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Caristi Type and Meir-Keeler Type Fixed Point Theorems

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Abstract. We generalize the Caristi fixed point theorem by employing a weaker form of continuity and show that contractive type mappings that satisfy the conditions of our theorem provide new solutions to the Rhoades' problem on continuity at fixed point. We also obtain a Meir-Keeler type fixed point theorem which gives a new solution to the Rhoades' problem on the existence of contractive mappings that admit discontinuity at the fixed point. We prove that our theorems characterize completeness of the metric space as well as Cantor's intersection property.

1. Introduction

In 1976 Caristi [7] proved the following fixed point theorem:

Theorem 1.1. *[7] Let (X, d) be a complete metric space and $f : X \to X$. If there exists a lower semi-continuous function $\phi : X \to [0, \infty)$ such that*

(*i*) $d(x, fx) \leq \phi(x) - \phi(fx), x \in X,$

then f has a fixed point.

Caristi's theorem has turned out to be a very important theorem and it has many applications. Several proofs (e.g. [14], [21]) and many generalizations of this result have been reported. A Caristi type mapping may have many fixed points (see Example 3 in [6]). In the present paper we prove a Caristi type fixed point theorem which contains the fixed point theorems due to Banach [1], Caristi [7], Kannan [18, 19] and Suzuki [37] as particular cases.

Fixed point theorems for contractive mappings which admit discontinuity at the fixed point and their applications to neural networks with discontinuous activation functions have emerged as a very active area of research (e.g. Bisht and Rakocevic [4, 5], Ozgur and Tas [27, 28], Rashid et al [33], Tas and Ozgur [38], Tas et al [39], Zheng and Wang [44]). The question of the existence of contractive mappings which admit discontinuity at the fixed point arose with the publication of two papers by Kannan [18, 19]. A Kannan type mapping as well as a multitude of generalized contractions not only admit discontinuity in their domain but could be discontinuous everywhere except at the fixed point. This can be seen from the following example:

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Example 1.2. *Let $X = [0, 2]$ and d be the Euclidean metric. Define $f : X \to X$ by*

$$fx = \frac{x}{4} \text{ if } x \text{ is rational}, \qquad fx = 0 \text{ if } x \text{ is irrational.}$$

Then f satisfies the Kannan contraction condition $d(fx, fy) \leq \frac{1}{3}[d(x, fx) + d(y, fy)]$ for all x, y in X and has a unique fixed point $x = 0$ at which f is continuous. However, f is discontinuous at every other point in X.

Such possibilities, as in Example 1.2, naturally led to the question whether there exists a contractive mapping which is discontinuous at its fixed point and this question emerged as an open problem. In 1988 Rhoades [35] examined in detail the continuity of a large number of contractive mappings at their fixed points and found that all the contractive definitions studied in [35] imply continuity at the fixed point. The question whether there exists a contractive definition which ensures the existence of a fixed point but which does not imply continuity at the fixed point was listed by Rhoades in [[35], p. 242] as an open problem. In continuation of the work of Rhoades [35], the question of continuity of contractive mappings at their fixed points was examined in more detail by Hicks and Rhoades [17]; and each contractive condition studied in the 24 theorems established by them was found continuous at the fixed point. Thus, the question of the existence of a contractive mapping which is discontinuous at its fixed point remained unresolved even after these detailed studies.

To obtain a solution of this problem by employing a constructive proof we need a contractive condition which ensures that (a) for each x in X the sequence of iterates $\{f^n x\}$ is a Cauchy sequence which converges to some point, say z, and (b) the limit z of the sequence of iterates is a fixed point of f. However, as seen in Example 2.5, there exist contractive definitions which ensure that for each x in X the sequence of iterates $\{f^n x\}$ is a Cauchy sequence which converges to some point, say z, but z may not be a fixed point. Theorem 2.4 given below pertains to such a contractive mapping. Thus, in order to ensure the existence of a fixed point under such contractive conditions some additional assumption is necessary. The additional condition may be some weaker form of continuity or a stronger form of the contractive condition. We will see that the additional condition need not imply continuity at the fixed point but it characterizes completeness of the metric space and Cantor's intersection property.

In 1999, Pant [29] resolved the question of continuity of contractive mappings at fixed points by proving the following theorem:

Theorem 1.3. *[29] Let f be a self mapping of a complete metric space (X, d) such that for any x, y in X*

(ii) Given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon < \max\{d(x, fx), d(y, fy)\} < \epsilon + \delta \Rightarrow d(fx, fy) \leq \epsilon,$$

(iii) $d(fx, fy) \leq \varphi\{\max\{d(x, fx), d(y, fy)\}$, where the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\varphi(t) < t$ for each $t > 0$.

Then f has a unique fixed point, say z. Moreover, f is continuous at z if and only if $\lim_{x \to z} \max\{d(x, fx), d(z, fz)\} = 0$.

The (ϵ, δ) condition (ii) is strictly weaker than the Meir-Keeler type condition:

Given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \max\{d(x, fx), d(y, fy)\} < \epsilon + \delta \Rightarrow d(fx, fy) < \epsilon.$$

Condition (ii) can be satisfied by contractive type mappings as well as non-expansive type mappings (see Pant and Pant [30]).

Recently some more solutions to the problem of continuity at fixed point and applications of such results to neural networks with discontinuous activation functions have been reported (e.g. Bisht and Pant [2, 3], Bisht and Rakocevic [4, 5], Ozgur and Tas [27, 28], Rashid et al [33], Tas and Ozgur [38], Tas et al [39], Zheng and Wang [44]). All the known solutions of the Rhoades' problem (e.g. [2, 3], [4, 5], [27, 28], [30], [31], [44]) employ condition (ii) or some generalized form of (ii) together with a condition similar to condition (iii) or some weaker form of continuity, e.g., orbital continuity, continuity of f^k or k-continuity for some $k > 1$.

However, it is not yet known whether there exists a mapping satisfying a Meir-Keeler [23] type contractive condition or some other stronger contractive condition which admits discontinuity at its fixed point. The significance of the existence of such contractive conditions lies in the fact that stronger conditions generally do not admit discontinuity at the fixed point (see Theorem 2.7 below).

In the present paper we introduce a weaker form of continuity which is a necessary and sufficient condition for the existence of fixed points of Caristi type and Meir-Keeler type mappings. By employing the new notion we obtain a generalization of the Caristi fixed point theorem and also obtain a Meir-Keeler type solution of the problem of continuity at the fixed point. Our theorems characterise completeness of the metric space and Cantor's intersection property.

Definition 1.4. *[30] A self-mapping f of a metric space X is called k-continuous, $k = 1, 2, 3, \ldots$, if $f^k x_n \to ft$ whenever $\{x_n\}$ is a sequence in X such that $f^{k-1}x_n \to t$.*

It was shown in [30] that continuity of f^k and k-continuity of f are independent conditions when $k > 1$; and continuity $\Rightarrow 2-continuity \Rightarrow 3-continuity \Rightarrow \ldots$ but not conversely. It is also easy to see that 1- continuity is equivalent to continuity.

Definition 1.5. *If f is a self-mapping of a metric space (X, d) then the set $O(x, f) = \{f^n x : n = 0, 1, 2, \ldots\}$ is called the orbit of f at x and f is called orbitally continuous if $u = \lim_i f^{m_i} x$ implies $fu = \lim_i f f^{m_i} x$.*

A continuous mapping is orbitally continuous but not conversely [8, 9]. A k-continuous mapping is obviously orbitally continuous. We now introduce a weaker form of the above definitions:

Definition 1.6. *A self-mapping f of a metric space (X, d) will be called weakly orbitally continuous if the set $\{y \in X : \lim_i f^{m_i} y = u \Rightarrow \lim_i f f^{m_i} y = fu\}$ is nonempty whenever the set $\{x \in X : \lim_i f^{m_i} x = u\}$ is nonempty.*

Example 1.7. *Let $X = [0, 2]$ equipped with the Euclidean metric. Define $f : X \to X$ by*

$$fx = \frac{(1+x)}{2} \text{ if } x < 1, \qquad fx = 0 \text{ if } 1 \le x < 2, \qquad f2 = 2.$$

Then $f^n 0 \to 1$ and $f(f^n 0) \to 1 \neq f1$. Therefore f is not orbitally continuous. However, f is weakly orbitally continuous. If we take $x = 2$ then $f^n 2 \to 2$ and $f(f^n 2) \to 2 = f2$ and, hence, f is weakly orbitally continuous. If we consider the sequence $\{f^n 0\}$ then for any integer $k \ge 1$, we have $f^{k-1}(f^n 0) \to 1$ and $f^k(f^n 0) \to 1 \neq f1$. This shows that f is not k-continuous.

Example 1.8. *Let $X = [0, \infty)$ equipped with the Euclidean metric. Define $f : X \to X$ by*

$$fx = 1 \text{ if } x \le 1, \qquad fx = \frac{x}{3} \text{ if } x > 1.$$

Then it is easy to see that f is orbitally continuous. Let $k \ge 1$ be any integer. Consider the sequence $\{x_n\}$ given by $x_n = 3^{k-1} + \frac{1}{n}$. Then $f^{k-1}x_n = 1 + \frac{1}{n3^{k-1}}, f^k x_n = \frac{1}{3} + \frac{1}{(n3^k)}$. This implies $f^{k-1}x_n \to 1, f^k x_n \to \frac{1}{3} \neq f1$ as $n \to \infty$. Hence f is not k-continuous.

Cromme and Diener [11] and Cromme [12] have proved results on approximate fixed points for discontinuous functions and have given applications of their results to neural nets, economic equilibria and analysis. We show that many such functions satisfy Caristi type condition or Meir-Keeler contractive conditions and possess fixed points. Fixed point theorems for discontinuous mappings have found wide applications, for example application of such theorems in the study of neural networks with discontinuous activation function is presently a very active area of research (e. g. Ding et al [13], Forti and Nistri [16], Nie and Zheng [24–26], Wu and Shan [43]).

2. Main Results

Theorem 2.1. *Let f be a self-mapping of a complete metric space (X, d) such that $d(fx, fy) \leq \max\{d(x, fx), d(y, fy)\}$ and*

(iv) given $\epsilon > 0$ there exists a $\delta > 0$ such that $\epsilon \leq \max\{d(x, fx), d(y, fy)\} < \epsilon + \delta \Rightarrow d(fx, fy) < \epsilon$.

Then f possesses a fixed point if and only if f is weakly orbitally continuous. Moreover, the fixed point is unique and f is continuous at the fixed point, say z, if and only if $\lim_{x\to z} \max\{d(x, fx), d(z, fz)\} = 0$ or, equivalently, $\lim_{x\to z} \sup d(fz, fx) = 0$.

Proof. It is obvious that f satisfies the contractive condition:

$$d(fx, fy) < \max\{d(x, fx), d(y, fy)\}, \tag{1}$$

whenever $\max\{d(x, fx), d(y, fy)\} > 0$. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = fx_{n-1}$, that is, $x_n = f^n x_0$. If $x_n = x_{n+1}$ for some n then $x_n = x_{n+1} = x_{n+2} = x_{n+3} \ldots$, that is, $\{x_n\} = \{f^n x_0\}$ is a Cauchy sequence and x_n is a fixed point of f. We can, therefore, assume that $x_n \neq x_{n+1}$ for each n. Then using (1) we get

$$\begin{aligned} d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) &< \max\{d(x_{n-1}, fx_{n-1}), d(x_n, fx_n)\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n). \end{aligned}$$

Thus $\{d(x_n, x_{n+1})\}$ is a strictly decreasing sequence of positive real numbers and, hence, tends to a limit $r \geq 0$. Suppose $r > 0$. Then there exists a positive integer N such that

$$n \geq N \Rightarrow r < d(x_n, x_{n+1}) < r + \delta(r). \tag{2}$$

This yields $r < \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = \max\{d(x_n, fx_n), d(x_{n+1}, fx_{n+1})\} < r + \delta(r)$, which by virtue of (iv) yields $d(fx_n, fx_{n+1}) = d(x_{n+1}, x_{n+2}) < r$. This contradicts (2). Hence $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Now if p is any positive integer then

$$\begin{aligned} d(x_n, x_{n+p}) &= d(fx_{n-1}, fx_{n+p-1}) \\ &< \max\{d(x_{n-1}, fx_{n-1}), d(x_{n+p-1}, fx_{n+p-1})\} \\ &= \max\{d(x_{n-1}, x_n), d(x_{n+p-1}, x_{n+p})\} = d(x_{n-1}, x_n). \end{aligned}$$

This implies that $d(x_n, x_{n+p}) \to 0$ since $d(x_{n-1}, x_n) \to 0$. Therefore, $\{x_n\} = \{f^n x_0\}$ is a Cauchy sequence. Since X is complete, there exists z in X such that $x_n \to z$. Moreover, for each integer $p \geq 1$, we have $f^p x_n \to z$. Also, using (1) it follows easily that $f^n y \to z$ for any y in X.
Suppose that f is weakly orbitally continuous. Since $f^n x_0 \to z$ for each x_0, by virtue of weak orbital continuity of f we get $f^n y_0 \to z$ and $f^{n+1} y_0 \to fz$ for some y_0 in X. This implies $z = fz$ since $f^{n+1} y_0 \to z$. Therefore z is a fixed point of f. Uniqueness of the fixed point follows easily.

Conversely, suppose that the mapping f possesses a fixed point, say z. Then $\{f^n z = z\}$ is a constant sequence such that $\lim_n f^n z = z$ and $\lim_n f^{n+1} z = z = fz$. Hence, f is weak orbitally continuous. It is also easy to verify that f is continuous at z if and only if $\lim_{x\to z} \max\{d(x, fx), d(z, fz)\} = 0$ or, equivalently, $\lim_{x\to z} \sup d(fz, fx) = 0$. This can alternatively be stated as:

$$f \text{ is discontinuous at } z \text{ if and only if } \lim_{x\to z} \sup d(fz, fx) > 0.$$

This proves the theorem. □

Example 2.2. *Let $X = [0, \infty)$ equipped with the usual metric and let $f : X \to X$ be defined by*

$$fx = \frac{x}{3} \qquad \textit{for each } x \textit{ in } X.$$

Then it easy to verify that X is complete, f satisfies (iv), f is continuous, and f has a unique fixed point $x = 0$.

Example 2.3. *Let $X = [0, 2]$ and d be the usual metric. Define $f : X \to X$ by*

$$fx = 1 \text{ if } 0 \le x \le 1, \qquad fx = x - 1 \text{ if } 1 < x \le 2.$$

Then f satisfies all the conditions of the above theorem and has a unique fixed point $z = 1$ at which f is discontinuous. The mapping f is orbitally continuous and, hence, weak orbitally continuous. The function f satisfies condition (iv) with $\delta(\epsilon) = 1 - \epsilon$ if $\epsilon < 1$ and $\delta(\epsilon) = \epsilon$ if $\epsilon \ge 1$. It is easy to see that $\lim_{x\to z} \max\{d(x, fx), d(z, fz)\}$ does not exist at $z = 1$, as required in the theorem. Also, $\lim_{x\to z} \sup d(fz, fx) = 1$.

Theorem 2.1 clearly establishes that if a self-mapping f of a complete metric space (X, d) satisfies condition (iv) then there exists a point, say z, in X such that for each x in X the sequence of iterates $\{f^n x\}$ converges to z. However, z is a fixed point if and only if f is weak orbitally continuous. Therefore, as a corollary of Theorem 2.1 we obtain the following:

Theorem 2.4. *Let f be a self-mapping of a complete metric space (X, d) such that*

(v) Given $\epsilon > 0$ there exists a $\delta > 0$ such that $\epsilon \le \max\{d(x, fx), d(y, fy)\} < \epsilon + \delta \Rightarrow d(fx, fy) < \epsilon$.

Then there exists a point z in X such that for each x in X the sequence of iterates $\{f^n x\}$ is a Cauchy sequence and $\lim_{n\to\infty} f^n x = z$.

The mappings in the next two examples satisfy conditions of Theorem 2.4 but do not have a fixed point.

Example 2.5. *Let $X = [0, 2]$ equipped with the Euclidean metric d. Define $f : X \to X$ by*

$$fx = \frac{1 + x}{2} \text{ if } x < 1, \qquad fx = 0 \text{ if } 1 \le x \le 2.$$

Then X is a complete metric space and f satisfies the contractive condition (v) with $\delta(\epsilon) = 1 - \epsilon$ for $\epsilon < 1$ and $\delta(\epsilon) = \epsilon$ for $\epsilon \ge 1$ but does not possess a fixed point. It is easy to verify that for each x in X, the sequence of iterates $\{f^n x\}$ is a Cauchy sequence and $f^n x \to 1$. It is easily seen that f is not weak orbitally continuous.

Example 2.6. *Let $X = [1, 2] \cup \{1 - \frac{1}{2^n} : n = 0, 1, 2, \ldots\}$ and d be the usual metric. Define $f : X \to X$ by*

$$fx = 0 \text{ if } 1 \le x \le 2, \qquad f(1 - \frac{1}{2^n}) = 1 - \frac{1}{2^{n+1}}, n = 0, 1, 2, \ldots.$$

Then the range of f is the countable set $f(X) = \{1 - \frac{1}{2^n} : n = 0, 1, 2, \ldots\}$ and f has no fixed point. The mapping f in this example satisfies the contractive condition (v) with $\delta(\epsilon) = \epsilon$ if $\epsilon \ge 1$ and $\delta(\epsilon) = \frac{1}{2^n} - \epsilon$ if $\frac{1}{2^{n+1}} \le \epsilon < \frac{1}{2n}, n = 0, 1, 2, \ldots.$

We now generalize Theorem 2.1 to obtain a Meir-Keeler type analogue of Theorems 2.1, 2.7 and 2.8 of Bisht and Rakocevic [4]. Given x, y in X and $0 \le a < 1$ let us denote:

$$m(x, y) = \max\{a\, d(x, fx) + (1 - a)\, d(y, fy), (1 - a)\, d(x, fx) + a\, d(y, fy)\}.$$

Theorem 2.7. *Let f be a self-mapping of a complete metric space (X, d) such that for all x, y in X we have*

(vi) Given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon \le m(x, y) < \epsilon + \delta \Rightarrow d(fx, fy) < \epsilon.$$

If $a = 0$, f has a unique fixed point whenever f is k-continuous or f^k is continuous for some $k \ge 1$ or f is weakly orbitally continuous. If $a > 0$, f possesses a unique fixed point at which f is continuous.

Proof. The proof follows on the lines of the proof of Theorem 2.1 above. If we take $a = 0$ in $m(x, y)$ then condition (vi) reduces to condition (iv) and Theorem 2.7 reduces to Theorem 2.1 above. When $a > 0$, the proof is similar to the case $a = 0$. As seen in Theorem 2.1 and Example 2.3 above, f need not be continuous at the fixed point if $a = 0$. We now show that f is continuous at the fixed point when $a > 0$. Suppose $a > 0$

and z is the fixed point of f. Let $\{x_n\}$ be any sequence in X such that $x_n \to z$ as $n \to \infty$. Then using (vi), for sufficiently large values of n we get

$$\begin{aligned} d(z, fx_n) = d(fz, fx_n) &< \max\{a\, d(z, fz) + (1-a)\, d(x_n, fx_n), (1-a)\, d(z, fz) + a\, d(x_n, fx_n)\} \\ &= \max\{(1-a)\, d(x_n, fx_n), a\, d(x_n, fx_n)\} \\ &\leq \max\{\epsilon_1 + (1-a)\, d(z, fx_n), \epsilon_2 + a\, d(z, fx_n)\}, \end{aligned}$$

where $\epsilon_1, \epsilon_2 \to 0$ as $n \to \infty$. This yields $a\, d(z, fx_n) < \epsilon_1$ or $(1-a)\, d(z, fx_n) < \epsilon_2$. Making $n \to \infty$ we get $\lim_{n\to\infty} fx_n = z = fz$. Hence f is continuous at the fixed point. □

Remark 2.8. *Theorem 2.7 shows that if a contractive condition admits discontinuity at the fixed point, a slightly stronger version of the condition may not.*

Remark 2.9. *As in Theorem 2.7 above, it can be shown that if $a > 0$ in Theorems 2.1, 2.7 and 2.8 of Bisht and Rakocevic [4] then the mapping is continuous at the fixed point.*

In the next theorem we obtain a generalization of the Caristi fixed point theorem.

Theorem 2.10. *Let f be a self-mapping of a complete metric space (X, d). Suppose $\phi : X \to [0, \infty)$ is a function such that for each x, y in X we have*

(*vii*) $d(x, fx) \leq \phi(x) - \phi(fx)$.

If f is weakly orbitally continuous or f^k is continuous or f is k-continuous for some $k \geq 1$, then f has a fixed point.

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ by $x_1 = fx_0, x_2 = fx_1, \ldots, x_n = fx_{n-1}, \ldots$, that is, $x_n = f^n x_0$. Then

$$d(x_0, x_1) = d(x_0, fx_0) \leq \phi(x_0) - \phi(fx_0) = \phi(x_0) - \phi(x_1).$$

Similarly

$$\begin{aligned} d(x_1, x_2) &\leq \phi(x_1) - \phi(x_2) \\ d(x_2, x_3) &\leq \phi(x_2) - \phi(x_3) \\ d(x_3, x_4) &\leq \phi(x_3) - \phi(x_4) \\ d(x_4, x_5) &\leq \phi(x_4) - \phi(x_5) \\ &\ldots \\ d(x_{n-1}, x_n) &\leq \phi(x_{n-1}) - \phi(x_n) \\ d(x_n, x_{n+1}) &\leq \phi(x_n) - \phi(x_{n+1}). \end{aligned}$$

Adding these inequalities we get

$$d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n) + d(x_n, x_{n+1}) \leq \phi(x_0) - \phi(x_{n+1}) \leq \phi(x_0).$$

Making $n \to \infty$ we get

$$\Sigma_{n=0}^{\infty} d(x_n, x_{n+1}) \leq \phi(x_0). \tag{3}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $t \in X$ such that $\lim_{n\to\infty} x_n = t$ and $\lim_{n\to\infty} f^p x_n = t$ for each $p \geq 1$. Suppose that f is weakly orbitally continuous. Since $\{f^n x_0\}$ converges for each x_0 in X, weak orbital continuity implies that there exists $y_0 \in X$ such that $f^n y_0 \to z$ and $f^{n+1} y_0 \to fz$ for some z in X. This implies that $z = fz$, that is, z is a fixed point of f. If f is k-continuous or f^k is continuous for some $k \geq 1$ then f is weakly orbitally continuous and the proof follows. This establishes the theorem. □

Example 2.11. *Let $X = [0, 2]$ equipped with the Euclidean metric. Define $f : X \to X$ by*

$$fx = \frac{(1 + x)}{2} \text{ if } x < 1, \qquad fx = 0 \text{ if } 1 \leq x < 2, \qquad f2 = 2.$$

Then it has been shown in Example 1.7 that f is weakly orbitally continuous but not k-continuous or orbitally continuous. Let us define $\phi : X \to [0, \infty)$ by

$$\phi(x) = 1 - x \text{ if } x < 1, \qquad \phi(x) = 1 + x \text{ if } x \geq 1.$$

Then $d(x, fx) \leq \phi(x) - \phi(fx)$ for each x in X, f satisfies all the conditions of 2.10 and has a fixed point $x = 2$.

Remark 2.12. *In the above example, $\lim_{x\to 1} \inf \phi(x) = 0 < \phi(1)$, that is, ϕ is not lower semi-continuous. This shows that f does not satisfy the conditions of Caristi's theorem. On the other hand, if a mapping f satisfies the conditions of Caristi's theorem then f has a fixed point, say z. This implies that $f^n z = z$ for each n, that is, $f^n z \to z$ and $f^{n+1} z \to z = fz$ as $n \to \infty$. Thus, f is weakly orbitally continuous and satisfies the conditions of Theorem 2.10. Therefore, Theorem 2.10 contains Caristi's theorem as a particular case.*

The next result is a particular case of Theorem 2.10 and it provides an answer to the question of continuity of contractive mappings at the fixed point in the form of a Caristi type fixed point theorem.

Corollary 2.13. *Let f be a contractive type self-mapping of a complete metric space (X, d). Suppose $\phi : X \to [0, \infty)$ is a function such that for each x in X we have*

(viii) $d(x, fx) \leq \phi(x) - \phi(fx)$.

If f is weakly orbitally continuous or f^k is continuous or f is k-continuous for some $k \geq 1$ then f possesses a unique fixed point.

The next example illustrates Corollary 2.13.

Example 2.14. *Let $X = (-\infty, \infty)$ equipped with Euclidean metric. Define $f : X \to X$ by*

$$fx = 1 \text{ if } x \leq 1, \qquad fx = 0 \text{ if } x > 1.$$

Then f satisfies the conditions of Corollary 2.13 and has a unique fixed point $x = 1$ at which f is discontinuous. The mapping f satisfies the contractive condition $d(fx, fy) < \max\{d(x, fx), d(y, fy)\}$ and satisfies condition (viii) with $\phi : X \to [0, \infty)$ defined by

$$\phi(x) = 1 - x \text{ if } x \leq 1, \qquad \phi(x) = 1 + x \text{ if } x > 1.$$

We now show that if a self-mapping f of a complete metric space (X, d) satisfies the Banach contraction condition then it also satisfies condition (vii) of Theorem 2.10. Similarly, it can also be shown that if a self-mapping f of a complete metric space satisfies the Kannan contraction condition or the Suzuki contraction condition then it satisfies condition (vii) of Theorem 2.10.

Theorem 2.15. *Suppose a self-mapping f of a complete metric space (X, d) satisfies the Banach contraction condition*

$$d(fx, fy) \leq a\, d(x, y), 0 \leq a < 1.$$

Then there exists a function $\phi : X \to [0, \infty)$ such that for each x in X we have

$$d(x, fx) \leq \phi(x) - \phi(fx).$$

Proof. For any x in X we have $d(fx, f^2x) \leq a\, d(x, fx)$, that is,

$$(\frac{1}{a})d(fx, f^2x) \leq d(x, fx).$$

By virtue of this inequality we get

$$
\begin{aligned}
d(x, fx) &= (\frac{1}{(1-a)})d(x, fx) - (\frac{a}{(1-a)})d(x, fx) \\
&\leq (\frac{1}{(1-a)})d(x, fx) - (\frac{a}{(1-a)})(\frac{1}{a})d(fx, f^2x) \\
&= (\frac{1}{(1-a)})d(x, fx) - (\frac{1}{(1-a)})d(fx, f^2x) \\
&= \phi(x) - \phi(fx),
\end{aligned}
$$

where $\phi : X \to [0, \infty)$ is defined by $\phi(x) = (\frac{1}{(1-a)})d(x, fx)$. □

We now show that the Caristi type fixed point theorem 2.10 as well as the Meir-Keeler type fixed point theorem 2.1 characterizes metric completeness. Several researchers have studied fixed point theorems that characterize metric completeness (e.g. Kirk [20], Liu [22], Park [32], Reich [34], Subrahmanayam [36], Suzuki [37], Weston [42]). Kirk [20] proved that Caristi's fixed theorem characterizes metric completeness. Subrahmanayam [36] proved that Kannan's theorem characterises metric completeness. Suzuki [37] proved a fixed point theorem that generalizes the Banach contraction theorem and characterises metric completeness. In view of an example given by Connel [10], the Banach contraction mapping theorem [1] does not characterise metric completeness. Park [32] gave some necessary and sufficient conditions for a metric space to be complete by combining some known characterizations of metric completeness.

There is, however, an essential difference between the next theorem and similar theorems (e. g. Subrahmanyam [36], Suzuki 37) giving characterization of completeness in terms of fixed point property for contractive type mappings. In [36] and [37] the contractive condition implies continuity at the fixed point; and completeness of the metric space X is equivalent to the existence of fixed point. On the other hand, the next theorem establishes that completeness of the space is equivalent to fixed point property for a larger class of mappings that includes continuous as well as discontinuous mappings.

Theorem 2.16. *If every k-continuous or weak orbitally continuous self-mapping of a metric space (X, d) satisfying condition (vii) of Theorem 2.10 or condition (iv) of Theorem 2.1 has a fixed point, then X is complete.*

Proof. Suppose that every k-continuous or weak orbitally continuous self-mapping of the metric space X satisfying condition (vii) of Theorem 2.10 or condition (iv) of Theorem 2.1 possesses a fixed point. We show that X is a complete metric space. If X is not complete, then there exists a Cauchy sequence in X, say $S = \{a_n : n = 1, 2, 3 \ldots\}$, consisting of distinct points which does not converge. Let $x \in X$ be given. Then, since x is not a limit point of the sequence S, we have $d(x, S - \{x\}) > 0$ and there exists a least positive integer, say $N(x)$, such that for each $m \geq N(x)$ we have

$$d(a_{N(x)}, a_m) < \frac{1}{2}d(x, a_{N(x)}). \tag{4}$$

Let us define a mapping $f : X \to X$ by $f(x) = a_{N(x)}$. Then, $f(x) \neq x$ for each x and, using (4), for any x, y in X we get

$$d(fx, fy) = d(a_{N(x)}, a_{N(y)}) < \frac{1}{2}d(x, a_N(x)) = d(x, fx) \text{ if } N(x) \leq N(y) \tag{5}$$

or

$$d(fx, fy) = d(a_{N(x)}, a_{N(y)}) < \frac{1}{2}d(y, a_N(y)) = d(y, fy) \text{ if } N(x) > N(y). \tag{6}$$

This implies that

$$d(fx, fy) < \frac{1}{2}\max\{d(x, fx), d(y, fy)\}. \tag{7}$$

In other words, given $\epsilon > 0$ we can select $\delta(\epsilon) = \epsilon$ such that

$$\epsilon \le \max\{d(x, fx), d(y, fy)\} < \epsilon + \delta \Rightarrow d(fx, fy) < \epsilon. \tag{8}$$

Similarly, taking $y = fx$ in (5) we get $N(fx) > N(x)$ and

$$d(fx, f^2x) = d(a_{N(x)}, a_{N(fx)}) < \frac{1}{2}d(x, a_{N(x)}) = \frac{1}{2}d(x, fx). \tag{9}$$

Now let us define a function $\phi : X \to [0, \infty)$ by $\phi(x) = 2d(x, fx)$. Then using (9) we get

$$\phi(x) - \phi(fx) = 2d(x, fx) - 2d(fx, f^2x) \ge 2d(x, fx) - d(x, fx) = d(x, fx). \tag{10}$$

It is clear from (8) that the mapping f satisfies condition (iv) of Theorem 2.1. Similarly, (10) shows that f satisfies condition (vii) of Theorem 2.10. Since the range of f is contained in the non-convergent Cauchy sequence $S = \{a_n\}$, there exists no sequence $\{x_n\}$ in X for which the condition $fx_n \to t \Rightarrow f^2x_n \to ft$ is violated. Therefore, f is a 2-continuous mapping. In a similar manner it follows that f is weak orbitally continuous. Thus, f is a 2-continuous as well as weak orbitally continuous self-mapping of X satisfying (iv) and (vii) which does not possess a fixed point. This contradicts our assumption. Hence X is complete. □

Examples 2.2 and 2.3 show that if f is k-continuous or f^k is continuous for some $k \ge 1$ or f is weak orbitally continuous and f satisfies condition (iv) then f possesses a unique fixed point if X is complete. The next example shows that the condition of completeness of X cannot be dropped.

Example 2.17. *Let $X = [0, 1) \cup (1, 2]$ and d be the usual metric. Define $f : X \to X$ by*

$$fx = \frac{(1 + x)}{2} \quad \text{if } 0 \le x < 1, \qquad fx = 0 \quad \text{if } 1 < x \le 2.$$

Then f satisfies the contractive conditions (iv) with $\delta(\epsilon) = 1 - \epsilon$ for $\epsilon < 1$ and $\delta(\epsilon) = \epsilon$ for $\epsilon \ge 1$ but f does not have a fixed point. The mapping f is continuous and, hence, f is k- continuous, f^k is continuous for each $k \ge 1$ and f is weak orbitally continuous.

Definition 2.18. *A metric space (X, d) is said to satisfy Cantor's intersection property if every sequence $\{F_n\}$ of nonempty closed subsets of X that satisfies $F_{n+1} \subseteq F_n$ and $\lim_{n\to\infty} diam(F_n) = 0$, has a nonempty intersection.*

If X is a complete metric space then Cantor's intersection property holds in X. We now show that Theorems 2.1 and 2.10 characterize Cantor's intersection property.

Theorem 2.19. *Let (X, d) be a metric space and f a self-mapping of X satisfying condition (iv) of Theorem 2.1 or condition (vii) of Theorem 2.10. Suppose X satisfies Cantor's intersection property and f^k is continuous or f is k-continuous for some $k \ge 1$ or f is weak orbitally continuous. Then f has a fixed point and f is continuous at z if and only if $\lim_{x\to z} \max\{d(x, fx), d(z, fz)\} = 0$.*

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = fx_{n-1}$ as in Theorem 2.1. Then proceeding on the lines of Theorem 2.1 it follows that $\{x_n\}$ is a Cauchy sequence. Define a sequence $\{S_n : n = 1, 2, 3, \ldots\}$ of nonempty subsets of X by $S_n = \{x_i : i \ge n\}$. Let F_n denote the closure of S_n in X, then for each n it is obvious that F_n is a nonempty closed subset of $X, F_{n+1} \subseteq F_n$ and $\lim_{n\to\infty} diam(F_n) = 0$. Since Cantor's intersection property holds in $X, \cap\{F_n\}$ consists of exactly one point, say z, which is obviously the limit of the Cauchy sequence $\{x_n\}$. Now, following the proof of Theorem 2.1 or Theorem 2.10 it is easy to prove that z is the fixed point of f and f is continuous at z if and only if $\lim_{x\to z} \max\{d(x, fx), d(z, fz)\} = 0$. This completes the proof. □

Theorem 2.20. *Let (X, d) be a metric space. If every k-continuous or weak orbitally continuous self-mapping f of X satisfying the condition (iv) of Theorem 2.1 or condition (vii) of Theorem 2.10 has a fixed point then X satisfies Cantor's intersection property.*

Proof. Suppose that every k-continuous or weak orbitally continuous self-mapping of X satisfying condition (iv) or condition (vii) possesses a fixed point. We assert that X satisfies Cantor's intersection theorem. For, if not, then there exists a sequence $\{F_n\}$ of nonempty closed subsets of X satisfying $F_{n+1} \subseteq F_n$ and $\lim_{n\to\infty} diam(F_n) = 0$ and having empty intersection. Construct a sequence $S = \{x_n\}$ in X such that $x_i \in F_i$. Since $\lim_{n\to\infty} diam(F_n) = 0$, given $\epsilon > 0$ there exists a positive integer N such that $n, m \geq N$ implies $d(x_n, x_m) < \epsilon$. Therefore $\{x_n\}$ is a Cauchy sequence. However, $S = \{x_n\}$ is a non-convergent Cauchy sequence since the sequence $\{F_n\}$ has empty intersection. As done in the proof of Theorem 2.16 we can now define a k-continuous as well as weakly orbitally continuous self-mapping f on X satisfying (iv) and (vii) which does not possess a fixed point. This contradicts our hypothesis. Therefore, Cantor's intersection theorem holds in X. □

Theorems 2.19 and 2.20 show that fixed point property for k-continuous or weak orbitally continuous self-mappings of a metric space (X, d) satisfying (iv) and (vii) characterizes Cantor's intersection property. If we combine Theorem 2.16 and Theorem 2.20 then we get the following theorem:

Theorem 2.21. *For a metric space (X, d), the following are equivalent:*

(a) X is complete

(b) X satisfies Cantor's intersection property

(c) every k-continuous or weak orbitally continuous self-mapping f of X satisfying the condition:
Given $\epsilon > 0$ there exist $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \max\{d(x, fx), d(y, fy)\} < \epsilon + \delta \Rightarrow d(fx, fy) < \epsilon,$$

has a fixed point

(d) every k-continuous or weak orbitally continuous self-mapping f of X such that there exists a function $\phi : X \to [0, \infty)$ satisfying the condition:

$$d(x, fx) \leq \phi(x) - \phi(fx),$$

has a fixed point.

3. Measure of Discontinuity

For real valued functions on a subset of real numbers, Wardowski [41] has introduced the notion of measure of discontinuity to measure how much a given function is discontinuous. Wardowski's definition can be extended for self-mappings of a metric space (X, d). Following Wardowski, a self-mapping f of a metric space (X, d) will be said to be continuous at x_0 with accuracy $\eta(\eta \geq 0)$ if given $\epsilon > \eta$ there exists $\delta > 0$ such that $d(fx, fx_0) < \epsilon$ whenever $d(x, x_0) < \delta$. As in [41], let us denote the set of all $\eta \geq 0$ such that f is continuous at x_0 with accuracy η by $disc(f, x_0)$.

Definition 3.1. *If $disc(f, x_0)$ is nonempty, then the measure of discontinuity of f at x_0, denoted by $\sigma(f, x_0)$, will be defined as the non-negative real number given by*

$$\sigma(f, x_0) = \min disc(f, x_0).$$

If $disc(f, x_0)$ is empty set, we will say $\sigma(f, x_0) = \infty$. If we apply this definition to the mapping f in Example 2.3 above then at the fixed point $z = 1$ we get $\sigma(f, z) = 1$. On the other hand, in Example 2.3, at $z = 1$ we obtained $\lim_{x\to z} \sup d(fz, fx) = 1$. Thus at the fixed point z, we have $\sigma(f, z) = \lim_{x\to z} \sup d(fz, fx)$. Therefore, measure of discontinuity at a point z can also be defined as $\lim_{x\to z} \sup d(fz, fx)$.

Wardowski [40] introduced notion of F-contractions. As a generalization of F-contractions Ozgur and Tas [27, 28] defined $F_C(m_{x_0})$ contractions and obtained some fixed circle theorems and their application to neural networks. Using Theorem 2.1 above and following the method in [27, 28] we can obtain analogues of such results in [27, 28].

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