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Fixed Points for $\alpha - \beta_E$-Geraghty Contractions on b-Metric Spaces and Applications to Matrix Equations

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Abstract. In this paper, we introduce the notion of $\alpha - \beta_E$-Geraghty contraction type mappings on b-metric spaces and prove the existence and uniqueness of fixed point for such mappings. These results are generalizations of the recent results in [Fulga and Proca, Fixed points for φ_E-Geraghty contractions, J. Nonlinear Sci. Appl. 10 (2017), 5125-5131]. We give some examples illustrating the presented results. An application on matrix equations and numerical algorithms are also provided.

1. Introduction

It is known that the Banach contraction principle is considered as a one of the most important theorems in the classical functional analysis. There are many generalizations of this theorem. The following generalization is due to Geraghty [13].

Theorem 1.1. *[13] Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. If T satisfies the following inequality:*

$$d(Tx, Ty) \leq \beta(d(x, y))\, d(x, y)$$

for all $x, y \in X$, where $\beta : [0, \infty) \to [0, 1)$ is a function which satisfies the condition

$$\lim_{n\to\infty} \beta(t_n) = 1 \quad \textit{implies} \quad \lim_{n\to\infty} t_n = 0,$$

then T has a unique fixed point $u \in X$ and $\{T^n x\}$ converges to u for each $x \in X$.

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In 2014, Popescu [22] studied the existence and uniqueness of a fixed point of generalized α-Geraghty contraction type mappings in complete metric spaces.

Definition 1.2. *[22] Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a function. A map $T : X \to X$ is called a generalized α-Geraghty contraction type map if there exists a function $\beta : [0, \infty) \to [0, 1)$ satisfying the following condition:*

$$\lim_{n\to\infty} \beta(t_n) = 1 \quad \textit{implies} \quad \lim_{n\to\infty} t_n = 0$$

such that for all $x, y \in X$,

$$\alpha(x, y)d(Tx, Ty)\beta(M_T(x, y))\, M_T(x, y),$$

where $M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$.

Definition 1.3. *[22] For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}$ be given mappings. We say that T is α-orbital admissible if for all $x \in X$, we have*

$$\alpha(x, Tx) \geq 1 \Rightarrow \alpha(Tx, T^2x) \geq 1.$$

Definition 1.4. *[22] Let $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}$ be given mappings. A mapping $T : X \to X$ is called a triangular α-orbital admissible if*

(T_1) T is α-orbital admissible;

(T_2) $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1 \Rightarrow \alpha(x, Ty) \geq 1,\ x, y \in X$.

Theorem 1.5. *[22] Let (X, d) be a complete metric space and $\alpha : X\times X \to \mathbb{R}$ be a function. Given the map $T : X \to X$. Suppose that the following conditions are satisfied:*

(1) *T is a generalized α-Geraghty contraction type mapping;*

(2) *T is a triangular α-orbital admissible mapping;*

(3) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 0$;*

(4) *T is continuous.*

Then T has a fixed point $z \in X$ and $\{T^n x_0\}$ converges to z.

Now, for $s \geq 1$, denote by $\mathcal{F}_s$ the family of functions $\beta : [0, \infty) \to [0, \frac{1}{s})$ satisfying the condition:

$$\lim_{n\to\infty} \beta(t_n) = \frac{1}{s} \quad \text{implies} \quad \lim_{n\to\infty} t_n = 0.$$

If $s = 1$, put $\mathcal{F} = \mathcal{F}_1$.

Note that the notion of b-metric space is introduced by Czerwik [10] as a generalization of metric spaces.

Definition 1.6. *[10] Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a function such that for all $x, y, z \in X$ and some $s \geq 1$,*

(1) $d(x, y) = 0 \Leftrightarrow x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

Then d is called a b-metric on X and (X, d, s) is called a b-metric space.

Many fixed point results have been presented in this setting (and its generalization). For more details, see [1, 3–5, 15, 17, 24, 25].

Definition 1.7. *[10] Let (X, d, s) be a b-metric space. Then*

(a) a sequence $\{x_n\}$ in X is said to be a Cauchy if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

(b) a sequence $\{x_n\}$ in X is said to be convergent to x if $\lim_{n,m\to\infty} d(x_n, x) = 0$.

(b) (X, d, s) is said to be complete if every Cauchy sequence $\{x_n\}$ in X is convergent.

We state the following lemma.

Lemma 1.8. *In a b-metric space (X, d), the limit for a convergent sequence is unique. If $x_n \to u$, we have for all $y \in X$*

$$\frac{1}{s} d(u, y) \le \limsup_{n\to\infty} d(x_n, y) \le s d(u, y).$$

In 2011, Dukić et al. [11] obtained the following result.

Theorem 1.9. *[11] Let (X, d, s) be a complete b-metric space and $T : X \to X$ be a mapping. Suppose that there exists $\beta \in \mathcal{F}_s$ such for all $x, y \in X$,*

$$d(Tx, Ty) \le \beta(d(x, y))\, d(x, y).$$

Then T has a unique fixed point $u \in X$ and $\{T^n x\}$ converges to u for all $x \in X$.

Very recently, Fulga and Proca [12] introduced the notion of φ_E-Geraghty contractions and established a fixed point result.

Definition 1.10. *[12] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an φ_E-Geraghty contraction on (X, d) if there exists $\varphi \in \mathcal{F}$ such that*

$$d(Tx, Ty) \le \varphi(E(x, y))\, E(x, y)$$

for all $x, y \in X$, where $E(x, y) = d(x, y) + |d(x, Tx) - d(y, Ty)|$.

Theorem 1.11. *[12] Let (X, d) be a complete metric space and $T : X \to X$ be an φ_E-Geraghty contraction. Then T has a unique fixed point $u \in X$ and $\{T^n x\}$ converges to u for all $x \in X$.*

For other results using variant Geraghty type contractions, see [2, 6, 7, 9, 21]. Now, we introduce the notion of $\alpha - \beta_E$-Geraghty contraction type mappings in the context of b-metric spaces.

Definition 1.12. *Let (X, d, s) be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. A mapping $T : X \to X$ is said to be an $\alpha - \beta_E$-Geraghty type contraction if there exists $\beta \in \mathcal{F}_s$ such that*

$$\alpha(x, y) \ge 1 \Rightarrow d(Tx, Ty) \le \beta(E(x, y))\, E(x, y) \tag{1}$$

for all $x, y \in X$, where $E(x, y) = d(x, y) + |d(x, Tx) - d(y, Ty)|$.

The aim of this paper is to prove fixed point theorems for above mappings. We get a generalization of Theorem 1.11. Our obtained results are supported by two examples and an application on matrix equations. The convergence of an iterative method is studied for two different initial approximative solutions.

2. Main results

The following theorem is a sufficient condition for the existence of a fixed point for an α-Geraghty contraction type mapping in b-metric spaces.

Theorem 2.1. *Let (X, d, s) be a complete b-metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. Given a map $T : X \to X$. Suppose that the following conditions are satisfied:*

(*i*) *T is an $\alpha - \beta_E$-Geraghty contraction type mapping;*

(*ii*) *T is a triangular α-orbital admissible mapping;*

(*iii*) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 0$;*

(*iv*) *T is continuous.*

Then T has a fixed point $z \in X$ and $\{T^n x_0\}$ converges to z.

Proof. By assumption (*iii*), there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. We define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1} = T^n x_0$ for all $n \geq 1$. Suppose that $x_n = x_{n+1} = Tx_n$ for some n, so the proof is completed. Consequently, throughout the proof, we assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. We denote by $d_n = d(x_{n-1}, x_n)$ for all $n \geq 1$.

We have $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$. Since T is α-orbital admissible, by induction we have

$$\alpha(x_n, x_{n+1}) = \alpha(T^n x_0, T^{n+1} x_0) \geq 1 \quad \text{for all } \ n \geq 0.$$

T is triangular α-orbital admissible, then

$$\alpha(x_n, x_{n+1}) \geq 1 \ \text{ and } \ \alpha(x_{n+1}, Tx_{n+1}) \geq 1 \Rightarrow \alpha(x_n, x_{n+2}) \geq 1.$$

By induction, we get

$$\alpha(x_n, x_m) \geq 1 \quad \text{for all } \ m > n \geq 0.$$

From (1), we have

$$0 < d_{n+1} = d(Tx_{n-1}, Tx_n) \leq \beta(E(x_{n-1}, x_n))\, E(x_{n-1}, x_n), \quad n \geq 1. \tag{2}$$

Note that

$$E(x_{n-1}, x_n) = d(x_{n-1}, x_n) + |d(x_{n-1}, Tx_{n-1}) - d(x_n, Tx_n)| = d_n + |d_n - d_{n+1}|.$$

So (2) becomes

$$d_{n+1} \leq \beta(d_n + |d_n - d_{n+1}|)\,(d_n + |d_n - d_{n+1}|). \tag{3}$$

Assume that there exists $n > 0$ such that $d_n \leq d_{n+1}$. By (3), we get

$$d_{n+1} \leq \beta(d_{n+1})\, d_{n+1} < s^{-1} d_{n+1},$$

which is a contradiction. Thus, for all $n \geq 0$, $d_{n+1} < d_n$. Therefore, (3) becomes

$$0 < d_{n+1} \leq \beta(2d_n - d_{n+1})\,(2d_n - d_{n+1}), \quad \forall n = 0, \cdots \tag{4}$$

The real sequence $\{d_n\}$ is decreasing, so there exists $t \geq 0$ such that $\lim_{n\to\infty} d_n = t$. Suppose that $t > 0$. Take $n \to \infty$ in (4) to write

$$s^{-1}t = s^{-1} \lim_{n\to\infty} d_{n+1} \leq \lim_{n\to\infty} d_{n+1} \leq \lim_{n\to\infty} [\beta(2d_n - d_{n+1})\,(2d_n - d_{n+1})] \leq s^{-1} \lim_{n\to\infty} (2d_n - d_{n+1}) = s^{-1} t.$$

We obtain

$$\lim_{n\to\infty}[\beta(2d_n - d_{n+1})\,(2d_n - d_{n+1})] = s^{-1}t.$$

Therefore

$$\lim_{n\to\infty}\beta(2d_n - d_{n+1}) = s^{-1}.$$

Since $\beta \in \mathcal{F}_s$, we get

$$t = \lim_{n\to\infty}(2d_n - d_{n+1}) = 0,$$

which is a contradiction. Thus

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{5}$$

We shall prove that $\{x_n\}$ is a Cauchy sequence. We argue by contradiction. Then, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ such that for every k

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon. \tag{6}$$

Moreover, corresponding to $n(k)$ we can choose $m(k)$ in such a way that it is the smallest integer with $m(k) > n(k)$ and satisfying (6). Then

$$d(x_{m(k)-1}, x_{n(k)}) < \varepsilon. \tag{7}$$

Since $\alpha(x_{n(k)}, x_{m(k)}) \geq 1$, it follows from (1) and (6)

$$s\varepsilon \leq sd(x_{n(k)}, x_{m(k)}) \leq s\beta(E((x_{n(k)-1}, x_{m(k)-1}))\,E((x_{n(k)-1}, x_{m(k)-1}) < E(x_{n(k)-1}, x_{m(k)-1}), \tag{8}$$

where

$$E(x_{n(k)-1}, x_{m(k)-1}) = d(x_{n(k)-1}, x_{m(k)-1}) + |d(x_{n(k)-1}, x_{n(k)}) - d(x_{m(k)-1}, x_{m(k)})|.$$

By the triangle inequality and (7), we get

$$\begin{aligned} E((x_{n(k)-1}, x_{m(k)-1}) &\leq sd((x_{n(k)-1}, x_{n(k)}) + sd((x_{n(k)}, x_{m(k)-1}) \\ &\quad + |d((x_{n(k)-1}, x_{n(k)}) - d(x_{m(k)-1}, x_{m(k)})| \\ &\leq s\varepsilon + sd((x_{n(k)-1}, x_{n(k)}) + |d((x_{n(k)-1}, x_{n(k)}) - d(x_{m(k)-1}, x_{m(k)})|. \end{aligned} \tag{9}$$

Combining (5), (8) and (9), we obtain

$$\lim_{k\to\infty} s\beta(E((x_{n(k)-1}, x_{m(k)-1}))\,E((x_{n(k)-1}, x_{m(k)-1}) = \lim_{k\to\infty} E((x_{n(k)-1}, x_{m(k)-1}) = s\varepsilon. \tag{10}$$

We deduce

$$\lim_{k\to\infty}\beta(E((x_{n(k)-1}, x_{m(k)-1})) = s^{-1}.$$

Since $\beta \in \mathcal{F}_s$, we have

$$\lim_{k\to\infty} E(x_{n(k)-1}, x_{m(k)-1}) = 0,$$

which is a contradiction with respect to (10). Thus $\{x_n\}$ is a Cauchy sequence in the complete b-metric space (X, d, s). So there exists $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0.$$

Since T is continuous, we have $z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n) = Tz$. It implies that z is a fixed point of T. Moreover, as $x_n = T^n x_0$, we get $\{T^n x_0\}$ converges to u. □

In the following theorem, we replace the continuity of the mapping T in Theorem 2.1 by another condition.

Theorem 2.2. *Let (X, d, s) be a complete b-metric space, $\alpha : X \times X \to \mathbb{R}$ be a function and $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

(i) *T is an $\alpha - \beta_E$-Geraghty contraction type mapping;*

(ii) *T is a triangular α-orbital admissible mapping;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 0$;*

(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.*

Then T has a fixed point $z \in X$ and $\{T^n x_0\}$ converges to z.

Proof. Following the lines in the proof of Theorem 2.1, we conclude that the sequence defined by $x_n = T^n x_0$ converges to $z \in X$. By using hypothesis (iv), we deduce that there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, z) \geq 1$ for all k. Since T is an $\alpha - \beta_E$-Geraghty contraction type mapping, we have for all k,

$$d(x_{n(k)+1}, Tz) \leq \beta(E((x_{n(k)}, z))\, E((x_{n(k)}, z), \tag{11}$$

where

$$E(x_{n(k)}, z) = d(x_{n(k)}, z) + |d(x_{n(k)}, x_{n(k)+1}) - d(z, Tz)|.$$

Suppose that $d(z, Tz) > 0$. By the triangle inequality and (11), we have for all k

$$s^{-1} d(z, Tz) - d(z, x_{n(k)+1}) \leq d(x_{n(k)+1}, Tz) \leq \beta(E((x_{n(k)}, z))\, E((x_{n(k)}, z) < s^{-1} E(x_{n(k)}, z).$$

Passing to limit as $k \to \infty$ in the above inequality, we get

$$\lim_{k\to\infty} \beta(E((x_{n(k)-1}, x_{m(k)-1}))\, E(x_{n(k)-1}, x_{m(k)-1}) = \lim_{k\to\infty} s^{-1} E(x_{n(k)}, z) = s^{-1} d(z, Tz). \tag{12}$$

We deduce that

$$\lim_{k\to\infty} \beta(E(x_{n(k)}, z)) = s^{-1}.$$

Since $\beta \in \mathcal{F}_s$, we have

$$\lim_{k\to\infty} E(x_{n(k)-1}, z) = 0,$$

which is a contradiction with respect to (12). Since $\beta \in \mathcal{F}_s$, we have

$$\lim_{k\to\infty} E(x_{n(k)-1}, x_{m(k)-1}) = 0,$$

which is a contradiction with respect to (10). Hence $d(z, Tz) = 0$, so z is a fixed point of T. Also, $\{T^n x_0\}$ converges to z. □

Now, we prove the uniqueness of such fixed point. For this, we need the following additional condition.
(U): For all $x, y \in Fix(T)$, we have $\alpha(x, y) \geq 1$, where $Fix(T)$ denotes the set of fixed points of T.

Theorem 2.3. *Adding condition (U) to the hypotheses of Theorem 2.1 (resp. Theorem 2.2), we obtain that z is the unique fixed point of T.*

Proof. We argue by contradiction, that is, there exist $z, w \in X$ such that $z = Tz$ and $w = Tw$ with $z \neq w$. By assumption (U), we have $\alpha(z, w) \geq 1$. So, by (1), we get

$$\begin{aligned} d(z, w) = d(Tz, Tw) &\leq \beta(E(z, w))E(z, w) < s^{-1}E(z, w) \\ &= s^{-1}[d(z, w) + |d(z, Tz) - d(w, Tw)|] = s^{-1}d(z, w), \end{aligned}$$

which is a contradiction. Hence $z = w$. □

Letting $\alpha(x, y) = 1$ in Theorem 2.2, we state the following corollary.

Corollary 2.4. *Let (X, d, s) be a complete b-metric space and let $T : X \to X$ be a map. Suppose there exists $\beta \in \mathcal{F}_s$ such that*

$$d(Tx, Ty) \leq \beta(E(x, y))\, E(x, y) \tag{13}$$

for all $x, y \in X$, where $E(x, y) = d(x, y) + |d(x, Tx) - d(y, Ty)|$. Then T has a unique fixed point $z \in X$ and $\{T^n x_0\}$ converges to z for any $x_0 \in X$.

We may also state the following two consequences.

Corollary 2.5. *Let (X, d, s) be a complete b-metric space and let $T : X \to X$ be such that*

$$d(Tx, Ty) \leq \frac{E(x, y)}{s + E(x, y)} \tag{14}$$

for all $x, y \in X$. Then T has a unique fixed point $z \in X$ and $\{T^n x_0\}$ converges to z for any $x_0 \in X$.

Proof. Take

$$\beta(t) = \begin{cases} \frac{1}{s+t} & \text{if } t > 0 \\ \frac{1}{s+1} & \text{if } t = 0. \end{cases}$$

Clearly, $\beta \in \mathcal{F}_s$. If $x \neq y$, $E(x, y) \neq 0$, so (14) becomes

$$d(Tx, Ty) \leq \beta(E(x, y)).E(x, y)$$

In the case $x = y$, we have $d(Tx, Ty) = E(x, y) = 0$ and so $d(Tx, Ty) \leq \beta(E(x, y)).E(x, y)$. Applying Corollary 2.4, the proof is completed. □

Corollary 2.6. *Let (X, d, s) be a complete b-metric space and let $T : X \to X$ be such that*

$$d(Tx, Ty) \leq qE(x, y) \tag{15}$$

for all $x, y \in X$, where $q \in (0, \frac{1}{s})$. Then T has a unique fixed point $z \in X$ and $\{T^n x_0\}$ converges to z, or all $x_0 \in X$. Moreover, we have

$$d(T^n x_0, z) \leq \gamma^{n-1} \frac{\lambda s}{1 - \gamma} d(Tx_0, x_0), \tag{16}$$

where

$$\gamma = \frac{2q}{1 + q}, \quad \lambda = \sum_{n \geq 1} s^{2n} \gamma^{2^{n-1}}.$$

Proof. It suffices to consider $\beta(t) = q$ for all $t \geq 0$ in Corollary 2.4. Let $x_0 \in X$ and $x_n = T^n x_0$. From (15),

$$d(x_n, x_{n+1}) \leq qE(x_{n-1}, x_n),$$

where $E(x_{n-1}, x_n) = d(x_{n-1}, x_n) + |d(x_{n-1}, x_n) - d(x_n, x_{n+1})|$. We know, for all $n \geq 1$

$$d(x_{n-1}, x_n) \geq d(x_n, x_{n+1}).$$

Therefore $E(x_{n-1}, x_n) = 2d(x_{n-1}, x_n) - d(x_n, x_{n+1})$ for all $n \geq 1$. We deduce

$$d(x_n, x_{n+1}) \leq \gamma d(x_{n-1}, x_n),$$

where $\gamma = \frac{2q}{1+q}$. Since $q \in (0, \frac{1}{s})$, that is $0 < q < 1$, we have $0 < \gamma < 1$.

Following the proof of Lemma 2.2 [18], we have for all $m \geq 1$

$$d(x_{n+1}, x_{n+m}) \leq \gamma^n \frac{\lambda d(Tx_0, x_0)}{1 - \gamma},$$

where

$$\lambda = \sum_{n \geq 1} s^{2n} \gamma^{2^{n-1}}.$$

Passing to $\lim sup$ as $m \to \infty$ in the above inequality, by Lemma 1.8, we obtain

$$d(x_n, z) \leq \gamma^{n-1} \frac{s\lambda d(Tx_0, x_0)}{1 - \gamma}.$$

□

Remark 2.7. *Following Lemma 2.2 [18], Corollary 2.6 remains valid for $q \in (0, 1)$. For related results, see [23].*

3. Examples

In this section, we present some examples.

Example 3.1. *Let $X = \mathbb{R}$ be endowed with the b-metric d given by $d(x, y) = (x - y)^2$ for all $x, y \in X$. Then (X, d, s) is a complete b-metric space with $s = 2$. Take $\beta(t) = \frac{1}{5}$ for all $t \geq 0$. Consider $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}$ as follows*

$$Tx = \begin{cases} x + 1, & x < -1 \\ 0, & -1 \leq x \leq 0 \\ -x, & 0 < x \leq 1 \\ -x^2, & x \geq 1, \end{cases} \qquad \alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [-1, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Mention that Theorem 1.9 is not applicable for any $\beta \in \mathcal{F}_s$. Indeed, by choosing $x = 0$ and $y = 1$, we have $d(x, y) = d(0, 1) = 1$ and $d(Tx, Ty) = d(0, -1) = 1$. If the condition $d(T0, T1) \leq \beta(d(0, 1))\, d(0, 1)$ is satisfied, then $1 \leq \beta(1)$, which is a contradiction. Moreover, Theorem 1.11 is also not applicable for $\beta(t) = \frac{1}{5}$ and $d(x, y) = |x - y|$. In fact, for $x = 2$ and $y = 4$, we have $E(x, y) = E(2, 4) = d(2, 4) + |d(2, T2) - d(4, T4)| = 16$ and $d(Tx, Ty) = d(T2, T4) = 12$. While, if the condition $d(T0, T1) \leq \beta(E(2, 4))\, E(2, 4)$ holds, then $\frac{3}{4} \leq \beta(16) = \frac{1}{5}$. It is a contradiction.

It is easy to prove that T is triangle α-orbital admissible. Also, T is continuous and for $x_0 = 1$, we have $\alpha(1, T1) = \alpha(1, -1) \geq 1$.

Now, we shall prove that T is an $\alpha - \beta_E$-Geraghty contraction type mapping. By symmetry of (1) and for $\alpha(x, y) \geq 1$

, we need the following cases:
Case 1: $x, y > 0$ *and* $x > y$. *Since* $E(x,y) = (x-y)^2 + 4(y^2 - x^2)$,

$$d(Tx,Ty) \leq \beta(E(x,y))\,E(x,y) \Leftrightarrow (x-y)^2 \leq \frac{1}{5}[(x-y)^2 + 4(y^2-x^2)]$$
$$\Leftrightarrow y - x \leq \frac{1}{5}[3x+5y] \Leftrightarrow x \geq 0.$$

Case 2: $x, y < 0$. *We have* $Tx = Ty = 0$. *Then*

$$d(Tx,Ty) = 0 \leq \beta(E(x,y))\,E(x,y).$$

Case 3: $x \leq 0$ *and* $y > 0$. *In this case, we have* $d(Tx,Ty) = y^2$, $E(x,y) = (x-y)^2 + |x^2 - 4y^2|$. *Then*

$$d(Tx,Ty) \leq \beta(E(x,y))\,E(x,y) \Leftrightarrow y^2 \leq \frac{1}{5}[(x-y)^2 + |x^2-4y^2|]. \tag{17}$$

To show this, we distinguish the following two subcases:
(i) *If* $2y \leq |x|$, *then (17) becomes*

$$5y^2 \leq (x-y)^2 + x^2 - 4y^2 \Leftrightarrow 4y^2 \leq x^2 - xy.$$

(ii) *If* $2y > |x|$, *then (17) becomes*

$$5y^2 \leq (x-y)^2 + 4y^2 - x^2 \Leftrightarrow 0 \leq -xy.$$

So (1) holds for all $x, y \in X$ *satisfying* $\alpha(x,y) \geq 1$. *All hypotheses of Theorem 2.1 are satisfied, so* T *has a fixed point, which is* $u = 0$.

The following example is inspired from [11, Example 3.9].

Example 3.2. *Let* $X = \{0,1,3\}$ *be endowed with the b-metric* d *given by* $d(x,y) = (x-y)^2$ *for all* $x, y \in X$. *Take* $\beta(t) = \frac{1}{2}e^{-\frac{t}{9}}$ *for all* $t > 0$ *and* $\beta(0) = \frac{1}{4}$. *Consider* $T : X \to X$ *as follows*

$$T0 = T1 = 1, \quad T3 = 0.$$

We have $E(0,3) = 17$, $E(1,3) = 13$. *Then*

$$d(T0,T1) = d(1,1) = 0 \leq \beta(E(0,1))\,E(0,1),$$

$$d(T0,T3) = d(1,0) = 1 \leq \frac{17}{2}e^{-\frac{17}{9}} = \beta(E(0,3))\,E(0,3),$$

$$d(T1,T3) = d(1,0) = 1 \leq \frac{13}{2}e^{-\frac{13}{9}} = \beta(E(1,3))\,E(1,3).$$

For $x = y$, *we have*

$$d(Tx,Ty) = 0 \leq \beta(E(x,y))\,E(x,y).$$

All hypotheses of Corollary 2.4 are satisfied, so T *has a unique fixed point, which is* $u = 1$.

4. Application on matrix equations

First, let $\mathcal{P}_n$ be the set of $n \times n$ Hermitian positive definite matrices. In this section, we will apply Corollary 2.6 to study the existence of $X \in \mathcal{P}_n$ solution of the nonlinear matrix equation:

$$X^2 = (AX^{\frac{-1}{2}}A^* + B)^{\frac{1}{3}} + C, \tag{18}$$

where B and C are an $n \times n$ positive semi definite matrix and A is a nonsingular $n \times n$ matrix. Here A^* denotes the conjugate transpose of the matrix A.

Mention that the problem (18) is equivalent to the research of $X \in \mathcal{P}_n$ such that

$$X = F(X) = [(AX^{\frac{-1}{2}}A^* + B)^{\frac{1}{3}} + C]^{\frac{1}{2}}, \tag{19}$$

that is, X is a fixed point of the mapping F.

In this study, we use the Thompson metric introduced by Thompson [26] for the study of solutions to systems of nonlinear matrix equations involving contractive mappings. We first review the Thompson metric on the open convex cone $\mathcal{P}(n)$ for $n \geq 2$, the set of all $n \times n$ Hermitian positive definite matrices. We endow $\mathcal{P}(n)$ with the Thompson metric defined by

$$d(A, B) = \max\{\log M(\frac{A}{B}), \log M(\frac{B}{A})\},$$

where $M(\frac{A}{B}) = \inf\{\lambda > 0,\ A \leq \lambda B\} = \lambda_{max}(B^{\frac{-1}{2}}AB^{\frac{1}{2}})$, the maximal eigenvalue of $B^{\frac{-1}{2}}AB^{\frac{1}{2}}$. Here, $X \leq Y$ means that $Y - X$ is positive semi definite and $X < Y$ means that $Y - X$ is positive definite. Thompson [26] (see also [19, 20]) has proved that $\mathcal{P}(n)$ is a complete metric space with respect to the Thompson metric d and

$$d(A, B) = \| \log(A^{\frac{-1}{2}}BA^{\frac{1}{2}})\|,$$

where $\|.\|$ stands for the spectral norm. The Thompson metric exists on any open normal convex cones of real Banach spaces [19, 26]; in particular, the open convex cone of positive definite operators of a Hilbert space. It is invariant under the matrix inversion and congruence transformations, that is,

$$d(A, B) = d(A^{-1}, B^{-1}) = d(MAM^*, MBM^*)$$

for any nonsingular matrix M. The other useful result is the nonpositive curvature property of the Thompson metric, that is,

$$d(X^r, Y^r) \leq rd(X, Y), \quad r \in [0, 1]$$

By the invariant properties of the metric, we have

$$d(MX^rM^*, MY^rM^*) \leq |r|d(X, Y), \quad r \in [-1, 1]$$

for all $X, Y \in \mathcal{P}(n)$ and nonsingular matrix M.

Lemma 4.1. *[16] For all $A, B, C, D \in \mathcal{P}(n)$, we have*

$$d(A + B, C + D) \leq \max\{d(A, C), d(B, D)\}.$$

In particular,

$$d(A + B, A + C) \leq d(B, C).$$

Let us consider the b-metric $\delta : \mathcal{P}_n \times \mathcal{P}_n \to [0, \infty)$ (with coefficient $s = 2$) such that

$$\delta(X, Y) = d^2(X, Y).$$

Theorem 4.2. *The problem (18) has a unique solution $X \in \mathcal{P}_n$. Moreover, for any $X(0) \in \mathcal{P}(n)$, the sequence $\{X(k)\}_{k\geq 0}$ defined by*

$$X(k+1) = [(AX(k)^{\frac{-1}{2}}A^* + B)^{\frac{1}{3}} + C]^{\frac{1}{2}}, \tag{20}$$

converges to X and the error estimation is

$$\delta(X(k), X) \leq (\frac{2}{145})^{n-1}\frac{290}{143}\lambda_0 E(X(1), X(0)), \tag{21}$$

where

$$\lambda_0 = \sum_{n\geq 1} 4^n(\frac{2}{145})^{2^{n-1}}.$$

Proof. We have

$$\begin{aligned}
\delta(F(X), F(Y)) =&\delta([(AX^{\frac{-1}{2}}A^* + B)^{\frac{1}{3}} + C]^{\frac{1}{2}}, [(AY^{\frac{-1}{2}}A^* + B)^{\frac{1}{3}} + C]^{\frac{1}{2}})\\
=&d^2([(AX^{\frac{-1}{2}}A^* + B)^{\frac{1}{3}} + C]^{\frac{1}{2}}, [(AY^{\frac{-1}{2}}A^* + B)^{\frac{1}{3}} + C]^{\frac{1}{2}})\\
\leq&\frac{1}{4}d^2((AX^{\frac{-1}{2}}A^* + B)^{\frac{1}{3}} + C, (AY^{\frac{-1}{2}}A^* + B)^{\frac{1}{3}} + C)\\
\leq&\frac{1}{4}d^2((AX^{\frac{-1}{2}}A^* + B)^{\frac{1}{3}}, (AY^{\frac{-1}{2}}A^* + B)^{\frac{1}{3}})\\
\leq&\frac{1}{36}d^2(AX^{\frac{-1}{2}}A^* + B, AY^{\frac{-1}{2}}A^* + B)\\
\leq&\frac{1}{36}d^2(AX^{\frac{-1}{2}}A^*, AY^{\frac{-1}{2}}A^*)\\
\leq&\frac{1}{144}d^2(X, Y)\\
=&\frac{1}{144}\delta(X, Y) = q\delta(X, Y) \leq qE(X, Y).
\end{aligned}$$

Applying Corollary 2.5, the mapping F has a unique fixed point $X \in \mathcal{P}(n)$. So that the problem (19) has a unique fixed point, that is, the nonlinear matrix equation (18) has a unique solution in $\mathcal{P}(n)$. Again, from Corollary 2.6 with $q = \frac{1}{144}$, we have the error estimate

$$\delta(X(k), X) \leq (\frac{2}{145})^{n-1}\frac{290}{143}\lambda_0 E(X(1), X(0)).$$

□

Now, we present numerical experiments illustrating the convergence algorithm in Theorem 4.2. For other similar results, see [8, 14].

Example: Take the 3×3 positive semi definite matrices B and C defined as

$$B = \begin{pmatrix} 1 & 0.95 & 0 \\ 0.95 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1.1 & 1.05 & 0 \\ 1.05 & 1.1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Mention that

$$Sp\{B\} = \{0, 0.05, 1.95\} \quad \text{and} \quad Sp\{C\} = \{0, 0.05, 2.15\}.$$

Take the 3×3 nonsingular matrix A

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Consider the residual error

$$R(X(k)) = \|X(k) - [(AX(k)^{\frac{-1}{2}}A^* + B)^{\frac{1}{3}} + C]^{\frac{1}{2}}\|.$$

Here, A is symmetric, so

$$R(X(k)) = \|X(k) - [(AX(k)^{\frac{-1}{2}}A + B)^{\frac{1}{3}} + C]^{\frac{1}{2}}\|.$$

Also (20) becomes

$$X(k+1) = [(AX(k)^{\frac{-1}{2}}A + B)^{\frac{1}{3}} + C]^{\frac{1}{2}}. \tag{22}$$

Case 1 (Diagonal matrix): Choose the positive definite matrix

$$X(0) = \begin{pmatrix} 1.25 & 0 & 0 \\ 0 & 1.3 & 0 \\ 0 & 0 & 1.35 \end{pmatrix}.$$

Using MATLAB (version 1) and considering the iterative method (22) with the above $X(0)$, after 10 iterations, one gets an approximation to the 3×3 positive definite solution $X(10)$ given by

$$X(10) = \begin{pmatrix} 1.4887 & 0.4584 & 0.0403 \\ 0.4584 & 1.4887 & 0.0403 \\ 0.0403 & 0.0403 & 1.0461 \end{pmatrix}.$$

Moreover, we obtain

$$R(X(10)) = 1.891e^{-11}.$$

Case 2 (Full matrix): On the other hand, choose the positive definite matrix

$$X(0) = \begin{pmatrix} 10 & 3.85 & -3.85 \\ 3.85 & 10 & 3.92 \\ -3.85 & 3.92 & 10 \end{pmatrix}.$$

Considering again the iterative method (22) with the above $X(0)$, after 10 iterations, one gets the same approximation to the 3×3 positive definite solution as Case 1, which is given by

$$X(10) = \begin{pmatrix} 1.4887 & 0.4584 & 0.0403 \\ 0.4584 & 1.4887 & 0.0403 \\ 0.0403 & 0.0403 & 1.0461 \end{pmatrix}$$

While, the residual error is

$$R(X(10)) = 1.1222e^{-10}.$$

This figure illustrates the convergence curve of the iterative method (20). Note that the curves are perfect lines, i.e., the algorithm (20) converges to the theoretical solution of (18). The residual errors are well given. Mention that the error for the full matrix (case 2) necessitates more calculus, without missing the notion of convergence.

Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

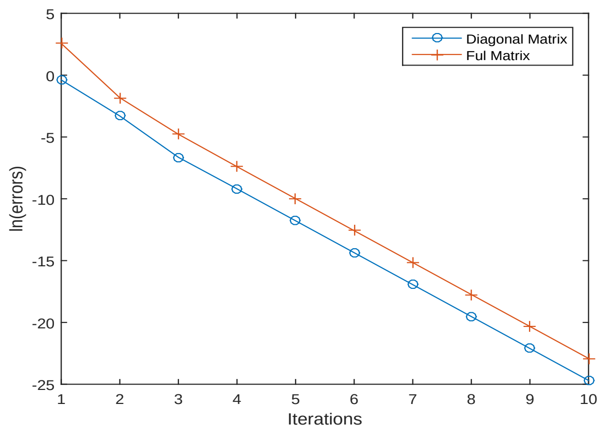

Figure: Convergence curve

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