



On Approximately Harmonic h -Convex Functions Depending on a Given Function

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Abstract. A new class of harmonic convex function depending on given functions which is called as "approximately harmonic h -convex functions" is introduced. With the discussion of special cases it is shown that this class unifies other classes of approximately harmonic h -convex function. Some associated integral inequalities with these new classes of harmonic convexity are also obtained. Several special cases of the main results are also discussed.

1. Introduction and Preliminaries

In modern analysis convexity and its various generalizations play a pivotal role through its numerous application. For some recent generalizations of classical convexity interested readers are referred to [3–5, 7, 14]. Another significance of convexity is its close relationship with theory of inequalities. Many famous inequalities are direct consequences of classical convexity. For more details see [1, 2, 6, 9–11]. Recently Noor et al. [12] introduced the notion of harmonic h -convex functions. This class not only generalizes the class of harmonic convex functions which was introduced and studied by Iscan [7], but also generalizes other classes of harmonic convexity for different suitable choices of the function $h(\cdot)$. The definition of the harmonic h -convex function runs as follows:

Definition 1.1 ([12]). Let $h : (0, 1) \rightarrow \mathbb{R}$ be a given function. A function $\mathcal{F} : \Omega \subset \mathbb{R}_+ \rightarrow \mathbb{R}$ is said to be harmonically h -convex function, if

$$\mathcal{F}\left(\frac{xy}{tx + (1-t)y}\right) \leq h(1-t)\mathcal{F}(x) + h(t)\mathcal{F}(y), \quad \forall x, y \in \Omega, t \in (0, 1).$$

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In this paper we will define a notion which is close to harmonic h -convex function and is named as "approximately harmonic h -convex functions." Before we proceed further let us recall some preliminary concepts and results. The following auxiliary result will play a significant role in obtaining some of the main results of the paper.

Lemma 1.2 ([7]). Let $\mathcal{F} : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a differentiable function on I^0 -the interior of I , $a, b \in I$, $a < b$ and $\mathcal{F}' \in L[a, b]$. Then

$$\frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx = \frac{ab(b-a)}{2} \int_0^1 \frac{1-2t}{A_t^2} \mathcal{F}'\left(\frac{ab}{A_t}\right) dt,$$

where $A_t = tb + (1-t)a$.

Let us recall the definitions of the Gamma function $\Gamma(\cdot)$ and Beta function $B(\cdot, \cdot)$ respectively.

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt,$$

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

It is known that [8]

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The integral form of the hypergeometric function is

$${}_2F_1(x, y; c; z) = \frac{1}{B(y, c-y)} \int_0^1 t^{y-1} (1-t)^{c-y-1} (1-zt)^{-x} dt$$

for $|z| < 1, c > y > 0$.

2. Approximately harmonic convexities

In this section, we define new class of "approximately harmonic h -convex functions" which is depending on a given function. We also discuss special cases of this class. First of all let $(\mathcal{X}, \|\cdot\|)$ be a real normed space. Ω denotes the nonempty harmonic convex subset of \mathcal{X} . $\mathfrak{d} : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ and $h : (0, 1) \rightarrow \mathbb{R}$ be the given function.

Definition 2.1. A function $\mathcal{F} : \Omega \rightarrow \mathbb{R}_+$ is said to be approximately harmonic h -convex function, if

$$\mathcal{F}\left(\frac{xy}{tx + (1-t)y}\right) \leq h(1-t)\mathcal{F}(x) + h(t)\mathcal{F}(y) + \mathfrak{d}(x, y), \quad \forall x, y \in \Omega, t \in (0, 1).$$

We now discuss some special cases of Definition .

I. If we take $\mathfrak{d}(x, y) = \epsilon(|x - y|)^\gamma$ for some $\epsilon \in \mathbb{R}$ and $\gamma > 1$ in Definition 2.1, we have a new definition of approximately harmonic convex function, which is called as " γ -paraharmonic h -convex function".

Definition 2.2. A function $\mathcal{F} : \Omega \rightarrow \mathbb{R}_+$ is said to be γ -paraharmonic h -convex function, if

$$\mathcal{F}\left(\frac{xy}{tx + (1-t)y}\right) \leq h(1-t)\mathcal{F}(x) + h(t)\mathcal{F}(y) + \epsilon(\|x - y\|)^\gamma, \quad \forall x, y \in \Omega, t \in (0, 1).$$

II. If we take $\mathfrak{d}(x, y) = \epsilon(|x - y|)$ for some $\epsilon \in \mathbb{R}$ in Definition 2.1, we have a new definition of approximately harmonic convex function, which is called as " ϵ -paraharmonic h -convex function".

Definition 2.3. A function $\mathcal{F} : \Omega \rightarrow \mathbb{R}_+$ is said to be ϵ -paraharmonic h -convex function, if

$$\mathcal{F}\left(\frac{xy}{tx + (1-t)y}\right) \leq h(1-t)\mathcal{F}(x) + h(t)\mathcal{F}(y) + \epsilon(\|x - y\|), \quad \forall x, y \in \Omega, t \in (0, 1).$$

III. If we take $d(x, y) = -\mu t(1-t)\left(\frac{1}{y} - \frac{1}{x}\right)^2$ for some $\mu > 0$ in Definition 2.1, we have a new definition of strongly harmonic h -convex function.

Definition 2.4. A function $\mathcal{F} : \Omega \rightarrow \mathbb{R}_+$ is said to be strongly harmonic h -convex function, if

$$\mathcal{F}\left(\frac{xy}{tx + (1-t)y}\right) \leq h(1-t)\mathcal{F}(x) + h(t)\mathcal{F}(y) - \mu t(1-t)\left(\frac{1}{y} - \frac{1}{x}\right)^2, \quad \forall x, y \in \Omega, t \in (0, 1).$$

IV. If we take $d(x, y) = \mu t(1-t)\left(\frac{1}{y} - \frac{1}{x}\right)^2$ for some $\mu > 0$ in Definition 2.1, we have a new definition of relaxed harmonic h -convex function.

Definition 2.5. A function $\mathcal{F} : \Omega \rightarrow \mathbb{R}_+$ is said to be relaxed harmonic h -convex function, if

$$\mathcal{F}\left(\frac{xy}{tx + (1-t)y}\right) \leq h(1-t)\mathcal{F}(x) + h(t)\mathcal{F}(y) + \mu t(1-t)\left(\frac{1}{y} - \frac{1}{x}\right)^2, \quad \forall x, y \in \Omega, t \in (0, 1).$$

V. If we take $d(x, y) = -t(1-t)F\left(\frac{xy}{x-y}\right)$ in Definition 2.1, we have the definition of strongly F -harmonic convex function [13].

Remark 2.6. Now one easily observe that for different suitable choices of function $h(\cdot)$, that are $h(t) = t, t^s, t^{-s}, t^{-1}$ and $h(t) = 1$, in Definitions 2.1, 2.2, 2.3, 2.4 and 2.5, we have other versions of these definitions for harmonic convex, harmonic s -convex of Breckner type, harmonic s -convex of Godunova-Levin-Dragomir type, Godunova-Levin type of harmonic convex and P -harmonic convex function respectively.

3. Main Results

In this section, we discuss the main results of the paper. For the sake of simplicity, we take $A_t = tb + (1-t)a$.

Theorem 3.1. Let $\mathcal{F} : \Omega \rightarrow \mathbb{R}$ be an integrable function. If \mathcal{F} is approximately harmonic h -convex function, then

$$\begin{aligned} & \frac{1}{2h\left(\frac{1}{2}\right)} \left[\mathcal{F}\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a} \int_a^b \frac{d\left(x, (a^{-1} + b^{-1} - x^{-1})^{-1}\right)}{x^2} dx \right] \\ & \leq \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \leq [\mathcal{F}(a) + \mathcal{F}(b)] \int_0^1 h(t) dt + d(a, b). \end{aligned}$$

Proof. Since \mathcal{F} is approximately harmonic h -convex function, then

$$\mathcal{F}\left(\frac{2xy}{x+y}\right) \leq h\left(\frac{1}{2}\right) [\mathcal{F}(x) + \mathcal{F}(y)] + d(x, y).$$

Using $x = \frac{ab}{ta + (1-t)b}$ and $y = \frac{ab}{(1-t)a + tb}$ in above inequality, we have

$$\begin{aligned} & \mathcal{F}\left(\frac{2ab}{a+b}\right) \\ & \leq h\left(\frac{1}{2}\right) \left[\mathcal{F}\left(\frac{ab}{ta + (1-t)b}\right) + \mathcal{F}\left(\frac{ab}{(1-t)a + tb}\right) \right] + d\left(\frac{ab}{ta + (1-t)b}, \frac{ab}{(1-t)a + tb}\right). \end{aligned}$$

Integrating above inequality with respect to t on $[0, 1]$, we have

$$\frac{1}{2h\left(\frac{1}{2}\right)} \left[\mathcal{F}\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a} \int_a^b \frac{\mathfrak{d}\left(x, (a^{-1} + b^{-1} - x^{-1})^{-1}\right)}{x^2} dx \right] \leq \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx. \quad (3.1)$$

Also

$$\mathcal{F}\left(\frac{ab}{ta + (1-t)b}\right) \leq h(1-t)\mathcal{F}(a) + h(t)\mathcal{F}(b) + \mathfrak{d}(a, b).$$

Integrating above inequality with respect to t on $[0, 1]$, we have

$$\frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \leq [\mathcal{F}(a) + \mathcal{F}(b)] \int_0^1 h(t) dt + \mathfrak{d}(a, b). \quad (3.2)$$

On summation of inequalities (3.1) and (3.2), we have the required result. \square

Now we shall discuss some special cases of Theorem 3.1.

Corollary 3.2. Let $\mathcal{F} : \Omega \rightarrow \mathbb{R}$ be an integrable function. If \mathcal{F} is approximately harmonic convex function, then

$$\begin{aligned} & \mathcal{F}\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a} \int_a^b \frac{\mathfrak{d}\left(x, (a^{-1} + b^{-1} - x^{-1})^{-1}\right)}{x^2} dx \\ & \leq \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \leq \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} + \mathfrak{d}(a, b). \end{aligned}$$

Corollary 3.3. Let $\mathcal{F} : \Omega \rightarrow \mathbb{R}$ be an integrable function. If \mathcal{F} is approximately harmonic s -convex function of Breckner type, then for $s \in [0, 1]$, then we have

$$\begin{aligned} & 2^{s-1} \left[\mathcal{F}\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a} \int_a^b \frac{\mathfrak{d}\left(x, (a^{-1} + b^{-1} - x^{-1})^{-1}\right)}{x^2} dx \right] \\ & \leq \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \leq \frac{\mathcal{F}(a) + \mathcal{F}(b)}{1+s} + \mathfrak{d}(a, b). \end{aligned}$$

Corollary 3.4. Let $\mathcal{F} : \Omega \rightarrow \mathbb{R}$ be an integrable function. If \mathcal{F} is approximately harmonic s -convex function of Godunova-Levin-Dragomir type, then for $s \in [0, 1)$, we have

$$\begin{aligned} & \frac{1}{2^{1+s}} \left[\mathcal{F}\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a} \int_a^b \frac{\mathfrak{d}\left(x, (a^{-1} + b^{-1} - x^{-1})^{-1}\right)}{x^2} dx \right] \\ & \leq \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \leq \frac{\mathcal{F}(a) + \mathcal{F}(b)}{1-s} + \mathfrak{d}(a, b). \end{aligned}$$

Corollary 3.5. Let $\mathcal{F} : \Omega \rightarrow \mathbb{R}$ be an integrable function. If \mathcal{F} is approximately harmonic P-convex function, then

$$\begin{aligned} & \mathcal{F}\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a} \int_a^b \frac{\mathfrak{d}\left(x, (a^{-1} + b^{-1} - x^{-1})^{-1}\right)}{x^2} dx \\ & \leq \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \leq [\mathcal{F}(a) + \mathcal{F}(b)] + \mathfrak{d}(a, b). \end{aligned}$$

Corollary 3.6. Let $\mathcal{F} : \Omega \rightarrow \mathbb{R}$ be an integrable function. If \mathcal{F} is strongly harmonic h-convex function, then

$$\begin{aligned} \frac{1}{2h\left(\frac{1}{2}\right)} \left[\mathcal{F}\left(\frac{2ab}{a+b}\right) - \frac{1}{30}\left(\frac{1}{b} - \frac{1}{a}\right)^2 \right] & \leq \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \\ & \leq [\mathcal{F}(a) + \mathcal{F}(b)] \int_0^1 h(t) dt + \frac{1}{6}\left(\frac{1}{b} - \frac{1}{a}\right)^2. \end{aligned}$$

Corollary 3.7. Let $\mathcal{F} : \Omega \rightarrow \mathbb{R}$ be an integrable function. If \mathcal{F} is strongly harmonic convex function, then

$$\mathcal{F}\left(\frac{2ab}{a+b}\right) - \frac{1}{30}\left(\frac{1}{b} - \frac{1}{a}\right)^2 \leq \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \leq \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} + \frac{1}{6}\left(\frac{1}{b} - \frac{1}{a}\right)^2.$$

Corollary 3.8. Let $\mathcal{F} : \Omega \rightarrow \mathbb{R}$ be an integrable function. If \mathcal{F} is strongly harmonic s-convex function of Breckner type, then

$$2^{s-1} \left[\mathcal{F}\left(\frac{2ab}{a+b}\right) - \frac{1}{30}\left(\frac{1}{b} - \frac{1}{a}\right)^2 \right] \leq \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \leq \frac{\mathcal{F}(a) + \mathcal{F}(b)}{1+s} + \frac{1}{6}\left(\frac{1}{b} - \frac{1}{a}\right)^2.$$

Corollary 3.9. Let $\mathcal{F} : \Omega \rightarrow \mathbb{R}$ be an integrable function. If \mathcal{F} is strongly harmonic s-convex function of Godunova-Levin-Dragomir type, then

$$\frac{1}{2^{1+s}} \left[\mathcal{F}\left(\frac{2ab}{a+b}\right) - \frac{1}{30}\left(\frac{1}{b} - \frac{1}{a}\right)^2 \right] \leq \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \leq \frac{\mathcal{F}(a) + \mathcal{F}(b)}{1-s} + \frac{1}{6}\left(\frac{1}{b} - \frac{1}{a}\right)^2.$$

Corollary 3.10. Let $\mathcal{F} : \Omega \rightarrow \mathbb{R}$ be an integrable function. If \mathcal{F} is strongly harmonic P-convex function, then

$$\frac{1}{2} \left[\mathcal{F}\left(\frac{2ab}{a+b}\right) - \frac{1}{30}\left(\frac{1}{b} - \frac{1}{a}\right)^2 \right] \leq \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \leq [\mathcal{F}(a) + \mathcal{F}(b)] + \frac{1}{6}\left(\frac{1}{b} - \frac{1}{a}\right)^2.$$

Remark 3.11. 1. It is worth to mention here that the above discussed special cases can easily be deduced from Theorem 3.1 by considering suitable choices of functions $h(\cdot)$ and $\mathfrak{d}(x, y)$.

2. One can easily obtain the results for relaxed harmonic convexity by choosing $\mathfrak{d}(x, y) = \mu t(1-t)\left(\frac{1}{y} - \frac{1}{x}\right)^2$ in Theorem 3.1.

Theorem 3.12. Let $\mathcal{F} : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a differentiable function on I^0 -the interior of I , $a, b \in I$, $a < b$ and $\mathcal{F}' \in L[a, b]$. If $|\mathcal{F}'|^q$ approximate harmonic h -convex function, then

$$\left| \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} C^{1-\frac{1}{q}}(a, b) [S_1(a, b; h)|\mathcal{F}(a)|^q + S_2(a, b; h)|\mathcal{F}(b)|^q + \mathfrak{d}(a, b)C(a, b)]^{\frac{1}{q}},$$

where $S_1(a, b; h) = \int_0^1 \frac{|1-2t|}{A_t^2} h(t) dt$ and $S_2(a, b; h) = \int_0^1 \frac{|1-2t|}{A_t^2} h(1-t) dt$.

Proof. Utilizing Lemma 1.2 and taking modulus on both sides, we have

$$\left| \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \right| = \left| \frac{ab(b-a)}{2} \int_0^1 \frac{1-2t}{A_t^2} \mathcal{F}'\left(\frac{ab}{A_t}\right) dt \right|.$$

Utilizing the property of modulus, we have

$$\left| \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} \int_0^1 \frac{|1-2t|}{A_t^2} \left| \mathcal{F}'\left(\frac{ab}{A_t}\right) \right| dt.$$

Using power-means inequality, we have

$$\left| \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} \left(\int_0^1 \frac{|1-2t|}{A_t^2} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 \frac{|1-2t|}{A_t^2} \left| \mathcal{F}'\left(\frac{ab}{A_t}\right) \right|^q dt \right)^{\frac{1}{q}}.$$

Since it is given that $|\mathcal{F}'|^q$ is approximate harmonic h -convex function, so we have

$$\begin{aligned} & \left| \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{2} C^{1-\frac{1}{q}}(a, b) \left(\int_0^1 \frac{|1-2t|}{A_t^2} [h(t)|\mathcal{F}(a)|^q + h(1-t)|\mathcal{F}(b)|^q + \mathfrak{d}(a, b)] dt \right)^{\frac{1}{q}} \\ & = \frac{ab(b-a)}{2} C^{1-\frac{1}{q}}(a, b) \left(|\mathcal{F}(a)|^q \int_0^1 \frac{|1-2t|}{A_t^2} h(t) dt + |\mathcal{F}(b)|^q \int_0^1 \frac{|1-2t|}{A_t^2} h(1-t) dt + \mathfrak{d}(a, b)C(a, b) \right)^{\frac{1}{q}}, \end{aligned}$$

here

$$\begin{aligned} C(a, b) & := \int_0^1 \frac{|1-2t|}{A_t^2} dt \\ & = a^{-2} \left[{}_2F_1\left(2, 2; 3; 1 - \frac{b}{a}\right) - {}_2F_1\left(2, 1; 2; 1 - \frac{b}{a}\right) + \frac{1}{2} {}_2F_1\left(2, 1; 3; \frac{1}{2}\left(1 - \frac{b}{a}\right)\right) \right]. \end{aligned} \tag{3.3}$$

This completes the proof. \square

Now we discuss some special cases of Theorem 3.12.

Corollary 3.13. Let $\mathcal{F} : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a differentiable function on I^0 -the interior of I , $a, b \in I$, $a < b$ and $\mathcal{F}' \in L[a, b]$. If $|\mathcal{F}'|^q$ approximate harmonic convex function, then

$$\left| \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} C^{1-\frac{1}{q}}(a, b) [S_3(a, b)|\mathcal{F}'(a)|^q + S_4(a, b)|\mathcal{F}'(b)|^q + d(a, b)C(a, b)]^{\frac{1}{q}},$$

where

$$S_3(a, b) = \int_0^1 \frac{|1-2t|}{A_t^2} t dt = a^{-2} \left[\frac{2}{3} {}_2F_1\left(2, 3; 4; 1 - \frac{b}{a}\right) - \frac{1}{2} {}_2F_1\left(2, 2; 3; 1 - \frac{b}{a}\right) + \frac{1}{12} {}_2F_1\left(2, 2; 4; \frac{1}{2}\left(1 - \frac{b}{a}\right)\right) \right],$$

$$S_4(a, b) = \int_0^1 \frac{|1-2t|}{A_t^2} (1-t) dt = a^{-2} \left[\frac{1}{3} {}_2F_1\left(2, 2; 4; 1 - \frac{b}{a}\right) - \frac{1}{2} {}_2F_1\left(2, 1; 3; 1 - \frac{b}{a}\right) + \frac{1}{2} {}_2F_1\left(2, 1; 3; \frac{1}{2}\left(1 - \frac{b}{a}\right)\right) \right],$$

respectively.

Corollary 3.14. Let $\mathcal{F} : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a differentiable function on I^0 -the interior of I , $a, b \in I$, $a < b$ and $\mathcal{F}' \in L[a, b]$. If $|\mathcal{F}'|^q$ approximate harmonic s -convex function of Breckner type, then

$$\left| \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} C^{1-\frac{1}{q}}(a, b) [S_5(a, b)|\mathcal{F}'(a)|^q + S_6(a, b)|\mathcal{F}'(b)|^q + d(a, b)C(a, b)]^{\frac{1}{q}},$$

where

$$S_5(a, b) = \int_0^1 \frac{|1-2t|}{A_t^2} t^s dt = a^{-2} \left[\frac{2}{s+2} {}_2F_1\left(2, s+2; s+3; 1 - \frac{b}{a}\right) - \frac{1}{s+1} {}_2F_1\left(2, s+1; s+2; 1 - \frac{b}{a}\right) + \frac{1}{2^s(s+1)(s+2)} {}_2F_1\left(2, s+1; s+3; \frac{1}{2}\left(1 - \frac{b}{a}\right)\right) \right],$$

$$S_6(a, b) = \int_0^1 \frac{|1-2t|}{A_t^2} (1-t)^s dt = a^{-2} \left[\frac{2}{(s+1)(s+2)} {}_2F_1\left(2, 2; s+3; 1 - \frac{b}{a}\right) - \frac{1}{s+1} {}_2F_1\left(2, 1; s+2; 1 - \frac{b}{a}\right) + \frac{1}{2} {}_2F_1\left(2, 1; 3; \frac{1}{2}\left(1 - \frac{b}{a}\right)\right) \right],$$

respectively.

Corollary 3.15. Let $\mathcal{F} : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a differentiable function on I^0 -the interior of I , $a, b \in I$, $a < b$ and $\mathcal{F}' \in L[a, b]$. If $|\mathcal{F}'|^q$ approximate harmonic s -convex function of Godunova-Levin-Dragomir type, then

$$\left| \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} C^{1-\frac{1}{q}}(a, b) [S_7(a, b)|\mathcal{F}(a)|^q + S_8(a, b)|\mathcal{F}(b)|^q + \mathfrak{d}(a, b)C(a, b)]^{\frac{1}{q}},$$

where

$$\begin{aligned} S_7(a, b) &= \int_0^1 \frac{|1-2t|}{A_t^2} t^{-s} dt \\ &= a^{-2} \left[\frac{2}{2-s} {}_2F_1\left(2, 2-s; 3-s; 1-\frac{b}{a}\right) - \frac{1}{1-s} {}_2F_1\left(2, 1-s; 2-s; 1-\frac{b}{a}\right) \right. \\ &\quad \left. + \frac{2^s}{(1-s)(2-s)} {}_2F_1\left(2, 1-s; 3-s; \frac{1}{2}\left(1-\frac{b}{a}\right)\right) \right], \end{aligned}$$

$$\begin{aligned} S_8(a, b) &= \int_0^1 \frac{|1-2t|}{A_t^2} (1-t)^{-s} dt \\ &= a^{-2} \left[\frac{1}{(1-s)(2-s)} {}_2F_1\left(2, 2; 3-s; 1-\frac{b}{a}\right) \right. \\ &\quad \left. - \frac{1}{1-s} {}_2F_1\left(2, 1; 2-s; 1-\frac{b}{a}\right) + \frac{1}{2} {}_2F_1\left(2, 1; 3; \frac{1}{2}\left(1-\frac{b}{a}\right)\right) \right], \end{aligned}$$

respectively.

Corollary 3.16. Let $\mathcal{F} : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a differentiable function on I^0 -the interior of I , $a, b \in I$, $a < b$ and $\mathcal{F}' \in L[a, b]$. If $|\mathcal{F}'|^q$ approximate harmonic P -convex function, then

$$\left| \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} C(a, b) [|\mathcal{F}(a)|^q + |\mathcal{F}(b)|^q + \mathfrak{d}(a, b)C(a, b)]^{\frac{1}{q}},$$

where $C(a, b)$ is given by (3.3).

Remark 3.17. The above discussed special cases can be obtained easily from Theorem 3.12 by taking suitable choices of function $h(\cdot)$.

Theorem 3.18. Let $\mathcal{F} : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a differentiable function on I^0 -the interior of I , $a, b \in I$, $a < b$ and $\mathcal{F}' \in L[a, b]$. If $|\mathcal{F}'|^q$ is approximate harmonic h -convex function, then

$$\left| \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} [S_9(a, b; h)|\mathcal{F}(a)|^q + S_{10}(a, b; h)|\mathcal{F}(b)|^q + S_{11}(a, b)\mathfrak{d}(a, b)]^{\frac{1}{q}},$$

where $S_9(a, b; h) = \int_0^1 \frac{1}{A_t^{2q}} h(t) dt$, $S_{10}(a, b; h) = \int_0^1 \frac{1}{A_t^{2q}} h(1-t) dt$ and $S_{11}(a, b) = \int_0^1 \frac{1}{A_t^{2q}} dt = {}_2F_1\left(-2q, 1; 2; 1-\frac{a}{b}\right)$.

Proof. Utilizing Lemma 1.2 and taking modulus on both sides, we have

$$\left| \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \right| = \left| \frac{ab(b-a)}{2} \int_0^1 \frac{1-2t}{A_t^2} \mathcal{F}'\left(\frac{ab}{A_t}\right) dt \right|.$$

Utilizing the property of modulus, we have

$$\left| \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} \int_0^1 \frac{|1-2t|}{A_t^2} \left| \mathcal{F}'\left(\frac{ab}{A_t}\right) \right| dt.$$

Using Hölder’s inequality and the fact that $|\mathcal{F}|^q$ is approximate harmonic h -convex function, we have

$$\begin{aligned} & \left| \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{2} \left(\int_0^1 |1-2t| dt \right)^{\frac{1}{p}} \left(\int_0^1 \frac{1}{A_t^{2q}} [h(t)|\mathcal{F}(a)|^q + h(1-t)|\mathcal{F}(b)|^q + \mathfrak{d}(a,b)] dt \right)^{\frac{1}{q}} \\ & = \frac{ab(b-a)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\int_0^1 \frac{1}{A_t^{2q}} [h(t)|\mathcal{F}(a)|^q + h(1-t)|\mathcal{F}(b)|^q] dt + S_{11}(a,b)\mathfrak{d}(a,b) \right)^{\frac{1}{q}}, \end{aligned}$$

where

$$S_{11}(a,b) = \int_0^1 \frac{1}{A_t^{2q}} dt = {}_2F_1\left(-2q, 1; 2; 1 - \frac{a}{b}\right). \tag{3.4}$$

This completes the proof. \square

Now we discuss some special cases of Theorem 3.18.

Corollary 3.19. Let $\mathcal{F} : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a differentiable function on I^0 -the interior of I , $a, b \in I$, $a < b$ and $\mathcal{F}' \in L[a, b]$. If $|\mathcal{F}|^q$ is approximate harmonic convex function, then

$$\begin{aligned} & \left| \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} (S_{12}(a,b)|\mathcal{F}(a)|^q + S_{13}(a,b)|\mathcal{F}(b)|^q + S_{11}(a,b)\mathfrak{d}(a,b))^{\frac{1}{q}}, \end{aligned}$$

where

$$S_{12}(a,b) = \int_0^1 \frac{t}{A_t^{2q}} dt = a^{-2} \left[{}_2F_1\left(-2q, 2; 3; 1 - \frac{b}{a}\right) \right],$$

$$\begin{aligned} S_{13}(a,b) &= \int_0^1 \frac{(1-t)}{A_t^{2q}} dt \\ &= 2a^{-2} \left[{}_2F_1\left(-2q, 1; 2; 1 - \frac{b}{a}\right) - {}_2F_1\left(-2q, 2; 3; 1 - \frac{b}{a}\right) \right], \end{aligned}$$

respectively.

Corollary 3.20. Let $\mathcal{F} : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a differentiable function on I^0 -the interior of I , $a, b \in I$, $a < b$ and $\mathcal{F}' \in L[a, b]$. If $|\mathcal{F}|^q$ is approximate harmonic s -convex function of Breckner type, then

$$\left| \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} (S_{14}(a, b)|\mathcal{F}(a)|^q + S_{15}(a, b)|\mathcal{F}(b)|^q + S_{11}(a, b)d(a, b))^{\frac{1}{q}},$$

where

$$S_{14}(a, b) = \int_0^1 \frac{t^s}{A_t^{2q}} dt = (s+1)a^{-2} \left[{}_2F_1 \left(-2q, 1+s; 2+s; 1 - \frac{b}{a} \right) \right],$$

and

$$S_{15}(a, b) = \int_0^1 \frac{(1-t)^s}{A_t^{2q}} dt = (s+1)a^{-2} \left[{}_2F_1 \left(-2q, 1; 2+s; 1 - \frac{b}{a} \right) \right],$$

respectively.

Corollary 3.21. Let $\mathcal{F} : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a differentiable function on I^0 -the interior of I , $a, b \in I$, $a < b$ and $\mathcal{F}' \in L[a, b]$. If $|\mathcal{F}|^q$ is approximate harmonic s -convex function of Godunova-Levin-Dragomir type, then

$$\left| \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} (S_{16}(a, b)|\mathcal{F}(a)|^q + S_{17}(a, b)|\mathcal{F}(b)|^q + S_{11}(a, b)d(a, b))^{\frac{1}{q}},$$

where

$$S_{16}(a, b) = \int_0^1 \frac{t^{-s}}{A_t^{2q}} dt = (1-s)a^{-2} \left[{}_2F_1 \left(-2q, 1-s; 2-s; 1 - \frac{b}{a} \right) \right],$$

$$S_{17}(a, b) = \int_0^1 \frac{(1-t)^{-s}}{A_t^{2q}} dt = (1-s)a^{-2} \left[{}_2F_1 \left(-2q, 1-s; 2-s; 1 - \frac{b}{a} \right) \right],$$

respectively.

Corollary 3.22. Let $\mathcal{F} : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a differentiable function on I^0 -the interior of I , $a, b \in I$, $a < b$ and $\mathcal{F}' \in L[a, b]$. If $|\mathcal{F}|^q$ is approximate harmonic P -convex function, then

$$\left| \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{\mathcal{F}(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} S_{11}^{\frac{1}{q}}(a, b) (|\mathcal{F}(a)|^q + |\mathcal{F}(b)|^q + d(a, b))^{\frac{1}{q}},$$

where $S_{11}(a, b)$ is given by (3.4).

Remark 3.23. The above discussed special cases can be obtained easily from Theorem 3.18 by taking suitable choices of function $h(\cdot)$.

Conclusion

We have introduced the notion of “approximately harmonic h -convex functions”. It has been observed that this generalization contains several other classes of approximately harmonic convexity. Some new integral inequalities of Hermite-Hadamard type are also obtained. We have also discussed some of the special cases of the results obtained in the main section of the paper. It is to remember that one can obtain new results for other types of approximate harmonic convexity by taking suitable choices of the function $\mathfrak{d}(\cdot, \cdot)$.

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