{1}{12}}\dfrac{u(s)}{\sqrt{\frac{1}{12}-s}}ds\right] + \dfrac{1}{\sqrt{\pi}}\displaystyle\int_0^t\dfrac{u(s)}{\sqrt{t-s}}ds, \\ y(t) = \tilde{a}\left[\dfrac{(\sqrt{\pi}+1)\sqrt{\pi+1}}{2\sqrt{\pi}}\displaystyle\int_0^{\frac{1}{4\pi}}\dfrac{v(s)}{\sqrt{\frac{1}{4\pi}-s}}ds \ominus \dfrac{2+\sqrt{3}}{\sqrt{\pi}}\displaystyle\int_0^{\frac{1}{12}}\dfrac{v(s)}{\sqrt{\frac{1}{12}-s}}ds\right] + \dfrac{1}{\sqrt{\pi}}\displaystyle\int_0^t\dfrac{v(s)}{\sqrt{t-s}}ds. \end{cases} \tag{49}$$

$$\begin{cases} x(t) = -a\left[\dfrac{1}{\sqrt{\pi}}\displaystyle\int_0^{\frac{1}{12}}\dfrac{u(s)}{\sqrt{\frac{1}{12}-s}}ds \ominus \dfrac{\sqrt{\pi+1}}{2\sqrt{\pi}}\displaystyle\int_0^{\frac{1}{4\pi}}\dfrac{u(s)}{\sqrt{\frac{1}{4\pi}-s}}ds\right] \ominus \dfrac{-1}{\sqrt{\pi}}\displaystyle\int_0^t\dfrac{u(s)}{\sqrt{t-s}}ds, \\ y(t) = -\tilde{a}\left[\dfrac{2+\sqrt{3}}{\sqrt{\pi}}\displaystyle\int_0^{\frac{1}{12}}\dfrac{v(s)}{\sqrt{\frac{1}{12}-s}}ds \ominus \dfrac{(\sqrt{\pi}+1)\sqrt{\pi+1}}{2\sqrt{\pi}}\displaystyle\int_0^{\frac{1}{4\pi}}\dfrac{v(s)}{\sqrt{\frac{1}{4\pi}-s}}ds\right] \ominus \dfrac{-1}{\sqrt{\pi}}\displaystyle\int_0^t\dfrac{v(s)}{\sqrt{t-s}}ds. \end{cases} \tag{50}$$

where $a = \dfrac{2}{4-\sqrt{\pi+1}}$, $\tilde{a} = \dfrac{2}{5+2\sqrt{3}-\pi-\sqrt{\pi^2+\pi}}$.

In this example, we have $g_1(t, x(t), y(t)) = \frac{\sqrt{\pi}}{4\sqrt{4t+1}}K$ and $g_2(t, x(t), y(t)) = 0$ are well-defined and continuous functions on $[0, +\infty)$, which satisfy hypothesis **(H2)** with $\sigma_{11} = \sigma_{21} = \sigma_{22} = \frac{1}{4\pi}, \sigma_{12} = \frac{\sqrt{\pi}}{4}$, and $h_1(t, u(t), v(t)) = 0$; $h_2(t, u(t), v(t)) = \frac{\sqrt{t}}{\pi\sqrt{4t+1}}u(t)$ are functions satisfying hypothesis **(H1)** with respective coefficients $\overline{b}_1 = \overline{c}_1 = \overline{c}_2 = \frac{1}{12}, \overline{b}_2 = \frac{1}{\pi}$.

In addition, we obtain

$$\beta_1 = \frac{\sqrt{\frac{1}{12}}}{\Gamma\left(\frac{3}{2}\right)}\left[1 + a\left(1 + \frac{\sqrt{\pi+1}}{2}\right)\right];\ \beta_2 = \frac{\sqrt{\frac{1}{12}}}{\Gamma\left(\frac{3}{2}\right)}\left[1 + \tilde{a}\left(2 + \sqrt{3} + \frac{\pi + 1 + \sqrt{\pi^2+\pi}}{2}\right)\right].$$

Then, the inequality

$$\sigma_{11}\beta_1 + \overline{b}_1 + \sigma_{22}\beta_2 + \overline{c}_2 < \min\{2, 1 + (\sigma_{11}\beta_1 + \overline{b}_1)(\sigma_{22}\beta_2 + \overline{c}_2) - (\sigma_{12}\beta_1 + \overline{b}_2)(\sigma_{21}\beta_2 + \overline{c}_1)\}$$

holds, which follows the spectral radius of the matrix $M_1 + M_2$ is less than 1. Therefore, applying Theorem 3.8 guarantees the unique global existence of mild fuzzy solution in type (i) of the problem (47) - (48).

Assume that $x(t)$, $y(t)$, $u(t)$, $v(t)$ can be written in the following parameter forms

$$\begin{array}{ll} [x(t)]^\alpha = [x_{1\alpha}(t), x_{2\alpha}(t)]; & [y(t)]^\alpha = [y_{1\alpha}(t), y_{2\alpha}(t)]; \\ [u(t)]^\alpha = [u_{1\alpha}(t), u_{2\alpha}(t)]; & [v(t)]^\alpha = [v_{1\alpha}(t), v_{2\alpha}(t)]. \end{array} \quad \alpha \in [0; 1]$$

Then, system (47) can be rewritten as following interval FDEs

$$\begin{cases} [u_{1\alpha}(t), u_{2\alpha}(t)] = \frac{\sqrt{\pi}}{4\sqrt{4t+1}}[\alpha, 2-\alpha], \\ [v_{1\alpha}(t), v_{2\alpha}(t)] = \frac{\sqrt{t}}{\pi\sqrt{4t+1}}[u_{1\alpha}(t), u_{2\alpha}(t)] \end{cases} \quad \alpha \in [0; 1] \tag{51}$$

We now make this result precisely by giving concretely formulas of (i) - mild fuzzy solution of the problem. Indeed, from the first equation of (49) and (51), we have

$$\begin{aligned} x_{1\alpha}(t) &= \frac{\alpha a}{4}\left[\frac{\sqrt{\pi+1}}{2}\int_0^{\frac{1}{4\pi}} \frac{ds}{\sqrt{4s+1}\sqrt{\frac{1}{4\pi}-s}} - \int_0^{\frac{1}{12}} \frac{ds}{\sqrt{4s+1}\sqrt{\frac{1}{12}-s}}\right] + \int_0^t \frac{\alpha ds}{4\sqrt{4s+1}\sqrt{t-s}} \\ &= \frac{\alpha a}{4}\left[\frac{\sqrt{\pi+1}}{2}\left(\frac{\pi}{2} - \arcsin\sqrt{\frac{\pi}{\pi+1}}\right) - \frac{\pi}{6}\right] + \frac{\alpha\sqrt{t}}{2\sqrt{4t+1}} = C_{1\alpha} + \frac{\alpha\sqrt{t}}{2\sqrt{4t+1}} \\ x_{2\alpha}(t) &= \frac{(2-\alpha)a}{4}\left[\frac{\sqrt{\pi+1}}{2}\int_0^{\frac{1}{4\pi}} \frac{ds}{\sqrt{4s+1}\sqrt{\frac{1}{4\pi}-s}} - \int_0^{\frac{1}{12}} \frac{ds}{\sqrt{4s+1}\sqrt{\frac{1}{12}-s}}\right] + \int_0^t \frac{(2-\alpha)ds}{4\sqrt{4s+1}\sqrt{t-s}} \\ &= \frac{(2-\alpha)a}{4}\left[\frac{\sqrt{\pi+1}}{2}\left(\frac{\pi}{2} - \arcsin\sqrt{\frac{\pi}{\pi+1}}\right) - \frac{\pi}{6}\right] + \frac{(2-\alpha)\sqrt{t}}{2\sqrt{4t+1}} = C_{2\alpha} + \frac{(2-\alpha)\sqrt{t}}{2\sqrt{4t+1}} \end{aligned}$$

Since $v_{1\alpha}(t) = \frac{\sqrt{t}\,u_{1\alpha}(t)}{\pi\sqrt{4t+1}} = \frac{4\alpha\sqrt{t}}{4t+1}; v_{2\alpha}(t) = \frac{\sqrt{t}\,u_{2\alpha}(t)}{\pi\sqrt{4t+1}} = \frac{4(2-\alpha)\sqrt{t}}{4t+1}$, then we receive

$$\begin{aligned} y_{1\alpha}(t) &= \tilde{a}\left[\frac{\pi+1+\sqrt{\pi^2+\pi}}{2\sqrt{\pi}}\int_0^{\frac{1}{4\pi}} \frac{v_{1\alpha}(s)ds}{\sqrt{\frac{1}{4\pi}-s}} - \frac{2+\sqrt{3}}{\sqrt{\pi}}\int_0^{\frac{1}{12}} \frac{v_{1\alpha}(s)ds}{\sqrt{\frac{1}{12}-s}}\right] + \frac{1}{\sqrt{\pi}}\int_0^t \frac{v_{1\alpha}(s)ds}{\sqrt{t-s}} \\ &= 4\alpha\tilde{a}\left[\frac{\sqrt{\pi+1}}{4} - \frac{\sqrt{\pi}}{8}\right] + \frac{4\alpha t}{4t+1+\sqrt{4t+1}} = \hat{C}_{1\alpha} + \frac{4\alpha t}{4t+1+\sqrt{4t+1}} \\ y_{2\alpha}(t) &= \tilde{a}\left[\frac{\pi+1+\sqrt{\pi^2+\pi}}{2\sqrt{\pi}}\int_0^{\frac{1}{4\pi}} \frac{v_{2\alpha}(s)ds}{\sqrt{\frac{1}{4\pi}-s}} - \frac{2+\sqrt{3}}{\sqrt{\pi}}\int_0^{\frac{1}{12}} \frac{v_{2\alpha}(s)ds}{\sqrt{\frac{1}{12}-s}}\right] + \frac{1}{\sqrt{\pi}}\int_0^t \frac{v_{2\alpha}(s)ds}{\sqrt{t-s}} \\ &= 4(2-\alpha)\tilde{a}\left[\frac{\sqrt{\pi+1}}{4} - \frac{\sqrt{\pi}}{8}\right] + \frac{4(2-\alpha)t}{4t+1+\sqrt{4t+1}} = \hat{C}_{2\alpha} + \frac{4(2-\alpha)t}{4t+1+\sqrt{4t+1}} \end{aligned}$$

Consider two fuzzy numbers $\mathscr{U}_1$, $\mathscr{U}_2$ given by their level sets as follows

$$\begin{cases}[\mathscr{U}_1]^\alpha = [C_{1\alpha}, C_{2\alpha}] = \frac{a}{4}\left[\frac{\sqrt{\pi+1}}{2}\left(\frac{\pi}{2} - \arcsin\sqrt{\frac{\pi}{\pi+1}}\right) - \frac{\pi}{6}\right][K]^\alpha, \\ [\mathscr{U}_2]^\alpha = [\hat{C}_{1\alpha}, \hat{C}_{2\alpha}] = 4\tilde{a}\left[\frac{\sqrt{\pi+1}}{4} - \frac{\sqrt{\pi}}{8}\right][K]^\alpha\end{cases}$$

Then, by applying Stacking Theorem, mild solution in type (i) of the problem is given by

$$\begin{cases}x(t) = \mathscr{U}_1 + \frac{\sqrt{t}}{2\sqrt{4t+1}}K \\ y(t) = \mathscr{U}_2 + \frac{4t}{4t+1+\sqrt{4t+1}}K\end{cases} \quad t \in J_\infty. \tag{52}$$

From (51), we have $\lim\limits_{t\to\infty}\rho_0(z(t), \Theta) = 0$, $z(t) = \begin{bmatrix}u(t) & v(t)\end{bmatrix}^T$. Moreover, it follows from (52) that $w(t) = \begin{bmatrix}x(t) & y(t)\end{bmatrix}^T$ is asymptotic to set $\mathcal{F}_0 = \begin{bmatrix}\mathscr{U}_1 & \mathscr{U}_2\end{bmatrix}^T$ as t tends to $+\infty$, and the unique (i) - mild fuzzy solution $w(t) = \begin{bmatrix}x(t) & y(t)\end{bmatrix}^T$ of the problem (47) - (48) is bounded.

In addition, the assumption about matrix having non-negative elements in Theorem 3.14 can be checked by using computer algebra programs. The obtained matrix is shown below

$$\mathcal{M} \approx \begin{bmatrix}0.14159 & 0.04922 \\ 0.05786 & 0.14158\end{bmatrix},$$

in which $\eta = 10^{-4}$, $\theta = 10^5$ and $Q(\theta, \frac{1}{2}) \approx 0.48$. Now, consider the following differential inequations system

$$\begin{bmatrix} d_\infty\left({}^C_{gH}\mathcal{D}^{\frac{1}{2}}_j\tilde{x}(t), \frac{K\sqrt{\pi}}{4\sqrt{4t+1}}\right) \\ d_\infty\left({}^C_{gH}\mathcal{D}^{\frac{1}{2}}_j\tilde{y}(t), \frac{\sqrt{t}}{\pi\sqrt{4t+1}}{}^C_{gH}\mathcal{D}^{\frac{1}{2}}_j\tilde{x}(t)\right)\end{bmatrix} \leq \begin{bmatrix}\epsilon_1 \\ \epsilon_2\end{bmatrix}, \tag{53}$$

we have that there exist fuzzy-valued functions $\Phi_1(t) = \frac{1}{4}K_1\epsilon_1$, $\Phi_2(t) = \frac{1}{4}K_1\epsilon_2$, in which K_1 is a triangular fuzzy number with level sets $[K_1]^\alpha = [2\alpha, 4-2\alpha]$ and $\epsilon_1, \epsilon_2 > 0$, such that

$$\begin{aligned} &(i)\ \begin{bmatrix}d_\infty(\Phi_1(t), \hat{0}) \\ d_\infty(\Phi_2(t), \hat{0})\end{bmatrix} \leq \begin{bmatrix}\epsilon_1 \\ \epsilon_2\end{bmatrix} \\ &(ii)\ \begin{cases}{}^C_{gH}\mathcal{D}^{\frac{1}{2}}_j\tilde{x}(t) = \frac{\sqrt{\pi}}{4\sqrt{4t+1}}K + \frac{1}{4}K_1\epsilon_1 \\ {}^C_{gH}\mathcal{D}^{\frac{1}{2}}_j\tilde{y}(t) = \frac{\sqrt{t}}{\pi\sqrt{4t+1}}{}^C_{gH}\mathcal{D}^{\frac{1}{2}}_j\tilde{x}(t) + \frac{1}{4}K_1\epsilon_2\end{cases}\end{aligned} \tag{54}$$

By using analogous arguments, we imply that

$$\begin{cases}x(t) = \mathscr{U}_1 + \frac{\sqrt{t}}{2\sqrt{4t+1}}K + \frac{\epsilon_1\sqrt{t}}{2\sqrt{\pi}}K_1 \\ y(t) = \mathscr{U}_2 + \frac{4t}{4t+1+\sqrt{4t+1}}K + \frac{\epsilon_2\sqrt{t}}{2\sqrt{\pi}}K_1\end{cases} \quad t \in J_\infty$$

is a mild fuzzy solution in type (i) of system (54) with nonlocal conditions (2) and it's easy to check that

$$\begin{bmatrix}D_{\frac{1}{12}}(x, \tilde{x}) \\ D_{\frac{1}{12}}(y, \tilde{y})\end{bmatrix} \leq \begin{bmatrix}\frac{1}{\sqrt{3\pi}} & 0 \\ 0 & \frac{1}{\sqrt{3\pi}}\end{bmatrix}\begin{bmatrix}\epsilon_1 \\ \epsilon_2\end{bmatrix}, \qquad \text{for } t \in [0, \frac{1}{12}],$$

$$\begin{bmatrix}\tilde{D}_{\frac{1}{12}}(x, \tilde{x}) \\ \tilde{D}_{\frac{1}{12}}(y, \tilde{y})\end{bmatrix} \leq \begin{bmatrix}\frac{e^{-10}10^{-2}}{\sqrt{5\pi e}} & 0 \\ 0 & \frac{e^{-10}10^{-2}}{\sqrt{5\pi e}}\end{bmatrix}\begin{bmatrix}\epsilon_1 \\ \epsilon_2\end{bmatrix}, \quad \text{for } t \in [\frac{1}{12}, \infty).$$

Therefore, by choosing matrix $\mathcal{M} = \begin{bmatrix}\frac{1}{\sqrt{3\pi}} & 0 \\ 0 & \frac{1}{\sqrt{3\pi}}\end{bmatrix}$, we immediately obtain $\begin{bmatrix}D(x, \tilde{x}) \\ D(y, \tilde{y})\end{bmatrix} \leq \mathcal{M}\begin{bmatrix}\epsilon_1 \\ \epsilon_2\end{bmatrix}$, which implies the problem is Ulam-Hyers stable in type (i).

5. Conclusions

The main goals of this paper have been to investigate the global existence of (i) (or (ii)) - mild fuzzy solutions of nonlocal problem for implicit fractional fuzzy differential systems under Caputo gH-fractional derivatives for two cases: in first case, our problem satisfies Hypotheses **(H1) - (H2)** and the other satisfies Hypotheses **(H1) - (H3)** . We have achieved these goals by using Perov's fixed point theorem and an extension of Krasnoselskii's fixed point theorem (see **Section 2**). Under assumptions of the problem, we have attained some results about qualitative properties of solutions, such as boundedness, decay, attractivity and some new concepts of stability for fractional fuzzy differential system. The next step in our future research is to investigate some controllability results for this problem.

6. Appendix

6.1. Some auxiliary lemmas

Lemma 6.1 ([25], Lemma 2.2.). *If A is a square matrix that converges to zero and the elements of an other square matrix B is small enough, then $A + B$ also converges to zero.*

Lemma 6.2. *Assume that the hypothesis **(H1)** is satisfied. Then the following estimation*

$$D(H_1(u,v), H_1(\overline{u},\overline{v})) \leq \overline{b}_1 D(u,\overline{u}) + \overline{c}_1 D(v,\overline{v})$$

holds for all $u, v, \overline{u}, \overline{v} \in C(J_\infty, E_c)$.

Proof. Let (u,v), $(\overline{u},\overline{v}) \in C(J_\infty, E_c) \times C(J_\infty, E_c)$ be arbitrary. Using the assumption (14), for $t \in [0, t_m]$, we deduce that

$$d_\infty(H_1(u,v)(t), H_1(\overline{u},\overline{v})(t)) \leq \overline{b}_1 d_\infty(u(t),\overline{u}(t)) + \overline{c}_1 d_\infty(v(t),\overline{v}(t)).$$

Then taking the supremum for $t \in [0, t_m]$ we obtain

$$D_{t_m}(H_1(u,v), H_1(\overline{u},\overline{v})) \leq \overline{b}_1 D_{t_m}(u,\overline{u}) + \overline{c}_1 D_{t_m}(v,\overline{v}) \leq \overline{b}_1 D(u,\overline{u}) + \overline{c}_1 D(v,\overline{v}). \tag{55}$$

For $t \in [t_m, \infty)$, we have

$$d_\infty(H_1(u,v)(t), H_1(\overline{u},\overline{v})(t)) \leq \overline{b}_1 d_\infty(u(t),\overline{u}(t)) + \overline{c}_1 d_\infty(v(t),\overline{v}(t))$$

Dividing both sides by $e^{\theta(t-\eta)}$ and taking supremum when $t \in [t_m, \infty)$ we have :

$$\tilde{D}_{t_m}(H_1(u,v), H_1(\overline{u},\overline{v})) \leq \overline{b}_1 \tilde{D}_{t_m}(u,\overline{u}) + \overline{c}_1 \tilde{D}_{t_m}(v,\overline{v}) \leq \overline{b}_1 D(u,\overline{u}) + \overline{c}_1 D(v,\overline{v}). \tag{56}$$

Combining (55), (56) one gets $D(H_1(u,v), H_1(\overline{u},\overline{v})) \leq \overline{b}_1 D(u,\overline{u}) + \overline{c}_1 D(v,\overline{v})$. □

Corollary 6.3. *Under assumption that $h_i(t, \hat{0}, \hat{0}) = \hat{0}$ $(i = 1, 2)$, by similar arguments as in Lemma 6.2, we have*

$$\begin{bmatrix} d_\infty(H_1(u,v)(t),\hat{0}) \\ d_\infty(H_2(u,v)(t),\hat{0}) \end{bmatrix} \lesssim_{\mathbb{R}^2} \begin{bmatrix} \overline{b}_1 & \overline{c}_1 \\ \overline{b}_2 & \overline{c}_2 \end{bmatrix} \begin{bmatrix} d_\infty(u(t),\hat{0}) \\ d_\infty(v(t),\hat{0}) \end{bmatrix} = M_1 \begin{bmatrix} d_\infty(u(t),\hat{0}) \\ d_\infty(v(t),\hat{0}) \end{bmatrix} \tag{57}$$

Remark 6.4. *For all $t \in [0, t_m]$ $(t_m < 1)$, we have*

$$\begin{aligned} d_\infty\Big({}^{RL}_{F}\mathcal{I}^q_{0^+}\phi(t), \hat{0}\Big) &= \frac{1}{\Gamma(q)} d_\infty\Big(\int_0^t (t-s)^{q-1}\phi(s)ds, \hat{0}\Big) \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} d_\infty(\phi(s),\hat{0})ds \\ &\leq \frac{D_{t_m}(\phi,\hat{0})}{\Gamma(q)} \int_0^t (t-s)^{q-1}ds = \frac{D_{t_m}(\phi,\hat{0})}{\Gamma(q+1)} t^q. \end{aligned}$$

Remark 6.5. *For all $t \in [t_m, \infty)$, we have*

$$\begin{aligned} d_\infty\Big({}^{RL}_{F}\mathcal{I}^q_{0^+}\phi(t), \hat{0}\Big) &\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} d_\infty(\phi(s), \hat{0}) ds \\ &\le \frac{1}{\Gamma(q)}\Big(\int_0^{t_m} (t-s)^{q-1} d_\infty(\phi(s), \hat{0}) ds + \int_{t_m}^t (t-s)^{q-1} d_\infty(\phi(s), \hat{0}) ds\Big) \\ &< \frac{D_{t_m}(\phi, \hat{0})}{\Gamma(q+1)} t_m^q + \tilde{D}_{t_m}(\phi, \hat{0})\tilde{Q}(\theta, q) e^{\theta(t-\eta)} \end{aligned}$$

Lemma 6.6. *Assume that the hypothesis **(H3)** is satisfied. Then the following estimations*

$$D_{t_m}(G_1(u,v), \hat{0}) \le b_1\beta_1 D_{t_m}(u, \hat{0}) + c_1\beta_2 D_{t_m}(v, \hat{0}) + e_1, \tag{58}$$

$$\begin{aligned} \tilde{D}_{t_m}(G_1(u,v), \hat{0}) &\le \Big(B_1\beta_1 D_{t_m}(u, \hat{0}) + C_1\beta_2 D_{t_m}(v, \hat{0}) + E_1\Big) e^{-3\theta(t-\eta)} \\ &\quad + \Big(B_1\tilde{Q}(\theta, q)\tilde{D}_{t_m}(u, \hat{0}) + C_1\tilde{Q}(\theta, q)\tilde{D}_{t_m}(v, \hat{0})\Big) e^{-2\theta(t-\eta)} \end{aligned} \tag{59}$$

hold for all $u, v \in C(J_\infty, E_c)$.

Proof. Without loss of generality, we suppose that $x, y \in \mathscr{C}^q_1(J_\infty, E_c)$ satisfy nonlocal condition (2) and $u, v \in C(J_\infty, E_c)$ are defined by setting (6). From (7) in Lemma 3.4 we have with $t \in [0, t_m]$

$$\begin{aligned} d_\infty(x(t), \hat{0}) &\le d_\infty\Big(a\sum_{k\in J_2} a_k {}^{RL}_{F}\mathcal{I}^q_{0^+}u(t_k), \hat{0}\Big) + d_\infty\Big(a\sum_{k\in J_1} a_k {}^{RL}_{F}\mathcal{I}^q_{0^+}u(t_k), \hat{0}\Big) + d_\infty\Big({}^{RL}_{F}\mathcal{I}^q_{0^+}u(t), \hat{0}\Big) \\ &\le \frac{D_{t_m}(u, \hat{0})}{\Gamma(q+1)}\Big[a\sum_{k=1}^m a_k t_k^q + t^q\Big] \\ &\le \frac{D_{t_m}(u, \hat{0})}{\Gamma(q+1)} t_m^q\Big[a\sum_{k=1}^m a_k + 1\Big] \\ &= \beta_1 D_{t_m}(u, \hat{0}). \end{aligned}$$

Similarly, one gets $d_\infty(y(t), \hat{0}) \le \beta_2 D_{t_m}(v, \hat{0})$. Applying hypothesis **(H3)**, we obtain

$$\begin{aligned} d_\infty(G_1(u,v)(t), \hat{0}) &\le b_1 d_\infty\Big(x(t), \hat{0}\Big) + c_1 d_\infty\Big(y(t), \hat{0}\Big) + e_1 \\ &\le b_1\beta_1 D_{t_m}(u, \hat{0}) + c_1\beta_2 D_{t_m}(v, \hat{0}) + e_1. \end{aligned}$$

By taking supremum, we receive the first estimation in (58). By doing same arguments for $t \in [t_m, \infty)$ and using Remark 6.5, we have

$$\begin{aligned} d_\infty(G_1(u,v)(t), \hat{0}) &\le \Big(B_1 d_\infty\Big(x(t), \hat{0}\Big) + C_1 d_\infty\Big(y(t), \hat{0}\Big) + E_1\Big) e^{-2\theta(t-\eta)} \\ &\le \Big(B_1\beta_1 D_{t_m}(u, \hat{0}) + C_1\beta_2 D_{t_m}(v, \hat{0}) + E_1\Big) e^{-2\theta(t-\eta)} \\ &\quad + \Big(B_1\tilde{Q}(\theta, q)\tilde{D}_{t_m}(u, \hat{0}) + C_1\tilde{Q}(\theta, q)\tilde{D}_{t_m}(v, \hat{0})\Big) e^{-\theta(t-\eta)}. \end{aligned} \tag{60}$$

Then dividing both sides of the inequality by $e^{\theta(t-\eta)}$ and taking supremum when $t \in [t_m, \infty)$, we obtain the inequality (59). □

Lemma 6.7. *Assume that hypothesis **(H2)** is satisfied. Then the following estimations*

$$D(G_i(u,v), G_i(\overline{u}, \overline{v})) \le \sigma_{i1}\beta_1 D(u, \overline{u}) + \sigma_{i2}\beta_2 D(v, \overline{v}),\ i = 1, 2 \tag{61}$$

hold for all $(u, v),\ (\overline{u}, \overline{v}) \in C(J_\infty, E_c) \times C(J_\infty, E_c)$.

Proof. Assume that x, $y \in \mathscr{C}_1^q(J_\infty, E_c)$ and satisfy the nonlocal condition (2) and u, $v \in C(J_\infty, E_c)$ are Caputo gH - derivatives in type 1 of x, y, respectively. From Lemma 3.4, it implies that x, y satisfies the integral forms (7). From $G_i(u, v)(t) = g_i(t, x(u(t), v(t)), y(u(t), v(t)))$ and **(H2)**, we imply that

$$\begin{aligned}
&d_\infty(G_i(u,v)(t),G_i(\overline{u},\overline{v})(t))\\
&\qquad = d_\infty\left(g_i\left(t, x(u(t), v(t)), y(u(t), v(t))\right), g_i\left(t, x(\overline{u}(t), \overline{v}(t)), y(\overline{u}(t), \overline{v}(t))\right)\right)\\
&\qquad \le \sigma_{i1}\left[d_\infty(x(u(t), v(t)), x(\overline{u}(t), \overline{v}(t))) + \sigma_{i2} d_\infty(y(u(t), v(t)), y(\overline{u}(t), \overline{v}(t)))\right]\\
&\qquad \le \sigma_{i1}\Big[d_\infty(A_1(u), A_1(\overline{u})) + d_\infty({}^{RL}_{F}\mathcal{I}^q_{0^+}u(t), {}^{RL}_{F}\mathcal{I}^q_{0^+}\overline{u}(t))\Big]\\
&\qquad\quad + \sigma_{i2}\Big[d_\infty(\tilde{A}_1(v), \tilde{A}_1(\overline{v})) + d_\infty({}^{RL}_{F}\mathcal{I}^q_{0^+}v(t), {}^{RL}_{F}\mathcal{I}^q_{0^+}\overline{v}(t))\Big]
\end{aligned} \tag{62}$$

Case 1. If $t \in [0, t_m]$ then we have

$$\begin{aligned}
d_\infty(A_1(u), A_1(\overline{u})) &\le a\sum_{k\in J_2} a_k\, d_\infty\Big({}^{RL}_{F}\mathcal{I}^q_{0^+}u(t_k), {}^{RL}_{F}\mathcal{I}^q_{0^+}\overline{u}(t_k)\Big) + a\sum_{k\in J_1} a_k\, d_\infty\Big({}^{RL}_{F}\mathcal{I}^q_{0^+}u(t_k), {}^{RL}_{F}\mathcal{I}^q_{0^+}\overline{u}(t_k)\Big)\\
&= a\sum_{k=1}^{m} a_k\, d_\infty\Big({}^{RL}_{F}\mathcal{I}^q_{0^+}u(t_k), {}^{RL}_{F}\mathcal{I}^q_{0^+}\overline{u}(t_k)\Big)\\
&\le a\sum_{k=1}^{m} a_k \frac{1}{\Gamma(q)}\int_0^{t_k}(t_k - s)^{q-1} d_\infty\Big(u(s), \overline{u}(s)\Big)ds\\
&\le \frac{a}{\Gamma(q+1)}\sum_{k=1}^{m} a_k t_k^q D_{t_m}\Big(u, \overline{u}\Big),
\end{aligned}$$

and

$$\begin{aligned}
d_\infty\left({}^{RL}_{F}\mathcal{I}^q_{0^+}u(t), {}^{RL}_{F}\mathcal{I}^q_{0^+}\overline{u}(t)\right) &\le \frac{1}{\Gamma(q)}\int_0^{t}(t - s)^{q-1} d_\infty\Big(u(s), \overline{u}(s)\Big)ds\\
&\le \frac{t^q}{\Gamma(q+1)} D_{t_m}\Big(u, \overline{u}\Big).
\end{aligned}$$

By doing the same arguments for the third and the forth terms in the right hand side of estimation (62), we obtain

$$d_\infty(\tilde{A}_1(v), \tilde{A}_1(\overline{v})) + d_\infty({}^{RL}_{F}\mathcal{I}^q_{0^+}v(t), {}^{RL}_{F}\mathcal{I}^q_{0^+}\overline{v}(t)) \le \frac{1}{\Gamma(q+1)}\left[\tilde{a}\sum_{k=1}^{m}\tilde{a}_k t_k^q + t^q\right] D_{t_m}\Big(v, \overline{v}\Big).$$

Thus, the inequality (62) becomes

$$d_\infty(G_i(u,v)(t), G_i(\overline{u},\overline{v})(t)) \le \frac{1}{\Gamma(q+1)}\left[\left(a\sum_{k=1}^{m} a_k t_k^q + t^q\right) D_{t_m}\Big(u, \overline{u}\Big) + \sigma_{i2}\left(\tilde{a}\sum_{k=1}^{m}\tilde{a}_k t_k^q + t^q\right) D_{t_m}\Big(v, \overline{v}\Big)\right].$$

Then, by taking supremum for $t \in [0, t_m]$, we obtain

$$\begin{aligned}
D_{t_m}(G_i(u,v), G_i(\overline{u},\overline{v})) &\le \sigma_{i1}\beta_1 D_{t_m}(u, \overline{u}) + \sigma_{i2}\beta_2 D_{t_m}(v, \overline{u})\\
&\le \sigma_{i1}\beta_1 D(u, \overline{u}) + \sigma_{i2}\beta_2 D(v, \overline{u}).
\end{aligned} \tag{63}$$

Case 2. If $t \in [t_m, \infty)$ then by using similar arguments, we also obtain

$$d_\infty(A_1(u), A_1(\overline{u})) \le \frac{a}{\Gamma(q+1)}\sum_{k=1}^{m} a_k t_k^q D_{t_m}\Big(u, \overline{u}\Big),$$

and

$$\begin{aligned}
d_\infty({}^{RL}_{F}\mathcal{I}^q_{0^+}u(t), {}^{RL}_{F}\mathcal{I}^q_{0^+}\overline{u}(t)) &\leq \frac{1}{\Gamma(q)}\left[\int_0^{t_m}(t-s)^{q-1}d_\infty(u(s),\overline{u}(s))ds + \int_{t_m}^t (t-s)^{q-1}d_\infty(u(s),\overline{u}(s))ds\right]\\
&\leq \frac{1}{\Gamma(q)}\left[\int_0^{t_m}(t-s)^{q-1}D_{t_m}(u,\overline{u})ds + \int_{t_m}^t (t-s)^{q-1}e^{\theta(s-\eta)}\tilde{D}_{t_m}(u,\overline{u})ds\right]\\
&< \frac{1}{\Gamma(q)}\left[\int_0^{t_m}(t_m-s)^{q-1}D_{t_m}(u,\overline{u})ds + \int_0^t (t-s)^{q-1}e^{\theta(s-\eta)}\tilde{D}_{t_m}(u,\overline{u})ds\right]\\
&< \frac{t_m^q}{\Gamma(q+1)}D_{t_m}(u,\overline{u}) + \frac{Q(\theta,q)e^{\theta(t-\eta)}}{\Gamma(q+1)}\tilde{D}_{t_m}(u,\overline{u}).
\end{aligned}$$

Then, the inequality (62) becomes

$$\begin{aligned}
d_\infty(G_i(u,v)(t),G_i(\overline{u},\overline{v})(t)) \leq{}& \frac{\sigma_{i1}}{\Gamma(q+1)}\left[\left(a\sum_{k=1}^m a_k t_k^q + t_m^q\right)D_{t_m}(u,\overline{u}) + e^{\theta(t-\eta)}Q(\theta,q)\tilde{D}_{t_m}(u,\overline{u})\right]\\
&+ \frac{\sigma_{i2}}{\Gamma(q+1)}\left[\left(\tilde{a}\sum_{k=1}^m \tilde{a}_k t_k^q + t_m^q\right)D_{t_m}(v,\overline{v}) + e^{\theta(t-\eta)}Q(\theta,q)\tilde{D}_{t_m}(v,\overline{v})\right].
\end{aligned}$$

Dividing both sides by $e^{\theta(t-\eta)}$ and taking supremum when $t \in [t_m,\infty)$, we obtain

$$\begin{aligned}
\tilde{D}_{t_m}(G_i(u,v),G_i(\overline{u},\overline{v})) <{}& \frac{\sigma_{i1}}{\Gamma(q+1)}\left[\left(a\sum_{k=1}^m a_k t_m^q + t_m^q\right)D_{t_m}(u,\overline{u}) + Q(\theta,q)\tilde{D}_{t_m}(u,\overline{u})\right]\\
&+ \frac{\sigma_{i2}}{\Gamma(q+1)}\left[\left(\tilde{a}\sum_{k=1}^m \tilde{a}_k t_m^q + t_m^q\right)D_{t_m}(v,\overline{v}) + Q(\theta,q)\tilde{D}_{t_m}(v,\overline{v})\right]\\
\leq{}& \frac{\sigma_{i1}}{\Gamma(q+1)}\left[a\sum_{k=1}^m a_k t_m^q + t_m^q + Q(\theta,q)\right]D(u,\overline{u})\\
&+ \frac{\sigma_{i2}}{\Gamma(q+1)}\left[\tilde{a}\sum_{k=1}^m \tilde{a}_k t_m^q + t_m^q + Q(\theta,q)\right]D(v,\overline{v}).
\end{aligned}$$

From Remark 6.9, we can choose a big enough number θ such that $t_m^q + Q(\theta,q) \leq 1$. Then, the above estimation becomes

$$\tilde{D}_{t_m}(G_i(u,v),G_i(\overline{u},\overline{v})) \leq \sigma_{i1}\beta_1 D(u,\overline{u}) + \sigma_{i2}\beta_2 D(v,\overline{u}). \tag{64}$$

Therefore, the inequality (61) can be implied directly from the estimations (63), (64). $\square$

Lemma 6.8 ([24], Lemma 2.3). *For all $\epsilon > 0$, the following estimation*

$$\int_0^t (t-s)^{q-1}e^{\lambda s}ds < \frac{e^{\lambda t}}{q}\left[2\left(\frac{C}{\lambda^{\frac{1}{1+\epsilon}}}\right)^{\frac{q}{2}} + \frac{1}{\lambda}\left(\frac{C}{\lambda^{\frac{1}{1+\epsilon}}}\right)^q\right] = \frac{e^{\lambda t}}{q}Q(\lambda,q)$$

holds, where $C > 0$ does not depend on λ, $t \in [0,\infty)$, $q \in (0,1]$.

Remark 6.9. *For $C > 0$ and $q \in (0,1]$, the value of $Q(\lambda,q)$ tends to 0 as $\lambda \to +\infty$.*

Lemma 6.10. *For $q \in (0,1]$ and $\lambda > q$, the following estimation*

$$\int_0^t (t-s)^{q-1}e^{-\lambda s}ds \leq J_t(\lambda,q)$$

holds, where $J_t(\lambda,q) := \frac{1}{\lambda} + e^{-\lambda(t-1)}\left(\frac{1}{q} - \frac{1}{\lambda}\right)$.

Proof. By changing variable $s_1 = t - s$, we have

$$\begin{aligned}
\int_0^t s_1^{q-1} e^{-\lambda(t-s_1)} ds_1 &\le e^{-\lambda t}\left(\int_0^1 s_1^{q-1} e^{\lambda s_1} ds_1 + \int_1^t s_1^{q-1} e^{\lambda s_1} ds_1\right) \\
&\le e^{-\lambda t}\left(e^{\lambda}\int_0^1 s_1^{q-1} ds_1 + \int_1^t e^{\lambda s_1} ds_1\right) \\
&\le e^{-\lambda t}\left(\frac{e^{\lambda t}}{\lambda} + \frac{e^{\lambda}(\lambda - q)}{\lambda q}\right) \\
&= \frac{1}{\lambda} + e^{-\lambda(t-1)}\left(\frac{1}{q} - \frac{1}{\lambda}\right) := J_t(\lambda, q).
\end{aligned}$$

□

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