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Integral Operator Acting on Weighted Dirichlet Spaces to Morrey Type Spaces

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Abstract. In this paper, we studied the boundedness and compactedness of integral operators from weighted Dirichlet spaces D_K to Morrey type spaces H_K^2. Carleson measure and essential norm were also considered.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$. As usual, let H^∞ be the set of bounded analytic functions in $\mathbb{D}$ and $\varphi_a(z) = (a-z)/(1-\bar{a}z)$.

The Hardy space H^p $(0 < p < \infty)$ is the spaces of all functions $f \in H(\mathbb{D})$ with

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Assume that $K : [0,\infty) \to [0,\infty)$ is a right-continuous and nondecreasing function.

We say that a function $f \in H^2$ belongs to Morrey type space H_K^2 if

$$\|f\|_{H_K^2}^2 = |f(0)|^2 + \sup_{I \subseteq \partial\mathbb{D}} \frac{1}{K(|I|)} \int_I |f(\zeta) - f_I|^2 \frac{d\zeta}{2\pi} < \infty,$$

where

$$f_I = \frac{1}{|I|} \int_I f(\zeta) \frac{d\zeta}{2\pi},\ I \subseteq \partial\mathbb{D}.$$

This space was introduced by H. Wulan and J. Zhou in [32]. When $K(t) = t$, it gives the $BMOA$ space, the space of those analytic functions f in the Hardy space H^p whose boundary functions have bounded mean oscillation on $\partial\mathbb{D}$. In the case $K(t) = t^\lambda$, $0 < \lambda < 1$, the space H_K^2 gives classical Morrey spaces $\mathcal{L}^{2,\lambda}$. Morrey spaces $\mathcal{L}^{2,\lambda}$ were introduced by Morrey in [21]. It has been studied extensively. We refer to [1, 2, 21, 31, 32].

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Let D_K denoted the space of function $f \in H(\mathbb{D})$ satisfies

$$\|f\|_{D_K}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \frac{1-|z|^2}{K(1-|z|^2)} dA(z) < \infty.$$

Clearly, D_K is a Hilbert space. In the case $K(t) = t^p$, $0 \le p < 1$, the space D_K gives the usual Dirichlet type space D_p. In particular, if $p = 1$ and $p = 0$, this gives the classical Dirichlet space $\mathcal{D}$ and Hardy space H^2. We refer to [25, 28, 29] for D_p spaces. The space D_K also has been extensively studied. For example, under some conditions on K, R. Kerman and E. Sawyer [15] characterized Carleson measures and multipliers of D_K in terms of a maximal operator. A. Aleman [5] proved that each element of the space D_K can be written as a quotient of two bounded functions in the same space. See [6, 19, 23, 24, 36] for more results on D_K spaces.

Throughout this paper, let weighted function K satisfies:

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty \tag{1.1}$$

and

$$\int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty, \tag{1.2}$$

where

$$\varphi_K(s) = \sup_{0 \le t \le 1} K(st)/K(t), \quad 0 < s < \infty.$$

By [13], there exists a small enough constant $c > 0$, such that $t^{-c}K(t)$ is nondecreasing and $K(t)t^{c-1}$ is nonincreasing. If K satisfies (1.2), we get $K(2t) \approx K(t)$ for $t > 0$ and we can assume that K is differentiable up to any desired order.

In this paper, the symbol $f \approx g$ means that $f \lesssim g \lesssim f$. We say that $f \lesssim g$ if there exists a constant C such that $f \le Cg$.

2. Preliminaries

In this section, we are going to give some auxiliary results. The following lemma can be found in [32, Theorem 3.1].

Lemma 1. *Let (1.1) and (1.2) hold for K. Then the following are equivalent.*

(1) $f \in H_K^2$;

(2)

$$\sup_{I \subseteq \partial\mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} |f'(z)|^2 (1-|z|^2) dA(z) < \infty;$$

(3)

$$\sup_{a \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(z)|^2 (1-|\varphi_a(z)|^2) dA(z) < \infty;$$

(4)

$$\sup_{a \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|\varphi_a(z)|} dA(z) < \infty;$$

(5)

$$\sup_{a \in \mathbb{D}} \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\partial\mathbb{D}} \frac{|f(\zeta)-f(a)|^2}{|\zeta-a|^2} \frac{|d\zeta|}{2\pi} < \infty;$$

(6)

$$\sup_{a \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \left(\int_{\partial\mathbb{D}} |f(\zeta)|^2 \frac{1-|a|^2}{|\zeta-a|^2} \frac{|d\zeta|}{2\pi} - |f(a)|^2 \right) < \infty.$$

Lemma 2. *Let (1.1) and (1.2) hold for K. Suppose that $f \in D_K$, then*

$$|f(z)| \lesssim \|f\|_{D_K} \sqrt{\frac{K(1-|z|^2)}{1-|z|^2}}, \quad z \in \mathbb{D}.$$

Proof. Noticed the fact that

$$|1-\bar{z}w| \approx 1-|z|^2 \approx 1-|w|^2, \;\; w \in D(z,r),$$

and

$$K(1-|z|^2) \approx K(1-|w|^2), \;\; w \in D(z,r),$$

where $D(z,r) = \{w : |\varphi_z(w)| < r\}$. Using the sub-mean value property of $|f'|^2$, we can deduce that

$$\begin{aligned}
|f'(z)|^2 \lesssim& \frac{1}{(1-|z|^2)^2}\int_{D(z,r)} |f'(w)|^2 dA(w)\\
\approx& \frac{K(1-|z|^2)}{(1-|z|^2)^3}\int_{D(z,r)} |f'(w)|^2 \frac{1-|w|^2}{K(1-|w|^2)} dA(w)\\
\leq& \frac{K(1-|z|^2)}{(1-|z|^2)^3}\int_{\mathbb{D}} |f'(w)|^2 \frac{1-|w|^2}{K(1-|w|^2)} dA(w).
\end{aligned}$$

Thus,

$$|f'(z)| \lesssim \|f\|_{D_K} \sqrt{\frac{K(1-|z|^2)}{(1-|z|^2)^3}}.$$

Since

$$|f(z)-f(0)| = \left| z\int_0^1 f'(zs)ds \right| \leq |z| \int_0^1 |f'(zs)|ds,$$

we can easy to get

$$\begin{aligned}
|f(z)-f(0)| \lesssim& \int_0^1 |f'(zs)| d(|z|s)\\
\lesssim& \|f\|_{D_K} \int_0^1 \sqrt{\frac{K(1-|z|s)}{(1-|z|s)^3}} d(|z|s)\\
\lesssim& \|f\|_{D_K} \int_0^{|z|} \sqrt{\frac{K(1-t)}{(1-t)^3}} dt\\
=& \|f\|_{D_K} \sqrt{K(1-|z|)} \int_0^{|z|} \sqrt{\frac{K(1-t)}{K(1-|z|)(1-t)^3}} dt.
\end{aligned}$$

Noted that K satisfies (1.2), by [13, Lemma 2.2], there exists a small $c > 0$ such that

$$\varphi_K(t) \lesssim t^{1-c}, \; t \geq 1.$$

Hence, we obtain

$$\begin{aligned}
|f(z)-f(0)| \lesssim& \|f\|_{D_K} \sqrt{K(1-|z|)} \int_0^{|z|} \sqrt{\frac{K(1-t)}{K(1-|z|)(1-t)^3}} dt\\
\lesssim& \|f\|_{D_K} \sqrt{K(1-|z|)} \int_0^{|z|} \sqrt{\left(\frac{(1-t)}{1-|z|}\right)^{1-c} \frac{1}{(1-t)^3}} dt\\
\lesssim& \|f\|_{D_K} \sqrt{\frac{K(1-|z|^2)}{(1-|z|^2)}}.
\end{aligned}$$

That is

$$|f(z)| \lesssim |f(0)| + \|f\|_{D_K} \sqrt{\frac{K(1-|z|^2)}{(1-|z|^2)}} \lesssim \|f\|_{D_K} \sqrt{\frac{K(1-|z|^2)}{(1-|z|^2)}}.$$

The proof is completed. □

Let us recall a useful theorem.

Lemma 3. *([39, Lemma 3.10]) Suppose that $\alpha > 0$, then we have*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^{\alpha}}{|1-\overline{a}z|^{2+\alpha}} dA(z) \lesssim \frac{1}{(1-|a|^2)^{\alpha}}.$$

Lemma 4. *Let (1.1) and (1.2) hold for K. Then*

$$f_w(z) = \sqrt{\frac{K(1-|w|)}{1-|w|}}\,(\varphi_w(z) - w) \in D_K$$

and

$$F_w(z) = \frac{(1-|w|)\sqrt{K(1-|w|)}}{(1-\overline{w}z)^{\frac{3}{2}}} \in D_K,$$

where $z,\ w \in \mathbb{D}$.

Proof. With an easy computation, by Lemma 3, we have

$$\begin{aligned}
&\int_{\mathbb{D}} |f_w'(z)|^2 \frac{1-|z|^2}{K(1-|z|^2)} dA(z)\\
=&\int_{\mathbb{D}} \frac{K(1-|w|^2)(1-|w|^2)}{|1-\overline{w}z|^4} \frac{1-|z|^2}{K(1-|z|^2)} dA(z)\\
\lesssim&\int_{\mathbb{D}} \frac{K(|1-\overline{w}z|)(1-|w|^2)}{|1-\overline{w}z|^4} \frac{1-|z|^2}{K(1-|z|^2)} dA(z)
\end{aligned}$$

Since K is nondecreasing and the fact that

$$\varphi_K(t) \lesssim t^{1-c},\ t \geq 1,$$

combined with Lemma 3, it follows that

$$\begin{aligned}
&\int_{\mathbb{D}} |f_w'(z)|^2 \frac{1-|z|^2}{K(1-|z|^2)} dA(z)\\
\lesssim&(1-|w|^2) \int_{\mathbb{D}} \frac{(|1-\overline{w}z|)^{1-c}(1-|z|^2)}{(1-|z|^2)^{1-c}|1-\overline{w}z|^4} dA(z) \lesssim 1.
\end{aligned}$$

That is $f_w \in D_K$. By similar calculation as above, we can deduce that

$$\begin{aligned}
&\int_{\mathbb{D}} |F_w'(z)|^2 \frac{1-|z|^2}{K(1-|z|^2)} dA(z)\\
=&(1-|w|^2)^2 \int_{\mathbb{D}} \left(\frac{(1-|z|^2)K(1-|w|^2)}{|1-\overline{w}z|^5 K(1-|z|^2)} \right) dA(z)\\
\lesssim&(1-|w|^2)^2 \int_{\mathbb{D}} \left(\frac{(1-|z|^2)K(|1-\overline{w}z|^2)}{|1-\overline{w}z|^5 K(1-|z|^2)} \right) dA(z)\\
\lesssim&(1-|w|^2)^2 \int_{\mathbb{D}} \left(\frac{(1-|z|^2)(|1-\overline{w}z|^2)^{1-c}}{|1-\overline{w}z|^5 (1-|z|^2)^{1-c}} \right) dA(z) \lesssim 1.
\end{aligned}$$

Thus, $F_w \in D_K$. The proof is completed. □

Let $S(I)$ be the Carleson box based on the interval $I \subset \partial\mathbb{D}$ with

$$S(I) = \{z \in \mathbb{C} : 1 - |I| \le |z| < 1 \ \ and \ \frac{z}{|z|} \in I\}.$$

If $I = \partial\mathbb{D}$, let $S(I) = \mathbb{D}$. For $0 < p < \infty$, we say that a non-negative measure μ on $\mathbb{D}$ is a p-Carleson measure if

$$\sup_{I \subseteq \partial\mathbb{D}} \frac{\mu(S(I))}{|I|^p} < \infty.$$

When $p = 1$, it gives the classical Carleson measure.

The following two lemmas which can be founded in [14] and [33, Theorem 4.1.1] respectively.

Lemma 5. *Suppose that μ is a non-negative measure on $\mathbb{D}$. Then μ is a Carleson measure if and only if the following inequality*

$$\int_{\mathbb{D}} |f(z)|^2 d\mu \lesssim \|f\|_{H^2}^2$$

holds for all $f \in H^2$. Moreover,

$$\sup_{\|f\|_{H^2}=1} \int_{\mathbb{D}} |f(z)|^2 d\mu \approx \sup_{I \subseteq \partial\mathbb{D}} \frac{\mu(S(I))}{|I|}.$$

Lemma 6. *Suppose that $f \in H(\mathbb{D})$, then $f \in BMOA$ if and only if the measure $\mu_f = |f'(z)|^2(1 - |z|^2)dA(z)$ is a Carleson measure. Moreover,*

$$\|f\|_{BMOA}^2 \approx |f(0)| + \sup_{I \subseteq \partial\mathbb{D}} \frac{\mu_f(S(I))}{|I|}.$$

3. Boundedness of I_g and T_g operators

For any $g \in H(\mathbb{D})$, the Volterra type operator T_g is defined as

$$T_g f(z) = \int_0^z f(w)g'(w)dw,$$

on the space of $f \in H(\mathbb{D})$. Another similar integral operator I_g is defined as

$$I_g f(z) = \int_0^z f'(w)g(w)dw.$$

There are many papers related to these operators, we refer to [3, 4, 10, 17, 27, 33].

Theorem 1. *Let (1.1) and (1.2) hold for K. Suppose that $g \in H(\mathbb{D})$, then I_g is bounded on H^2_K if and only if $g \in H^\infty$. Moreover, the operator norm satisfies $\|I_g\| = \sup_{z\in\mathbb{D}} |g(z)|$.*

Proof. Since $g \in H(\mathbb{D})$, then $g \circ \varphi_w \in H(\mathbb{D})$. By sub-mean value property of $|g \circ \varphi_w|^2$, we get

$$|g(w)|^2 \lesssim \int_{\mathbb{D}} |(g \circ \varphi_w)(z)|^2(1 - |z|^2)dA(z).$$

If I_g is bounded from D_K to H^2_K, using the function f_w as in Lemma 4, combine with Lemma 1 and subharmonic property of $|g\circ\varphi_w|^2$, we easy to calculate that

$$\begin{aligned}\infty >&\|I_g f_w\|^2_{H^2_K}\\ \gtrsim&\sup_{a\in\mathbb{D}}\frac{1-|a|^2}{K(1-|a|^2)}\int_{\mathbb{D}}|f'_w(z)|^2|g(z)|^2\left(1-|\varphi_a(z)|^2\right)dA(z)\\ \gtrsim&\frac{1-|w|^2}{K(1-|w|^2)}\int_{\mathbb{D}}|f'_w(z)|^2|g(z)|^2\left(1-|\varphi_w(z)|^2\right)dA(z)\\ \gtrsim&\int_{\mathbb{D}}|g(z)|^2|\varphi'_w(z)|^2\left(1-|\varphi_w(z)|^2\right)dA(z)\\ =&\int_{\mathbb{D}}|(g\circ\varphi_w)(\eta)|^2(1-|\eta|^2)dA(\eta)\gtrsim|g(w)|^2.\end{aligned}$$

Since $w\in\mathbb{D}$ is arbitrary, we have

$$\infty > \|I_g f_w\|^2_{H^2_K}\gtrsim\|g\|^2_{H^\infty}.$$

On the other hand. If $g\in H^\infty$, by [13, Lemma 2.2], using

$$\varphi_K(t)\lesssim t^{1-c},\ t\geq 1.$$

We can deduce that

$$\begin{aligned}&\frac{1-|a|^2}{K(1-|a|^2)}\int_{\mathbb{D}}|f'(z)|^2|g(z)|^2\left(1-|\varphi_a(z)|^2\right)dA(z)\\ \lesssim&\|g\|^2_{H^\infty}\int_{\mathbb{D}}|f'(z)|^2\frac{(1-|a|^2)^2K(1-|z|^2)}{|1-\bar{a}z|^2K(1-|a|^2)}\frac{1-|z|^2}{K(1-|z|^2)}dA(z)\\ \lesssim&\|g\|^2_{H^\infty}\int_{\mathbb{D}}|f'(z)|^2\frac{(1-|a|^2)^2K(|1-\bar{a}z|)}{|1-\bar{a}z|^2K(1-|a|^2)}\frac{1-|z|^2}{K(1-|z|^2)}dA(z)\\ \lesssim&\|g\|^2_{H^\infty}\int_{\mathbb{D}}|f'(z)|^2\frac{(1-|a|^2)^2(|1-\bar{a}z|)^{1-c}}{|1-\bar{a}z|^2(1-|a|^2)^{1-c}}\frac{1-|z|^2}{K(1-|z|^2)}dA(z)\\ \lesssim&\|g\|^2_{H^\infty}\|f\|^2_{D_K}.\end{aligned}$$

The proof is completed. □

Theorem 2. *Let (1.1) and (1.2) hold for K. Suppose that $g\in H(\mathbb{D})$, then T_g is bounded from D_K to H^2_K if and only if $g\in BMOA$. Moreover, the operator norm satisfies $\|T_g\|=\|g\|_{BMOA}$.*

Proof. For any $I\in\partial\mathbb{D}$, let $w=(1-|I|)\zeta\in\mathbb{D}$, where ζ is the center of I. Then

$$1-|w|\approx|1-\overline{w}z|\approx|I|,\ \ z\in S(I).$$

Thus, we also have

$$K(1-|w|)\approx K(|I|),\ \ z\in S(I).$$

If T_g is bounded from D_K to H^2_K and F_w is defined as in Lemma 4. By Lemma 1, we have

$$\begin{aligned}&\frac{1}{|I|}\int_{S(I)}|g'(z)|^2(1-|z|^2)dA(z)\\ \lesssim&\frac{1}{K(|I|)}\int_{S(I)}|F_w(z)|^2|g'(z)|^2(1-|z|^2)dA(z)\\ \lesssim&\frac{1}{K(|I|)}\int_{S(I)}|(T_gF_w)'(z)|^2(1-|z|^2)dA(z)\\ \lesssim&\|T_gF_w\|^2_{H^2_K}<\infty.\end{aligned}$$

Thus, $g \in BMOA$.

On the other hand, suppose that $g \in BMOA$ and $f \in D_K$, we have

$$\begin{aligned}&\frac{1}{K(|I|)}\int_{S(I)}|(T_g f)'(z)|^2(1-|z|^2)dA(z)\\ =&\frac{1}{K(|I|)}\int_{S(I)}|f(z)|^2|g'(z)|^2(1-|z|^2)dA(z)\\ \lesssim& A+B,\end{aligned}$$

where

$$A =: \frac{1}{K(|I|)}\int_{S(I)}|f(w)|^2|g'(z)|^2(1-|z|^2)dA(z)$$

and

$$B =: \frac{1}{K(|I|)}\int_{S(I)}|f(z)-f(w)|^2|g'(z)|^2(1-|z|^2)dA(z).$$

By Lemma 3, it follows that

$$|f(w)| \lesssim \frac{\|f\|_{D_K}\sqrt{K(1-|w|^2)}}{\sqrt{1-|w|^2}} \approx \frac{\|f\|_{D_K}\sqrt{K(|I|)}}{\sqrt{|I|}},\ w \in S(I).$$

Combine with Lemma 7, it easy to have

$$A \lesssim \|f\|^2_{D_K}\|g\|^2_{BMOA}.$$

Since

$$\frac{1-|z|^2}{|I|} \lesssim 1-|\varphi_w(z)|^2,\ z \in S(I),$$

then

$$\begin{aligned}B \lesssim&\frac{|I|}{K(|I|)}\int_{S(I)}|f(z)-f(w)|^2|g'(z)|^2(1-|\varphi_w(z)|^2)dA(z)\\ \lesssim&\frac{|I|}{K(|I|)}\int_{S(I)}|f\circ\varphi_w(\eta)-f(w)|^2|(g\circ\varphi_w)'(\eta)|^2(1-|\eta|^2)dA(\eta)\\ \lesssim&\frac{1-|w|^2}{K(1-|w|^2)}\int_{\mathbb{D}}|f\circ\varphi_w(\eta)-f(w)|^2|(g\circ\varphi_w)'(\eta)|^2(1-|\eta|^2)dA(\eta).\end{aligned}$$

Since $g \in BMOA$, then $g\circ\varphi_w \in BMOA$ and $|(g\circ\varphi_w)'(\eta)|^2(1-|\eta|^2)dA(\eta)$ is a Carleson measure by Lemma 6. Since $f \in D_K \subseteq H^2$, then $(f\circ\varphi_w)(\eta)-f(w) \in H^2$. Combining this with Lemma 5 and Littlewood-Paley identity (see [14, page 236]) gives

$$\begin{aligned}B \lesssim&\frac{1-|w|^2}{K(1-|w|^2)}\|g\circ\varphi_w\|^2_{BMOA}\int_0^{2\pi}|f\circ\varphi_w(e^{i\theta})-f(w)|^2d\theta\\ \lesssim&\frac{1-|w|^2}{K(1-|w|^2)}\|g\circ\varphi_w\|^2_{BMOA}\int_{\mathbb{D}}|f'(z)|^2(1-|\varphi_w(z)|^2)dA(z)\\ \lesssim&\|g\|^2_{BMOA}\int_{\mathbb{D}}|f'(z)|^2\frac{(1-|w|^2)^2K(1-|z|^2)}{|1-\bar{a}z|^2K(1-|w|^2)}\frac{1-|z|^2}{K(1-|z|^2)}dA(z)\\ \lesssim&\|g\|^2_{BMOA}\int_{\mathbb{D}}|f'(z)|^2\frac{(1-|w|^2)^2K(|1-\overline{w}z|)}{|1-\bar{a}z|^2K(1-|w|^2)}\frac{1-|z|^2}{K(1-|z|^2)}dA(z)\\ \lesssim&\|g\|^2_{BMOA}\int_{\mathbb{D}}|f'(z)|^2\frac{(1-|w|^2)^2(|1-\bar{a}z|)^{1-c}}{|1-\overline{w}z|^2(1-|a|^2)^{1-c}}\frac{1-|z|^2}{K(1-|z|^2)}dA(z)\\ \lesssim&\|g\|^2_{BMOA}\|f\|^2_{D^2_K},\end{aligned}$$

Hence,

$$\|T_g f\|_{H_K^2}^2 \lesssim A + B \lesssim \|g\|_{BMOA}^2 \|f\|_{D_K}^2.$$

The proof is completed. □

For $g \in H(\mathbb{D})$, the multiplication operator M_g is defined by $M_g f(z) = f(z)g(z)$. It is easy to see that M_g is related with I_g and T_g by

$$M_g f(z) = f(0)g(0) + I_g f(z) + T_g f(z).$$

Corollary 1. *Let (1.1) and (1.2) hold for K. Suppose that $g \in H(\mathbb{D})$, then M_g is bounded from D_K to H_K^2 if and only if $g \in H^\infty$.*

Proof. Suppose M_g is bounded from D_K to H_K^2, consider the function F_w is defined as in Lemma 4. Using Lemma 2, it gives

$$\begin{aligned}\left|\frac{(1-|w|)\sqrt{K(1-|w|)}}{(1-\overline{w}z)^{\frac{3}{2}}}g(z)\right| &\lesssim \frac{\|M_g F_w\|_{H_K^2}\sqrt{K(1-|z|^2)}}{\sqrt{1-|z|^2}} \\ &\lesssim \frac{\|M_g\|\sqrt{K(1-|z|^2)}}{\sqrt{1-|z|^2}}.\end{aligned}$$

Let $z = w$. We have

$$|g(w)| \lesssim \|M_g\|.$$

Since $w \in \mathbb{D}$ is arbitrary, we deduce that $g \in H^\infty$. The other side is obvious. The proof is completed. □

4. Essential Norm

Let X be a Banach space and T is a bounded linear operator on X. The essential norm of T is defined as follows,

$$\|T\|_e = inf\{\|T - S\| : S \text{ are compact operator on } X\}.$$

It is the distance of T from the closed ideals of compact operators. Since T is compact if and only if $\|T\|_e = 0$, the estimate of $\|T\|_e$ indicates the condition for T to be compact. In this note, we estimate the norm of I_g,J_g.

Let X and Y be two Banach spaces with $X \subset Y$. If $f \in Y$, then the distance from f to X is defined as

$$\mathrm{dist}_Y(f, X) = \inf_{g \in X} \|f - g\|_Y.$$

Theorem 3. *Suppose $g \in H(\mathbb{D})$ and K satisfy the conditions (1) and (2). If I_g is bounded from D_K to H_K^2, then*

$$\|I_g\|_e \approx \sup_{z \in \mathbb{D}} |g(z)|.$$

Proof. For compact operators S, it follows from

$$\|I_g\|_e = \inf_S \|I_g - S\| \leq \|I_g\| \lesssim \sup_{z \in \mathbb{D}} |g(z)|.$$

On the other hand, we choose the sequence $\{w_n\} \subset \mathbb{D}$ such that $|w_n| \to 1$. we define

$$f_n(z) = \sqrt{\frac{K(1-|w_n|^2)}{1-|w_n|^2}}\,(\varphi_{w_n}(z) - w_n)\,, \quad z \in \mathbb{D}.$$

It follows from the proof of lemma 4 that $\|f_n\|_{D_K} \lesssim 1$. It is easily to check that f_n converges to zero uniformly on any compact subsets of $\mathbb{D}$. Then $\|Sf_n\|_{H^2_K} \to 0$ as $n \to \infty$ for any compact operator S on D_K to H^2_K. Since

$$\begin{aligned}\|I_g - S\| &\gtrsim \limsup_{n\to\infty} \|(I_g - S)f_n\|_{H^2_K}\\ &\geq \limsup_{n\to\infty}(\|I_g f_n\|_{H^2_K} - \|Sf_n\|_{H^2_K})\\ &= \limsup_{n\to\infty} \|I_g f_n\|_{H^2_K}\end{aligned}$$

and

$$\begin{aligned}\|I_g f_n\|_{H^2_K} &\approx \sup_{a\in\mathbb{D}}\left(\frac{1-|a|^2}{K(1-|a|^2)}\int_{\mathbb{D}} |f_n'(z)|^2|g(z)|^2\left(1-|\varphi_a(z)|^2\right)dA(z)\right)^{\frac{1}{2}}\\ &\geq \left(\frac{1-|w_n|^2}{K(1-|w_n|^2)}\int_{\mathbb{D}} \frac{K(1-|w_n|^2)(1-|w_n|^2)}{|1-\overline{w_n}z|^4}|g(z)|^2\left(1-|\varphi_{w_n}(z)|^2\right)dA(z)\right)^{\frac{1}{2}}\\ &= \left(\int_{\mathbb{D}} |\varphi_{w_n}'(z)|^2|g(z)|^2\left(1-|\varphi_{w_n}(z)|^2\right)dA(z)\right)^{\frac{1}{2}}\\ &= \left(\int_{\mathbb{D}} |\left(g\circ\varphi_{w_n}\right)(z)|^2\left(1-|z|^2\right)dA(z)\right)^{\frac{1}{2}}\\ &\gtrsim g(w_n).\end{aligned}$$

Since $w_n \in \mathbb{D}$ is arbitrary, we have

$$\|I_g\|_e \gtrsim \sup_{z\in\mathbb{D}} g(z).$$

The proof is completed. □

Here and afterward we denote $g_r(z) = g(rz)$ with $0 < r < 1$.

Lemma 7. ([16, Lemma 3]) *Suppose $g \in BMOA$. Then*

$$dist(g, VMOA) \approx \limsup_{|a|\to 1} \|g - g_r\|_{BMOA} \approx \limsup_{|a|\to 1} \|g\circ\sigma_a - g(a)\|_{H^2}.$$

Lemma 8. *Suppose $g \in BMOA$ and K satisfy the conditions (1) and (2). Then $J_{g_r} : D_K \to H^2_K$ is compact.*

Proof. Let $\{f_n\}$ be function sequence such that $\|f_n\|_{D_K} \leq 1$ and $f_n \to 0$ uinformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. We need only to show that

$$\lim_{n\to\infty} \|J_{g_r} f_n\|_{H^2_K} = 0.$$

Since $\|g_r\|_{BMOA} \lesssim \|g\|_{BMOA}$ ([37, Lemma 1]), for all $z \in \mathbb{D}$

$$|g_r'(z)| \lesssim \frac{\|g\|_{BMOA}}{1-r^2}.$$

Thus

$$\begin{aligned}\|J_{g_r} f_n\|_{H^2_K} &\approx \sup_{a\in\mathbb{D}}\left(\frac{1-|a|^2}{K(1-|a|^2)}\int_{\mathbb{D}} |f_n(z)|^2|g_r'(z)|^2\left(1-|\varphi_a(z)|^2\right)dA(z)\right)^{\frac{1}{2}}\\ &\lesssim \frac{\|g\|_{BMOA}}{1-r^2}\sup_{a\in\mathbb{D}}\left(\frac{1-|a|^2}{K(1-|a|^2)}\int_{\mathbb{D}} |f_n(z)|^2\left(1-|\varphi_a(z)|^2\right)dA(z)\right)^{\frac{1}{2}}\\ &\lesssim \frac{\|g\|_{BMOA}}{1-r^2}\left(\int_{\mathbb{D}} |f_n(z)|^2\frac{1-|z|^2}{K(1-|z|^2)}\left(\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^2K(1-|z|^2)}{K(1-|a|^2)|1-\bar{a}z|^2}\right)dA(z)\right)^{\frac{1}{2}}\\ &\lesssim \frac{\|g\|_{BMOA}}{1-r^2}\left(\int_{\mathbb{D}} |f_n(z)|^2\frac{1-|z|^2}{K(1-|z|^2)}dA(z)\right)^{\frac{1}{2}}.\end{aligned}$$

The last inequality similar to Theorem 1 and 2. Note that $\|f_n\|_{D_K} \leq 1$ and by lemma 2, the argument is then finished by the Dominated Convergence Theorem. □

Theorem 4. *Suppose $g \in BMOA$ and K satisfy the conditions (1) and (2). Then $J_g : D_K \to H^2_K$ satisfies*

$$\|J_g\|_e \approx \operatorname{dist}(g, VMOA) \approx \limsup_{|a|\to 1} \|g \circ \sigma_a - g(a)\|_{H^2}.$$

Proof. Let $\{I_n\}$ be the subarc sequence of $\partial\mathbb{D}$, such that $|I_n| \to 0$ as $n \to \infty$, $w_n = (1 - |I_n|)\zeta_n \in \mathbb{D}$, where ζ_n is the center of I_n. $n = 1, 2, ...$ Then

$$1 - |w_n| \approx |1 - \overline{w_n} z| \approx |I_n|, \ \ z \in S(I_n).$$

Thus, by double condition and nondecreasing of weighted function K, we know that

$$K(1 - |w_n|) \approx K(|I_n|), \ \ z \in S(I_n).$$

Take

$$h_n(z) = \frac{(1 - |w_n|^2)\sqrt{K(1 - |w_n|^2)}}{(1 - \overline{w_n} z)^{\frac{3}{2}}}, \quad z \in \mathbb{D}.$$

Then $h_n \to 0$ uniformly on the compact subsets of $\mathbb{D}$ as $n \to \infty$ and $\|h_n\|_{D_K} \lesssim 1$ by the proof of Lemma 2. Thus, for any compact operator S from D_K to H^2_K, we have

$$\lim_{n\to\infty} \|Sh_n\|_{H^2_K} \to 0.$$

Therefore

$$\begin{aligned}
\|J_g - S\| &\gtrsim \lim_{n\to\infty} \sup \left(\|J_g h_n\|_{H^2_K} - \|Sh_n\|_{H^2_K}\right) \\
&= \limsup_{n\to\infty} \|J_g h_n\|_{H^2_K} \\
&\approx \limsup_{n\to\infty} \left(\frac{1}{K(|I_n|)} \int_{S(I_n)} |(J_g h_n)'(z)|^2 (1 - |z|^2) dA(z)\right)^{\frac{1}{2}} \\
&= \limsup_{n\to\infty} \left(\frac{1}{K(|I_n|)} \int_{S(I_n)} |h_n(z)|^2 |g'(z)|^2 (1 - |z|^2) dA(z)\right)^{\frac{1}{2}} \\
&\approx \limsup_{n\to\infty} \left(\frac{1}{|I_n|} \int_{S(I_n)} |g'(z)|^2 (1 - |z|^2) dA(z)\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $\{I_n\}$ is arbitrary, we have

$$\|J_g\|_e \gtrsim \limsup_{|I|\to 0} \left(\frac{1}{|I|} \int_{S(I)} |g'(z)|^2 (1 - |z|^2) dA(z)\right)^{\frac{1}{2}}.$$

It follows from the proof of Lemma 3.4 [27], for $g \in BMOA$,

$$\limsup_{|a|\to 1} \|g \circ \sigma_a - g(a)\|_{H^2} \approx \limsup_{|I|\to 0} \left(\frac{1}{|I|} \int_{S(I)} |g'(z)|^2 (1 - |z|^2) dA(z)\right)^{\frac{1}{2}}.$$

Hence

$$\|J_g\|_e \gtrsim \limsup_{|a|\to 1} \|g \circ \sigma_a - g(a)\|_{H^2}.$$

On the other hand, by Lemma 8, $J_{g_r} : D_K \to H^2_K$ is compact operator. Combining this with Theorem 2 and the linearity of J_g respect to g implies

$$\|J_g\|_e \le \|J_g - J_{g_r}\| = \|J_{g-g_r}\| \approx \|g - g_r\|_{BMOA}.$$

Hence

$$\|J_g\|_e \lesssim \limsup_{|r|\to 1} \|g - g_r\|_{BMOA} \approx \limsup_{|a|\to 1} \|g \circ \sigma_a - g(a)\|_{H^2}$$

by Lemma 7. The proof is completed. □

5. Carleson measure for D_K

Let T^K_μ be the spaces of function $f \in H(\mathbb{D})$ for which

$$\sup_{I\subseteq\partial\mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} |f(z)|^2 d\mu(z) < \infty.$$

Theorem 5. *Let K satisfied (1) and (2). Let μ be a nonnegative Borel measure on $\mathbb{D}$.*

(a) The inclusion mapping $I : D_K \to T^K_\mu$ is bounded if and only if μ is a Carleson measure.

(b) The inclusion mapping $I : D_K \to T^K_\mu$ is compact if and only if μ is a vanishing Carleson measure.

Proof. Suppose that the identity operator $I : D_K \to T^K_\mu$ is bounded. For any given arc $I \subseteq \partial\mathbb{D}$, set

$$f_I(z) = \frac{1 - |w|^2 \sqrt{K(1 - |w|^2)}}{(1 - \overline{w}z)^{3/2}},$$

where $w = (1 - |I|)\xi$ and ξ is the center point of I. We see that $f_I \in D_K$ and $\|f_I\|^2_{D_K} \lesssim 1$. In addition, it is easy to see that

$$|1 - \overline{w}z| \approx 1 - |w|^2 \approx |I|, \quad z \in S(I).$$

So

$$|f_I(z)| \approx \sqrt{\frac{K(|I|)}{|I|}}$$

when $z \in S(I)$. By the boundedness of $I : D_K \to T^K_\mu$, we have

$$\|f_I\|^2_{T^K_\mu} = \sup_{I\subseteq\mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} |f_I(z)|^2 d\mu(z) < \infty,$$

i.e.,

$$\sup_{I\subseteq\mathbb{D}} \frac{\mu(S(I))}{|I|} < \infty.$$

Hence μ is a Carleson measure.

Conversely, assume that μ is a Carleson measure. For any given $I \subseteq \partial\mathbb{D}$, denote by $w = (1 - |I|)\xi$, where ξ is the midpoint of I. For any $f \in D_K$, Lemma 2 gives

$$|f(w)| \lesssim \sqrt{\frac{K(|I|)}{|I|}} \|f\|_{D_K}.$$

Since μ is a Carleson measure, combine with

$$\|g\|^2_{H^2} \approx |g(0)|^2 + \int_{\mathbb{D}} |g'(z)|^2 (1 - |z|^2) dA(z)$$

and

$$\int_{S(I)} \left|g\right|^2 d\mu(z) \le \|\mu\|^2 \|g\|^2_{H^2},$$

we deduce that

$$\begin{aligned}
&\frac{1}{K(|I|)} \int_{S(I)} |f(z)|^2 d\mu(z) \\
&\lesssim \frac{1}{K(|I|)} \left(\int_{S(I)} |f(z) - f(w)|^2 d\mu(z) + |f(w)|^2 \mu(S(I)) \right) \\
&\lesssim \frac{(1-|w|)^2}{K(1-|w|)} \int_{S(I)} \left| \frac{f(z)-f(w)}{1-\overline{w}z} \right|^2 d\mu(z) + \|\mu\|^2 \|f\|^2_{D_K} \\
&\lesssim \|\mu\|^2 \left(\|f\|^2_{D_K} + \frac{(1-|w|^2)^2}{K(1-|w|)} \int_{\mathbb{D}} \left| \left(\frac{f(z)-f(w)}{1-\overline{w}z} \right)' \right|^2 (1-|z|^2) dA(z) \right).
\end{aligned}$$

Notice the fact that

$$\left| \left(\frac{f(z)-f(w)}{1-\overline{w}z} \right)' \right| = \left| \frac{d}{dz} \left(\frac{f(z)-f(w)}{1-\overline{w}z} \right) \right| \lesssim \frac{|f'(z)|}{|1-\overline{w}z|} + \frac{|f(z)-f(w)|}{|1-\overline{w}z|^2},$$

we obtain

$$\begin{aligned}
&\frac{1}{K(|I|)} \int_{S(I)} |f(z)|^2 d\mu(z) \\
&\lesssim \|\mu\|^2 \left(\|f\|^2_{D_K} + \frac{(1-|w|^2)}{K(1-|w|^2)} \int_{\mathbb{D}} \left| \frac{f(z)-f(w)}{1-\overline{w}z} \right|^2 (1-|\sigma_w(z)|^2) dA(z) \right) \\
&+ \|\mu\|^2 \frac{(1-|w|)}{K(1-|w|)} \int_{\mathbb{D}} |f'(z)|^2 (1-|\sigma_w(z)|^2) dA(z) \\
&\lesssim \|\mu\|^2 \left(\|f\|^2_{D_K} + \frac{(1-|w|)}{K(1-|w|)} \int_{\mathbb{D}} |f'(z)|^2 (1-|\sigma_w(z)|^2) dA(z) \right) \\
&\lesssim \|\mu\|^2 \|f\|^2_{D_K}.
\end{aligned}$$

The last second inequality following the proof of [18, Lemma 1]. Hence $I : D_K \to T^K_\mu$ is bounded.

(2) First we assume that the identity operator $I : D_K \to T^K_\mu$ is compact. Let $\{I_n\}$ be a sequence arcs with $\lim_{n\to\infty} |I_n| = 0$. Denote by $w_n = (1-|I_n|)\xi_n$, where ξ_n is the midpoint of arc I_n. Set

$$f_n(z) = \frac{1-|w_n|^2 \sqrt{K(1-|w_n|^2)}}{(1-\overline{w_n}z)^{3/2}},$$

The estimate in the proof of (a) gives that $f_n \in D_K$ and $\|f_n\|_{D_K} \lesssim 1$. It is easy to see that $\{f_n\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. Then

$$\frac{\mu(S(I_n))}{|I_n|} \lesssim \frac{1}{K(|I_n|)} \int_{S(I_n)} |f_n(z)|^2 d\mu(z) \lesssim \|f_n\|^2_{T^K_\mu} \to 0,$$

as $n \to \infty$. Since I_n is arbitrary, we see that μ is a vanishing Carleson measure.

Conversely, assume that μ is a vanishing Carleson measure. We also assume that $\|f_n\|_{D_K} \lesssim 1$ and $\{f_n\}$ converge to 0 uniformly on compact subsets of $\mathbb{D}$. Note that if μ is a vanishing Carleson measure, then from [18, Lemma 4],

$$\lim_{r\to 1} \sup_{I \subseteq \partial\mathbb{D}} \frac{(\mu-\mu_r)(S(I))}{|I|} = 0$$

i.e.,

$$\|\mu - \mu_r\|^2 \to 0, r \to 1,$$

where $\mu_r(z) = \mu(z)$ for $|z| < r$ and $\mu_r(z) = 0$ for $r \le |z| < 1$. Then

$$\begin{aligned}
&\frac{1}{K(|I|)} \int_{S(I)} |f_n(z)|^2 d\mu(z) \\
&\lesssim \frac{1}{K(|I|)} \int_{S(I)} |f_n(z)|^2 d\mu_r(z) + \frac{1}{K(|I|)} \int_{S(I)} |f_n(z)|^2 d(\mu - \mu_r)(z) \\
&\lesssim \frac{1}{K(|I|)} \int_{S(I)} |f_n(z)|^2 d\mu_r(z) + \|\mu - \mu_r\|^2 \|f_n\|^2_{D_K} \\
&\lesssim \frac{1}{K(|I|))} \int_{S(I)} |f_n(z)|^2 d\mu_r(z) + \|\mu - \mu_r\|^2.
\end{aligned}$$

Letting $n \to \infty$ and then $r \to 1$, we have $\lim_{n\to\infty} \|f_n\|_{T^K_\mu} = 0$. Therefore $I : D_K \to T^K_\mu$ is compact. The proof is completed. □

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