Filomat 33:12 (2019), 3669–3682
https://doi.org/10.2298/FIL1912669D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Ruscheweyh-Type Harmonic Functions with Correlated Coefficients

Jacek Dziok[a]

[a] *Institute of Mathematics, University of Rzeszów, ul. Prof. Pigonia 1, 35-310 Rzeszów, Poland*

Abstract. In the paper we introduce the classes of functions with correlated coefficients defined by generalized Ruscheweyh derivatives. We also define the dual set for harmonic functions and show that the classes of functions can be presented as dual sets. Moreover, by using extreme points theory, we obtain estimations of classical convex functionals on the defined classes of functions. Some applications of the main results are also considered.

1. Introduction

Harmonic functions are famous for their use in the study of minimal surfaces and also play important roles in a variety of problems in applied mathematics. Harmonic functions have been studied by differential geometers such as Choquet [4], Kneser [18], Lewy [20], and Rado [23]. Recent interest in harmonic complex functions has been triggered by geometric function theorists Clunie and Sheil-Small [5]. Let $\mathcal{H}$ denote the family of continuous complex-valued functions which are harmonic in the open unit disk $\mathbb{U} := \mathbb{U}(1)$, where $\mathbb{U}(r) := \{z \in \mathbb{C} : |z| < r\}$, and let $\mathcal{A}$ denote the class of functions which are analytic in $\mathbb{U}$. Every harmonic function $f \in \mathcal{H}$ has a unique representation

$$f = h + \overline{g}, \tag{1}$$

where h and g are analytic in $\mathbb{U}$ and $g(0) = 0$. A result of Lewy [20] shows that f is locally univalent and sense-preserving if and only if

$$|h'(z)| > |g'(z)| \quad (z \in \mathbb{U}). \tag{2}$$

We denote by $\mathcal{H}_0$ the family of functions $f \in \mathcal{H}$ of the form

$$f(z) = h + \overline{g} = z + \sum_{n=2}^{\infty} \left(a_n z^n + \overline{b_n z^n} \right) \quad (h, g \in \mathcal{A}, z \in \mathbb{U}) \tag{3}$$

which are univalent and sense-preserving in $\mathbb{U}$.

2010 *Mathematics Subject Classification*. Primary 30C55; Secondary 30C45
Keywords. harmonic functions, subordination, starlike functions, Ruscheweyh derivatives, dual set, correlated coefficients
Received: 07 November 2017; Revised: 10 June 2018; Accepted: 21 August 2018
Communicated by Miodrag Mateljević
Research supported by the Centre for Innovation and Transfer of Natural Sciences and Engineering Knowledge, University of Rzeszów.
Email address: jdziok@ur.edu.pl (Jacek Dziok)

Let $\mathbb{N}_k := \{k, k+1, ...\}$, $\mathbb{N} := \mathbb{N}_1$. For functions $f_1, f_2 \in \mathcal{H}$ of the form

$$f_l(z) = \sum_{n=0}^{\infty} \left(a_{l,n} z^n + \overline{b_{l,n} z^n} \right) \ (z \in \mathbb{U}, l \in \mathbb{N}) \tag{4}$$

we define *convolution* of f_1 and f_2 by

$$(f_1 * f_2)(z) = \sum_{n=0}^{\infty} \left(a_{1,n} a_{2,n} z^n + \overline{b_{1,n} b_{2,n} z^n} \right) \quad (z \in \mathbb{U}).$$

We say that a function $f \in \mathcal{H}_0$ is harmonic starlike in $\mathbb{U}(r)$ if f maps the circle $\partial\mathbb{U}(r)$ onto a closed curve that is starlike with respect to the origin *i.e.*

$$\frac{\partial}{\partial t}\left(\arg f\left(re^{it} \right) \right) > 0 \ \ (0 \le t \le 2\pi)$$

or equivalently

$$\operatorname{Re} \frac{D_{\mathcal{H}} f(z)}{f(z)} > 0 \quad (|z| = r),$$

where

$$D_{\mathcal{H}} f(z) := zh'(z) - \overline{zg'(z)} \quad (z \in \mathbb{U}).$$

Ruscheweyh [25] introduced an operator $\mathcal{D}^{\lambda} : \mathcal{A} \to \mathcal{A}$, defined by the convolution:

$$\mathcal{D}^{\lambda} f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) \quad (\lambda \neq -1;\ f \in \mathcal{A}), \tag{5}$$

which implies that

$$\mathcal{D}^{n} f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!} \quad (n \in \mathbb{N}_0).$$

Let $D_{\mathcal{H}}^{\lambda,\tau} : \mathcal{H} \to \mathcal{H}$ denote the linear operator defined for a function $f = h + \overline{g} \in \mathcal{H}$ by

$$D_{\mathcal{H}}^{\lambda,\tau} := \mathcal{D}^{\lambda} h + \tau \overline{\mathcal{D}^{\lambda} g} \quad (|\tau| = 1).$$

The operator $D_{\mathcal{H}}^{\lambda,\tau}$ for $\tau = (-1)^n$ was investigated in [21] (see also [7, 10, 12, 14, 28]).

We say that a function $f \in \mathcal{H}$ is *subordinate* to a function $F \in \mathcal{H}$, and write $f(z) \prec F(z)$ (or simply $f \prec F$) if there exists a complex-valued function ω which maps $\mathbb{U}$ into oneself with $\omega(0) = 0$, such that $f(z) = F(\omega(z)) \quad (z \in \mathbb{U})$.

Let A and B be two distinct complex parameters.We denote by $\mathcal{S}_{\mathcal{H}}^{\lambda,\tau}(A, B)$ the class of functions $f \in \mathcal{H}_0$ such that

$$\frac{D_{\mathcal{H}}\left(D_{\mathcal{H}}^{\lambda,\tau} f \right)(z)}{D_{\mathcal{H}}^{\lambda,\tau} f(z)} \prec \frac{1 + Az}{1 + Bz}. \tag{6}$$

Also, by $\mathcal{R}_{\mathcal{H}}^{\lambda,\tau}(A, B)$ we denote the class of functions $f \in \mathcal{H}_0$ such that

$$\frac{D_{\mathcal{H}}^{\lambda,\tau} f(z)}{z} \prec \frac{1 + Az}{1 + Bz}.$$

In particular, if we put $\lambda = n \in \mathbb{N}_0$, $\tau = (-1)^n$, then we obtain the classes

$$\mathcal{S}^n_{\mathcal{H}}(A,B) := \mathcal{S}^{n,(-1)^n}_{\mathcal{H}}(A,B),\ \mathcal{R}^n_{\mathcal{H}}(A,B) := \mathcal{R}^{n,(-1)^n}_{\mathcal{H}}(A,B)$$

related to the harmonic Ruscheweyh derivatives $\mathcal{D}^n_{\mathcal{H}}f$ (see [10]). The classes $\mathcal{S}_{\mathcal{H}}(A,B) := \mathcal{S}^0_{\mathcal{H}}(A,B)$, $\mathcal{K}_{\mathcal{H}}(A,B) := \mathcal{S}^1_{\mathcal{H}}(A,B)$ and $\mathcal{R}_{\mathcal{H}}(A,B) := \mathcal{R}^1_{\mathcal{H}}(A,B)$ are defined in [8] (see also [9]) with restrictions $-B \le A < B \le 1$.

The object of this papers to obtain some necessary and sufficient conditions for the above defined classes. Some topological properties and extreme points of the classes are also considered. By using extreme points theory we obtain coefficients estimates, distortion theorems, integral mean inequalities for these classes of functions.

2. Dual sets

Let $\mathcal{V} \subset \mathcal{H}$, $\mathbb{U}_0 := \mathbb{U} \setminus \{0\}$. Motivated by Ruscheweyh [24] we define the dual set of $\mathcal{V}$ by

$$\mathcal{V}^* := \left\{ f \in \mathcal{H}_0 : \bigwedge_{q \in \mathcal{V}} (f * q)(z) \neq 0 \quad (z \in \mathbb{U}_0) \right\}.$$

The object of this section is to show that the defined classes of functions can be presented as dual sets.

Theorem 2.1.

$$\mathcal{S}^{\lambda,\tau}_{\mathcal{H}}(A,B) = \{\psi_\xi : |\xi| = 1\}^*,$$

where

$$\begin{aligned} \psi_\xi(z) \quad : \quad = z\frac{(B-A)\xi + (1+\lambda+\lambda B\xi + A\xi)z}{(1-z)^{\lambda+2}} \\ -\tau\overline{z}\frac{2+(A+B)\xi - (1-\lambda-\lambda B\xi + A\xi)\overline{z}}{(1-\overline{z})^{\lambda+2}} \quad (z \in \mathbb{U}). \end{aligned} \tag{7}$$

Proof. Let $f \in \mathcal{H}_0$ be of the form (1). Then $f \in \mathcal{S}^{\lambda,\tau}_{\mathcal{H}}(A,B)$ if and only if it satisfies (6) or equivalently

$$\frac{D_{\mathcal{H}}\left(D^{\lambda,\tau}_{\mathcal{H}} f\right)(z)}{D^{\lambda,\tau}_{\mathcal{H}} f(z)} \neq \frac{1+A\xi}{1+B\xi} \quad (z \in \mathbb{U}_0,\ |\xi| = 1). \tag{8}$$

Since

$$D_{\mathcal{H}}\left(D^{\lambda,\tau}_{\mathcal{H}} h\right)(z) = h(z) * \frac{z}{(1-z)^{\lambda+1}} * \frac{z}{(1-z)^2} = h(z) * \frac{z+\lambda z^2}{(1-z)^{\lambda+2}},$$

the above inequality yields

$$\begin{aligned} & (1+B\xi)\, D_{\mathcal{H}}\left(D^{\lambda,\tau}_{\mathcal{H}} f\right)(z) - (1+A\xi)\, D^{\lambda,\tau}_{\mathcal{H}} f(z) \\ = \ & (1+B\xi)\, D_{\mathcal{H}}\left(D^{\lambda,\tau}_{\mathcal{H}} h\right)(z) - (1+A\xi)\, D^{\lambda,\tau}_{\mathcal{H}} h(z) \\ & -\tau\left[(1+B\xi)\, \overline{D_{\mathcal{H}}\left(D^{\lambda,\tau}_{\mathcal{H}} g\right)(z)} + (1+A\xi)\, \overline{D^{\lambda,\tau}_{\mathcal{H}} h(z)}\right] \\ = \ & h(z) * \left(\frac{(1+B\xi)(1+\lambda z) z}{(1-z)^{\lambda+2}} - \frac{(1+A\xi) z}{(1-z)^{\lambda+1}}\right) \\ & -\tau\overline{g(z)} * \left(\frac{(1+B\xi)\left(1+\overline{\lambda}\overline{z}\right)\overline{z}}{(1-\overline{z})^{\lambda+2}} + \frac{(1+A\xi)\overline{z}}{(1-\overline{z})^{\lambda+1}}\right) \\ = \ & f(z) * \psi_\xi(z) \neq 0 \quad (z \in \mathbb{U}_0,\ |\xi| = 1). \end{aligned}$$

Thus, $f \in \mathcal{S}^{\lambda,\tau}_{\mathcal{H}}(A,B)$ if and only if $f(z) * \psi_\xi(z) \neq 0$ for $z \in \mathbb{U}_0$, $|\xi| = 1$ *i.e.* $\mathcal{S}^{\lambda,\tau}_{\mathcal{H}}(A,B) = \{\psi_\xi : |\xi| = 1\}^*$. □

Using an argument similar to that given in the proof of Theorem 2.1, we can prove the following Theorem 2.2.

Theorem 2.2.

$$\mathcal{R}^{\lambda,\tau}_{\mathcal{H}}(A,B) = \{\delta_\xi : |\xi| = 1\}^*,$$

where

$$\delta_\xi(z) := z\frac{1 + B\xi - (1 + A\xi)(1 - z)^{\lambda+1}}{(1 - z)^{\lambda+1}} + \tau\overline{z}\frac{1 + B\xi}{(1 - \overline{z})^{\lambda+1}} \quad (z \in \mathbb{U}). \tag{9}$$

If we put $\lambda = n \in \mathbb{N}_0$, $\tau = (-1)^n$, in Theorems 2.1 and 2.2 we obtain the following results (see [8]).

Theorem 2.3.

$$\mathcal{S}^n_{\mathcal{H}}(A,B) = \{\rho_\xi : |\xi| = 1\}^*, \tag{10}$$

where

$$\begin{aligned}\rho_\xi(z) \quad : \quad &= z\frac{(B - A)\xi + (1 + n + (nB + A)\xi)z}{(1 - z)^{n+2}} \\ &- (-1)^n\overline{z}\frac{2 + (A + B)\xi + (n - 1 + (nB - A)\xi)\overline{z}}{(1 - \overline{z})^{n+2}} \quad (z \in \mathbb{U}).\end{aligned} \tag{11}$$

Theorem 2.4.

$$\mathcal{R}^n_{\mathcal{H}}(A,B) = \{\delta_\xi : |\xi| = 1\}^*,$$

where

$$\delta_\xi(z) := z\frac{1 + B\xi - (1 + A\xi)(1 - z)^{n+1}}{(1 - z)^{n+1}} + (-1)^n\overline{z}\frac{1 + B\xi}{(1 - \overline{z})^{n+1}} \quad (z \in \mathbb{U}).$$

Moreover, if we get $n = 0$ and $n = 1$ in Theorem 2.3 and $n = 1$ in Theorem 2.4 we obtain the following results (see [8]).

Theorem 2.5.

$$\mathcal{S}_{\mathcal{H}}(A,B) = \{\psi_\xi : |\xi| = 1\}^*,$$

where

$$\psi_\xi(z) := z\frac{(B - A)\xi + (1 + A\xi)z}{(1 - z)^2} - \overline{z}\frac{2 + (A + B)\xi - (1 + A\xi)\overline{z}}{(1 - \overline{z})^2} \quad (z \in \mathbb{U}).$$

Theorem 2.6.

$$\mathcal{K}_{\mathcal{H}}(A,B) = \{\psi_\xi : |\xi| = 1\}^*,$$

where

$$\psi_\xi(z) := z\frac{(B - A)\xi + (2 + A\xi + B\xi)z}{(1 - z)^3} - \overline{z}\frac{2 + (A + B)\xi + (B - A)\xi\overline{z}}{(1 - \overline{z})^3} \quad (z \in \mathbb{U}).$$

Theorem 2.7.

$$\mathcal{R}_{\mathcal{H}}(A,B) = \{\delta_\xi : |\xi| = 1\}^*,$$

where

$$\delta_\xi(z) := z\frac{1 + B\xi - (1 + A\xi)(1 - z)^2}{(1 - z)^2} - \overline{z}\frac{1 + B\xi}{(1 - \overline{z})^2} \quad (z \in \mathbb{U}).$$

3. Correlated coefficients

Let us consider the function $\varphi \in \mathcal{H}$ of the form

$$\varphi = u + \overline{v}, \ \ u(z) = \sum_{n=0}^{\infty} u_n z^n, \ \ v(z) = \sum_{n=1}^{\infty} v_n z^n \quad (z \in \mathbb{U})\,. \tag{12}$$

We say that a function $f \in \mathcal{H}$ of the form (3) has coefficients correlated with the function φ, if

$$u_n a_n = -|u_n|\,|a_n|\,,\ v_n b_n = |v_n|\,|b_n| \quad (n \in \mathbb{N}_2)\,. \tag{13}$$

In particular, if there exists a real number η such that

$$\varphi(z) = \frac{z}{1-e^{i\eta}z} + \frac{e^{-2i\eta}\overline{z}}{1-\overline{e^{i\eta}z}} = \sum_{n=1}^{\infty}\left(e^{i(n-1)\eta}z^n + e^{-i(n+1)\eta}\overline{z}^n\right) \quad (z \in \mathbb{U})\,,$$

then we obtain functions with varying coefficients defined by Jahangiri and Silverman [15] (see also [9]). Moreover, if we take

$$\varphi(z) = 2\mathrm{Re}\frac{z}{1-z} = \sum_{n=1}^{\infty}\left(z^n + \overline{z}^n\right) \quad (z \in \mathbb{U})\,,$$

then we obtain functions with negative coefficients introduced by Silverman [27]. These functions were intensively investigated by many authors (for example, see [6, 8–11, 13, 15, 17, 30]).

Let $\mathcal{T}^{\lambda,\tau}(\eta)$ denote the class of functions $f \in \mathcal{H}$ with coefficients correlated with respect to the function

$$\varphi(z) = \frac{z}{(1-e^{i\eta}z)^{\lambda+1}} + \frac{\tau e^{-2i\eta}\overline{z}}{(1-\overline{e^{i\eta}z})^{\overline{\lambda}+1}} \quad (z \in \mathbb{U})\,. \tag{14}$$

Moreover, let us define

$$\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta; A, B) := \mathcal{T}^{\lambda,\tau}(\eta) \cap \mathcal{S}_{\mathcal{H}}^{\lambda,\tau}(A,B),\ \mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(\eta; A, B) := \mathcal{T}^{\lambda,\tau}(\eta) \cap \mathcal{R}_{\mathcal{H}}^{\lambda,\tau}(A,B),$$

where $\eta; A, B$ are real parameters with $B > \max\{0, A\}$.

Let $f \in \mathcal{H}$ be of the form (3). Thus, by (5) we have

$$D_{\mathcal{H}}^{\lambda} f(z) = z + \sum_{n=2}^{\infty} \lambda_n a_n z^n + \tau \sum_{n=2}^{\infty} \overline{\lambda_n b_n}\,\overline{z}^n \quad (z \in \mathbb{U})\,,$$

where

$$\lambda_1 := 1,\ \lambda_n := \frac{(\lambda+1)\cdot \ldots \cdot (\lambda+n-1)}{(n-1)!} \quad (n \in \mathbb{N}_2)\,. \tag{15}$$

Moreover, let us assume

$$|\lambda_n| \geq |\lambda_2| \geq 1,\ B > \max\{0, A\}. \tag{16}$$

Theorem 3.1. *If a function* $f \in \mathcal{H}$ *of the form* (3) *satisfies the condition*

$$\sum_{n=2}^{\infty}\left(|\alpha_n|\,|a_n| + \left|\beta_n\right|\,|b_n|\right) \leq B - A, \tag{17}$$

where

$$\alpha_n = \lambda_n \{n(1+B) - (1+A)\}\,,\ \ \beta_n = \lambda_n \{n(1+B) + (1+A)\}\,, \tag{18}$$

then $f \in \mathcal{S}_{\mathcal{H}}^{\lambda,\tau}(A,B)$.

Proof. It is clear that the theorem is true for the function $f(z) \equiv z$. Let $f \in \mathcal{H}_0$ be a function of the form (3) and let there exist $n \in \mathbb{N}_2$ such that $a_n \neq 0$ or $b_n \neq 0$. By (16) we have

$$\frac{|\alpha_n|}{B-A} \geq n, \ \frac{|\beta_n|}{B-A} \geq n \quad (n \in \mathbb{N}_2). \tag{19}$$

Thus, by (17) we get

$$\sum_{n=2}^{\infty} (n|a_n| + n|b_n|) \leq 1 \tag{20}$$

and

$$\begin{aligned} |h'(z)| - |g'(z)| &\geq 1 - \sum_{n=2}^{\infty} n|a_n||z|^n - \sum_{n=2}^{\infty} n|b_n||z|^n \geq 1 - |z| \sum_{n=2}^{\infty} (n|a_n| + n|b_n|) \\ &\geq 1 - \frac{|z|}{B-A} \sum_{n=2}^{\infty} \left(|\alpha_n||a_n| + |\beta_n||b_n|\right) \geq 1 - |z| > 0 \quad (z \in \mathbb{U}). \end{aligned}$$

Therefore, by (2) the function f is locally univalent and sense-preserving in $\mathbb{U}$. Moreover, if $z_1, z_2 \in \mathbb{U}$, $z_1 \neq z_2$, then.

$$\left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right| = \left|\sum_{l=1}^{n} z_1^{l-1} z_2^{n-l}\right| \leq \sum_{l=1}^{n} |z_1|^{l-1} |z_2|^{n-l} < n \quad (n \in \mathbb{N}_2).$$

Hence, by (20) we have

$$\begin{aligned} |f(z_1) - f(z_2)| &\geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\ &= \left|z_1 - z_2 - \sum_{n=2}^{\infty} a_n \left(z_1^n - z_2^n\right)\right| - \left|\sum_{n=2}^{\infty} \overline{b_n \left(z_1^n - z_2^n\right)}\right| \\ &\geq |z_1 - z_2| - \sum_{n=2}^{\infty} |a_n| \left|z_1^n - z_2^n\right| - \sum_{n=2}^{\infty} |b_n| \left|z_1^n - z_2^n\right| \\ &= |z_1 - z_2| \left(1 - \sum_{n=2}^{\infty} |a_n| \left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right| - \sum_{n=2}^{\infty} |b_n| \left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right|\right) \\ &> |z_1 - z_2| \left(1 - \sum_{n=2}^{\infty} n|a_n| - \sum_{n=2}^{\infty} n|b_n|\right) \geq 0. \end{aligned}$$

This leads to the univalence of f *i.e.* $f \in \mathcal{S_H}$. Therefore, $f \in \mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(A,B)$ if and only if there exists a complex-valued function ω, $\omega(0) = 0$, $|\omega(z)| < 1$ $(z \in \mathbb{U})$ such that

$$\frac{D_{\mathcal{H}}\left(D_{\mathcal{H}}^{\lambda,\tau} f\right)(z)}{D_{\mathcal{H}}^{\lambda,\tau} f(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)} \quad (z \in \mathbb{U}),$$

or equivalently

$$\left|\frac{D_{\mathcal{H}}\left(D_{\mathcal{H}}^{\lambda\,\tau} f\right)(z) - D_{\mathcal{H}}^{\lambda\,\tau} f(z)}{BD_{\mathcal{H}}\left(D_{\mathcal{H}}^{\lambda\,\tau} f\right)(z) - A\left(D_{\mathcal{H}}^{\lambda\,\tau} f(z)\right)(z)}\right| < 1 \quad (z \in \mathbb{U}). \tag{21}$$

Thus for $z \in \mathbb{U} \setminus \{0\}$ it suffices to show that

$$\left| D_{\mathcal{H}}\left(D_{\mathcal{H}}^{\lambda,\tau} f\right)(z) - D_{\mathcal{H}}^{\lambda,\tau} f(z) \right| - \left| B D_{\mathcal{H}}\left(D_{\mathcal{H}}^{\lambda,\tau} f\right)(z) - D_{\mathcal{H}}^{\lambda,\tau} f(z) \right| < 0.$$

Indeed, letting $|z| = r$ $(0 < r < 1)$ we have

$$\begin{aligned}
&\left| D_{\mathcal{H}}\left(D_{\mathcal{H}}^{\lambda,\tau} f\right)(z) - D_{\mathcal{H}}^{\lambda,\tau} f(z) \right| - \left| B D_{\mathcal{H}}\left(D_{\mathcal{H}}^{\lambda,\tau} f\right)(z) - D_{\mathcal{H}}^{\lambda,\tau} f(z) \right| \\
&= \left| \sum_{n=2}^{\infty} (n-1)\lambda_n a_n z^n - \sum_{n=2}^{\infty} (n+1)\tau \overline{\lambda_n b_n} \overline{z}^n \right| \\
&- \left| (B-A)z + \sum_{n=2}^{\infty} (Bn-A)\lambda_n a_n z^n + \sum_{n=2}^{\infty} (Bn+A)\tau \overline{\lambda_n b_n} \overline{z}^n \right| \\
&\le \sum_{n=2}^{\infty} (n-1)|\lambda_n a_n| r^n + \sum_{n=2}^{\infty} (n+1)|\lambda_n b_n| r^n - (B-A)r \\
&+ \sum_{n=2}^{\infty} (Bn-A)|\lambda_n a_n| r^n + \sum_{n=2}^{\infty} (Bn+A)|\lambda_n b_n| r^n \\
&\le r\left\{ \sum_{n=2}^{\infty} \left(|\alpha_n||a_n| + \left|\beta_n\right||b_n|\right) r^{n-1} - (B-A) \right\} < 0.
\end{aligned}$$

whence $f \in \mathcal{S}_{\mathcal{H}}^{\lambda,\tau}(A,B)$. □

The next theorem, shows that the condition (17) is also the sufficient condition for a function $f \in \mathcal{H}$ of correlated coefficients to be in the class $\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta; A, B)$.

Theorem 3.2. *Let $f \in \mathcal{T}^{\lambda,\tau}(\eta)$ be a function of the form* (3). *Then $f \in \mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta; A, B)$ if and only if the condition* (17) *holds true.*

Proof. In view of Theorem 3.1 we need only to show that each function $f \in \mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta; A, B)$ satisfies the coefficient inequality (17). If $f \in \mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta; A, B)$, then f is of the form (3) satisfying (13) for which (21) must hold, or equivalently

$$\left| \frac{\sum_{n=2}^{\infty} (n-1)\lambda_n a_n z^n - (n+1)\tau \overline{\lambda_n b_n} \overline{z}^n}{(B-A)z + \sum_{n=2}^{\infty} \left\{ (Bn-A)\lambda_n a_n z^n - (Bn+A)\tau \overline{\lambda_n b_n} \overline{z}^n \right\}} \right| < 1 \quad (z \in \mathbb{U}).$$

Therefore, by (14) and (13) for $z = re^{i\eta}$ $(0 \le r < 1)$, we obtain

$$\frac{\sum_{n=2}^{\infty} (n-1)|\lambda_n||a_n| + (n+1)|\lambda_n||b_n| r^{n-1}}{(B-A) - \sum_{n=2}^{\infty} \left\{ (Bn-A)|\lambda_n||a_n| + (Bn+A)|\lambda_n||b_n| r^{n-1} \right\}} < 1. \tag{22}$$

It is clear that the denominator of the left hand side cannot vanish for $r \in \langle 0, 1)$. Moreover, it is positive for $r = 0$, and in consequence for $r \in \langle 0, 1)$. Thus, by (22) we have

$$\sum_{n=2}^{\infty} \left(|\alpha_n||a_n| + \left|\beta_n\right||b_n|\right) r^{n-1} < B - A \quad (0 \le r < 1). \tag{23}$$

The sequence of partial sums $\{S_n\}$ associated with the series $\sum_{n=2}^{\infty}\left(|\alpha_n|\,|a_n|+\left|\beta_n\right|\,|b_n|\right)$ is non-decreasing sequence. Moreover, by (23) it is bounded by $B-A$. Hence, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=2}^{\infty}\left(|\alpha_n|\,|a_n|+\left|\beta_n\right|\,|b_n|\right)=\lim_{n\to\infty}S_n\le B-A,$$

which yields the assertion (17). □

The following result may be proved in much the same way as Theorem 3.2.

Theorem 3.3. *Let $f\in\mathcal{H}_0$ be a function of the form* (15)*. Then $f\in\mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)$ if and only if*

$$\sum_{n=2}^{\infty}|\lambda_n|\left(|a_n|+|b_n|\right)\le\frac{B-A}{1+B}.$$

By Theorems 3.2 and 3.3 we have the following corollary.

Corollary 3.4. *Let $a=\frac{1+A}{1+B}$ and*

$$\begin{aligned}\phi(z) &= z+\sum_{n=2}^{\infty}\left(\frac{1}{n-a}z^n+\frac{n}{n+a}\overline{z}^n\right)\quad(z\in\mathbb{U}), &(24)\\ \omega(z) &= z+\sum_{n=2}^{\infty}\left((n-a)z^n+(n+a)\overline{z}^n\right)\quad(z\in\mathbb{U}).\end{aligned}$$

Then

$$\begin{aligned}f &\in \mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)\Leftrightarrow f*\phi\in\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B),\\ f &\in \mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)\Leftrightarrow f*\omega\in\mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B).\end{aligned}$$

In particular,

$$\mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(\eta;-1,B)=\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;-1,B).$$

4. Topological properties

We consider the usual topology on $\mathcal{H}$ defined by a metric in which a sequence $\{f_n\}$ in $\mathcal{H}$ converges to f if and only if it converges to f uniformly on each compact subset of $\mathbb{U}$. It follows from the theorems of Weierstrass and Montel that this topological space is complete.

Let $\mathcal{F}$ be a subclass of the class $\mathcal{H}$. A functions $f\in\mathcal{F}$ is called *an extreme point of $\mathcal{F}$* if the condition

$$f=\gamma f_1+(1-\gamma)f_2\quad(f_1,f_2\in\mathcal{F},\ 0<\gamma<1)$$

implies $f_1=f_2=f$. We shall use the notation $E\mathcal{F}$ to denote the set of all extreme points of $\mathcal{F}$. It is clear that $E\mathcal{F}\subset\mathcal{F}$.

We say that $\mathcal{F}$ is *locally uniformly bounded* if for each r, $0<r<1$, there is a real constant $M=M(r)$ so that

$$\left|f(z)\right|\le M\quad(f\in\mathcal{F},\ |z|\le r).$$

We say that a class $\mathcal{F}$ is *convex* if

$$\gamma f+(1-\gamma)g\in\mathcal{F}\qquad(f,g\in\mathcal{F},\ 0\le\gamma\le1).$$

Moreover, we define *the closed convex* hull of $\mathcal{F}$ as the intersection of all closed convex subsets of $\mathcal{H}$ that contain $\mathcal{F}$. We denote the closed convex hull of $\mathcal{F}$ by $\overline{co}\mathcal{F}$.

A real-valued functional $\mathcal{J}:\mathcal{H}\to\mathbb{R}$ is called *convex* on a convex class $\mathcal{F}\subset\mathcal{H}$ if

$$\mathcal{J}(\gamma f+(1-\gamma)g)\le\gamma\mathcal{J}(f)+(1-\gamma)\mathcal{J}(g)\quad(f,g\in\mathcal{F},\ 0\le\gamma\le1).$$

The Krein-Milman theorem (see [19]) is fundamental in the theory of extreme points. In particular, it implies the following lemma.

Lemma 4.1. *[8, pp.45] Let $\mathcal{F}$ be a non-empty compact convex subclass of the class $\mathcal{H}$ and $\mathcal{J}:\mathcal{H}\to\mathbb{R}$ be a real-valued, continuous and convex functional on $\mathcal{F}$. Then*

$$\max\{\mathcal{J}(f):f\in\mathcal{F}\}=\max\{\mathcal{J}(f):f\in E\mathcal{F}\}.$$

Since $\mathcal{H}$ is a complete metric space, Montel's theorem implies the following lemma.

Lemma 4.2. *A class $\mathcal{F}\subset\mathcal{H}$ is compact if and only if $\mathcal{F}$ is closed and locally uniformly bounded.*

Theorem 4.3. *The class $\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)$ is convex and compact subset of $\mathcal{H}$.*

Proof. Let $f_1,f_2\in\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)$ be functions of the form (4), $0\le\gamma\le1$. Since

$$\gamma f_1(z)+(1-\gamma)f_2(z)=z+\sum_{n=2}^{\infty}\left\{(\gamma a_{1,n}+(1-\gamma)a_{2,n})z^n+\overline{(\gamma b_{1,n}+(1-\gamma)b_{2,n})z^n}\right\},$$

and by Theorem 3.2 we have

$$\begin{aligned}&\sum_{n=2}^{\infty}\left\{|\alpha_n|\left|\gamma a_{1,n}+(1-\gamma)a_{2,n}\right|+\left|\beta_n\right|\left|\gamma b_{1,n}+(1-\gamma)b_{2,n}z^n\right|\right\}\\ \le\ &\gamma\sum_{n=2}^{\infty}\left\{\left|\alpha_na_{1,n}\right|+\left|\beta_nb_{1,n}\right|\right\}+(1-\gamma)\sum_{n=2}^{\infty}\left\{\left|\alpha_na_{2,n}\right|+\left|\beta_nb_{2,n}\right|\right\}\\ \le\ &\gamma(B-A)+(1-\gamma)(B-A)=B-A,\end{aligned}$$

the function $\phi=\gamma f_1+(1-\gamma)f_2$ belongs to the class $\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)$. Hence, the class is convex. Furthermore, for $f\in\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)$, $|z|\le r,\ 0<r<1$, we have

$$\left|f(z)\right|\le r+\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n\le r+\sum_{n=2}^{\infty}\left(|\alpha_n||a_n|+\left|\beta_n\right||b_n|\right)\le r+(B-A). \tag{25}$$

Thus, we conclude that the class $\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)$ is locally uniformly bounded. By Lemma 4.2, we only need to show that it is closed *i.e.* if $f_l\in\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)$ $(l\in\mathbb{N})$ and $f_l\to f$, then $f\in\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)$. Let f_l and f be given by (4) and (3), respectively. Using Theorem 3.2 we have

$$\sum_{n=2}^{\infty}\left(\left|\alpha_na_{l,n}\right|+\left|\beta_nb_{l,n}\right|\right)\le B-A\quad(l\in\mathbb{N}). \tag{26}$$

Since $f_l\to f$, we conclude that $\left|a_{l,n}\right|\to|a_n|$ and $\left|b_{l,n}\right|\to|b_n|$ as $l\to\infty$ $(n\in\mathbb{N})$. The sequence of partial sums $\{S_n\}$ associated with the series $\sum_{n=2}^{\infty}\left(|\alpha_na_n|+\left|\beta_nb_n\right|\right)$ is a non-decreasing sequence. Moreover, by (26) it is bounded by $B-A$. Therefore, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=2}^{\infty}\left(|\alpha_na_n|+\left|\beta_nb_n\right|\right)=\lim_{n\to\infty}S_n\le B-A.$$

This gives the condition (17), and, in consequence, $f\in\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)$, which completes the proof. □

Theorem 4.4.

$$E\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)=\{h_n:\ n\in\mathbb{N}\}\cup\{g_n:\ n\in\mathbb{N}_2\},$$

where

$$h_1(z)=z,\ h_n(z)=z-\frac{B-A}{\alpha_n e^{i(n-1)\eta}}z^n,\ g_n(z)=z+\frac{\overline{\tau}(B-A)}{\overline{\beta_n}e^{-i(n+1)\eta}}\overline{z}^n\ \ (z\in\mathbb{U}).\tag{27}$$

Proof. Suppose that $0<\gamma<1$ and

$$g_n=\gamma f_1+(1-\gamma)f_2,$$

where $f_1,f_2\in\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)$ are functions of the form (4). Then, by (17) we have $\left|b_{1,n}\right|=\left|b_{2,n}\right|=\frac{B-A}{\left|\beta_n\right|}$, and, in consequence, $a_{1,l}=a_{2,l}=0$ for $l\in\mathbb{N}_2$ and $b_{1,l}=b_{2,l}=0$ for $l\in\mathbb{N}_2\diagdown\{n\}$. It follows that $g_n=f_1=f_2$, and consequently $g_n\in E\mathcal{S}_{\mathcal{T}}^{*}(\eta;A,B)$. Similarly, we verify that the functions h_n of the form (27) are the extreme points of the class $\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)$. Now, suppose that a function f belongs to the set $E\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)$ and f is not of the form (27). Then there exists $m\in\mathbb{N}_2$ such that

$$0<|a_m|<\frac{B-A}{|\alpha_m|}\ \text{ or }\ 0<|b_m|<\frac{B-A}{\left|\beta_m\right|}.$$

If $0<|a_m|<\frac{B-A}{|\alpha_m|}$, then putting

$$\gamma=\frac{|\alpha_m a_m|}{B-A},\ \varphi=\frac{1}{1-\gamma}(f-\gamma h_m),$$

we have that $0<\gamma<1$, $h_m\neq\varphi$ and

$$f=\gamma h_m+(1-\gamma)\varphi.$$

Thus, $f\notin E\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)$. Similarly, if $0<|b_m|<\frac{B-A}{\left|\beta_n\right|}$, then pouting

$$\gamma=\frac{\left|\beta_m b_m\right|}{B-A},\ \phi=\frac{1}{1-\gamma}(f-\gamma g_m),$$

we have that $0<\gamma<1$, $g_m\neq\phi$ and

$$f=\gamma g_m+(1-\gamma)\phi.$$

It follows that $f\notin E\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)$, and the proof is completed. □

5. Applications

It is clear that if the class

$$\mathcal{F}=\{f_n\in\mathcal{H}:\ n\in\mathbb{N}\},$$

is locally uniformly bounded, then

$$\overline{co}\mathcal{F}=\left\{\sum_{n=1}^{\infty}\gamma_n f_n:\quad \sum_{n=1}^{\infty}\gamma_n=1,\ \gamma_n\geq 0\ \ (n\in\mathbb{N})\right\}.\tag{28}$$

Thus, by Theorem 7 we have the following corollary.

Corollary 5.1.

$$\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)=\left\{\sum_{n=1}^{\infty}(\gamma_n h_n+\delta_n g_n):\ \sum_{n=1}^{\infty}(\gamma_n+\delta_n)=1\ \ (\delta_1=0,\gamma_n,\delta_n\geq 0)\right\},$$

where h_n, g_n *are defined by* (27).

For each fixed value of $m,n,\lambda\in\mathbb{N},\ z\in\mathbb{U}$, the following real-valued functionals are continuous and convex on $\mathcal{H}$:

$$\mathcal{J}(f)=|a_n|,\ \mathcal{J}(f)=|b_n|,\ \mathcal{J}(f)=\left|f(z)\right|\ \mathcal{J}(f)=\left|D_{\mathcal{H}}f(z)\right|\quad (f\in\mathcal{H}). \tag{29}$$

Moreover, for $\gamma\geq 1,\ 0<r<1$, the real-valued functional

$$\mathcal{J}(f)=\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^{\gamma}d\theta\right)^{1/\gamma}\quad (f\in\mathcal{H}) \tag{30}$$

is also continuous and convex on $\mathcal{H}$.

Therefore, by Lemma 4.1 and Theorem 7 we have the following corollaries.

Corollary 5.2. *Let* $f\in\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)$ *be a function of the form* (15). *Then*

$$|a_n|\leq\frac{B-A}{|\alpha_n|},\ |b_n|\leq\frac{B-A}{\left|\beta_n\right|}\quad (n\in\mathbb{N}_2), \tag{31}$$

where α_n,β_n *are defined by* (18). *The result is sharp. The functions* h_n, g_n *of the form* (27) *are the extremal functions.*

Corollary 5.3. *Let* $f\in\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)$, $|z|=r<1$. *Then*

$$\begin{aligned} r-\frac{B-A}{|\lambda_2|(1+2B-A)}r^2 &\leq \left|f(z)\right|\leq r+\frac{B-A}{|\lambda_2|(1+2B-A)}r^2,\\ r-\frac{2(B-A)}{|\lambda_2|(1+2B-A)}r^2 &\leq \left|D_{\mathcal{H}}f(z)\right|\leq r+\frac{2(B-A)}{|\lambda_2|(1+2B-A)}r^2, \end{aligned}$$

where λ_2 *is defined by* (15). *The result is sharp. The function* h_2 *of the form* (27) *is the extremal function.*

Corollary 5.4. *Let* $0<r<1,\ \gamma\geq 1$. *If* $f\in\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)$, *then*

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^{\gamma}d\theta &\leq \frac{1}{2\pi}\int_0^{2\pi}\left|h_2(re^{i\theta})\right|^{\lambda}d\theta,\\ \frac{1}{2\pi}\int_0^{2\pi}\left|D_{\mathcal{H}}f(z)\right|^{\gamma}d\theta &\leq \frac{1}{2\pi}\int_0^{2\pi}\left|D_{\mathcal{H}}h_2(re^{i\theta})\right|^{\gamma}d\theta, \end{aligned}$$

where h_2 *is the function defined by* (27).

The following covering result follows from Corollary 5.3.

Corollary 5.5. *If* $f\in\mathcal{S}_{\mathcal{T}}^{\lambda,\tau}(\eta;A,B)$, *then* $\mathbb{U}(r)\subset f(\mathbb{U})$, *where*

$$r=1-\frac{B-A}{|\lambda_2|(1+2B-A)}.$$

By using Corollary 3.4 and the results above we obtain corollaries listed below.

Corollary 5.6. *The class* $\mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(\eta; A, B)$ *is convex and compact subset of* $\mathcal{H}$. *Moreover,*

$$E\mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(\eta; A, B) = \{h_n : n \in \mathbb{N}\} \cup \{g_n : n \in \mathbb{N}_2\}$$

and

$$\mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(\eta; A, B) = \left\{ \sum_{n=1}^{\infty} (\gamma_n h_n + \delta_n g_n) : (\gamma_n + \delta_n) = 1 \ (\delta_1 = 0, \gamma_n, \delta_n \geq 0) \right\},$$

where $h_1(z) = z$, *and*

$$h_n(z) = z - \frac{(B-A)\, e^{i(1-n)\eta}}{(1+B)\,\lambda_n} z^n, \ g_n(z) = z + \frac{(B-A)\, e^{i(n+1)\eta}}{(1+B)\,\tau\overline{\lambda_n}} \overline{z}^n \ \ (z \in \mathbb{U}). \tag{32}$$

Corollary 5.7. *Let* $f \in \mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(\eta; A, B)$ *be a function of the form* (3). *Then*

$$|a_n| \leq \frac{B-A}{(1+B)\,|\lambda_n|}, \ |b_n| \leq \frac{B-A}{(1+B)\,|\lambda_n|} \quad (n \in \mathbb{N}_2),$$

$$r - \frac{B-A}{(1+B)\,|\lambda_2|} r^2 \leq |f(z)| \leq r + \frac{B-A}{(1+B)\,|\lambda_2|} r^2 \quad (|z| = r < 1),$$

$$r - \frac{2\,(B-A)}{(1+B)\,|\lambda_2|} r^2 \leq |D_{\mathcal{H}} f(z)| \leq r + \frac{2\,(B-A)}{(1+B)\,|\lambda_2|} r^2 \quad (|z| = r < 1),$$

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^{\gamma} d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \left| h_2(re^{i\theta}) \right|^{\lambda} d\theta,$$

$$\frac{1}{2\pi} \int_0^{2\pi} \left| D_{\mathcal{H}} f(re^{i\theta}) \right|^{\gamma} d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \left| D_{\mathcal{H}} h_2(re^{i\theta}) \right|^{\gamma} d\theta,$$

where λ_n *is defined by* (15). *The results are sharp. The functions* h_n, g_n *of the form* (32) *are the extremal functions.*

Corollary 5.8. *Let us assume* (16). *If* $f \in \mathcal{R}_{\mathcal{T}}^{\lambda,\tau}(\eta; A, B)$, *then* $\mathbb{U}(r) \subset f(\mathbb{U})$, *where*

$$r = 1 - \frac{B-A}{(1+B)\,|\lambda_2|}.$$

The classes $\mathcal{S}_{\mathcal{H}}^{n}(A, B)$ and $\mathcal{R}_{\mathcal{H}}^{n}(A, B)$ are related to harmonic starlike functions, harmonic convex functions and harmonic Janowski functions.

The classes $\mathcal{S}_{\mathcal{H}}(\alpha) := \mathcal{S}_{\mathcal{H}}^{0}(2\alpha - 1, 1)$ and $\mathcal{K}_{\mathcal{H}}(\alpha) := \mathcal{S}_{\mathcal{H}}^{1}(2\alpha - 1, 1)$ were investigated by Jahangiri [13] (see also [2, 22]). They are the classes of starlike and convex functions of order α, respectively. The classes $N_{\mathcal{H}}(\alpha) := \mathcal{R}_{\mathcal{H}}^{1}(2\alpha - 1, 1)$ and $R_{\mathcal{H}}(\alpha) := \mathcal{R}_{\mathcal{H}}^{2}(2\alpha - 1, 1)$ were studied in [1] (see also [17]). Finally, the classes $\mathcal{S}_{\mathcal{H}} := \mathcal{S}_{\mathcal{H}}(0)$ and $\mathcal{K}_{\mathcal{H}} := \mathcal{K}_{\mathcal{H}}(0)$ are the classes of functions which are starlike and convex in $\mathbb{U}(r)$, respectively, for all $r \in (0, 1\rangle$. We should notice that the classes $\mathcal{S}(A, B) := \mathcal{S}_{\mathcal{H}}(A, B) \cap \mathcal{A}$ and $\mathcal{R}(A, B) := \mathcal{R}_{\mathcal{H}}(A, B) \cap \mathcal{A}$ were introduced by Janowski [16].

By Theorem 2.3 or Theorem 2.4 for the classes defined above we obtain corollaries listed below (see [7]).

Corollary 5.9.

$$\mathcal{S}_{\mathcal{H}}(\alpha) = \{\psi_{\xi} : |\xi| = 1\}^{*},$$

where

$$\psi_{\xi}(z) := z \frac{2\,(1-\alpha)\,\xi + (1 - \xi + 2\alpha\xi)\, z}{(1-z)^2} - \overline{z} \frac{2 + 2\alpha\xi - (1 - \xi + 2\alpha\xi)\,\overline{z}}{(1-\overline{z})^2} \quad (z \in \mathbb{U}).$$

Corollary 5.10.

$$\mathcal{K}_{\mathcal{H}}(\alpha) = \{\psi_\xi : |\xi| = 1\}^*,$$

where

$$\psi_\xi(z) := z\frac{(1-\alpha)\xi + (1+\alpha\xi)z}{(1-z)^3} + \overline{z}\frac{1+\alpha\xi + (1-\alpha)\xi\overline{z}}{(1-\overline{z})^3} \quad (z \in \mathbb{U}).$$

Corollary 5.11.

$$N_{\mathcal{H}}(\alpha) = \{\delta_\xi : |\xi| = 1\}^*,$$

where

$$\delta_\xi(z) := z\frac{2(1-\alpha)\xi + (2\alpha\xi - \xi + 1)\left(z^2 - 2z\right)}{(1-z)^2} - \overline{z}\frac{1+\xi}{(1-\overline{z})^2} \quad (z \in \mathbb{U}).$$

Corollary 5.12.

$$\mathcal{S}_{\mathcal{H}} = \{\psi_\xi : |\xi| = 1\}^*,$$

where

$$\psi_\xi(z) := z\frac{2\xi + (1-\xi)z}{(1-z)^2} - \overline{z}\frac{2-(1-\xi)\overline{z}}{(1-\overline{z})^2} \quad (z \in \mathbb{U}).$$

Corollary 5.13.

$$\mathcal{K}_{\mathcal{H}} = \{\psi_\xi : |\xi| = 1\}^*,$$

where

$$\psi_\xi(z) := z\frac{\xi + z}{(1-z)^3} + \overline{z}\frac{1+\xi\overline{z}}{(1-\overline{z})^3} \quad (z \in \mathbb{U}).$$

The class $\mathcal{S}^{\lambda}_{\mathcal{H}}(\tau, n; A, B)$ generalizes the classes of starlike functions of complex order. The class $\mathcal{CS}_{\mathcal{H}}(\gamma) := \mathcal{S}_{\mathcal{H}}(1-2\gamma, 1) \quad (\gamma \in \mathbb{C} \setminus \{0\})$ was defined by Yalçin and Öztürk [29]. In particular, if we put $\gamma := \frac{1-\alpha}{1+e^{i\eta}}$, then we obtain the class $\mathcal{RS}_{\mathcal{H}}(\alpha, \eta) := \mathcal{S}_{\mathcal{H}}\left(\frac{2\alpha-1+e^{i\eta}}{1+e^{i\eta}}, 1\right)$ studied by Yalçin *et al.* [30]. It is the class of functions $f \in \mathcal{H}_0$ such that

$$\operatorname{Re}\left\{\left(1+e^{i\eta}\right)\frac{D_{\mathcal{H}} f(z)}{f(z)} - e^{i\eta}\right\} > \alpha \quad (z \in \mathbb{U},\ \eta \in \mathbb{R}).$$

Thus, by Theorem 2.3 we have the following two corollaries (see [7]).

Corollary 5.14.

$$\mathcal{CS}_{\mathcal{H}}(\gamma) = \{\psi_\xi : |\xi| = 1\}^*,$$

where

$$\psi_\xi(z) := z\frac{2\gamma\xi + (1+\xi-2\gamma\xi)z}{(1-z)^2} - \overline{z}\frac{2+2(1-\gamma)\xi - (1+\xi-2\gamma\xi)\overline{z}}{(1-\overline{z})^2} \quad (z \in \mathbb{U}). \tag{33}$$

Corollary 5.15.

$$\mathcal{CS}^{n}_{\mathcal{H}}(\alpha, \eta) = \{\psi_\xi : |\xi| = 1\}^*,$$

where ψ_ξ *is defined by* (33) *with* $\gamma := \frac{1-\alpha}{1+e^{i\eta}}$.

Remark 5.16. *By varying the parameters in the defined classes of functions we can obtain new and also well-known results (see for example [1]-[3], [6]-[17], [22] and [27]-[30]).*

References

[1] O. Ahuja, Connections between various subclasses of planar harmonic mappings involving hypergeometric functions, Appl. Math. Comput. 198 (2008) 305–316.
[2] O.P. Ahuja, J.M. Jahangiri, Certain multipliers of univalent harmonic functions, Appl. Math. Lett. 18 (2005) 1319–1324.
[3] H.A. Al-Kharsani, R.A. Al-Khal, Univalent harmonic functions, JIPAM. J. Inequal. Pure Appl. Math. 8 (2007) Article 59 8 pp.
[4] G. Choquet, Sur un type de transformation analytique généralisant la représentation conforme et définie au moyen de fonctions harmoniques, Bull. Sci. Math. 89 (1945) 156-165.
[5] J. Clunie, T. Sheil Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A I Math. 9 (1984) 3–25.
[6] M. Darus, K. Al-Shaqsi, K. On certain subclass of harmonic univalent functions, J. Anal. Appl. 6 (2008) 17–28.
[7] J. Dziok, Classes of harmonic functions associated with Ruscheweyh derivatives, Publications de'l Institut Mathematique 106 (120) (2019) 1–10..
[8] J. Dziok, On Janowski harmonic functions, J. Appl. Anal. 21 (2015) 99–107.
[9] J. Dziok, J.M. Jahangiri, H. Silverman, Harmonic functions with varying coefficients, J. Inequal. Appl. (2016) 2016:139.
[10] J. Dziok, M. Darus, J. Sokół, T. Bulboaca, Generalizations of starlike harmonic functions, C. R. Math. Acad. Sci. Paris 354 (2016) 13-18.
[11] J. Dziok, H.M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, Integral Transforms Spec. Funct. 14 (2003) 7-18.
[12] A. Ebadian, A. Tehranchi, On certain classes of harmonic p-valent functions by applying the Ruscheweyh derivatives, Filomat 23 (2009) 91–101.
[13] J.M. Jahangiri, Harmonic functions starlike in the unit disk, J. Math. Anal. Appl. 235 (1999) 470–477.
[14] J.M. Jahangiri, G. Murugusundaramoorthy, K. Vijaya, Starlikeness of harmonic functions defined by Ruscheweyh derivatives, J. Indian Acad. Math. 26 (2004) 191–200.
[15] J. M. Jahangiri, H. Silverman, Harmonic univalent functions with varying arguments, Int. J. Appl. Math. 8 (2002) 267–275 .
[16] W. Janowski, Some extremal problems for certain families of analytic functions I, Ann. Polon. Math. 28 (1973) 297–326.
[17] S.Y. Karpuzogullari, M. Öztürk, M. Yamankaradeniz, A subclass of harmonic univalent functions with negative coefficients, Appl. Math. Comput. 142 (2003) 469–476.
[18] H. Kneser, Losung der Aufgabe 41, Jahresber. Deutsch. Math.-Verein. 36 (1926) 123–124.
[19] M. Krein and D. Milman, On the extreme points of regularly convex sets, Studia Mathematica 9 (1940) 133-138.
[20] H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, Bull. Amer. Math. Soc. 42 (1936) 689–692.
[21] G. Murugusundaramoorthy, K. Vijaya, On certain classes of harmonic functions involving Ruscheweyh derivatives, Bull. Calcutta Math. Soc. 96 (2004) 99–108.
[22] M. Öztürk, S. Yalçin, M. Yamankaradeniz, Convex subclass of harmonic starlike functions, Appl. Math. Comput. 154 (2004) 449–459.
[23] T. Rado, Aufgabe 41, Jahresber. Deutsch. Math.-Vermin. 35, 49 (1926).
[24] S. Ruscheweyh, Convolutions in geometric function theory, Sem. Math. Sup. 83, Les Presses del'Université de Montréal, 1982.
[25] S. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math Soc. 49 (1975) 109–115.
[26] T. Sheil-Small, Constants for planar harmonic mappings, J. London Math. Soc. 2 (1990) 237–248.
[27] H. Silverman, Harmonic univalent functions with negative coefficients, J. Math. Anal. Appl. 220 (1998) 283-289.
[28] S. Yalçın, S.B. Joshi, E. Yaşar, On certain subclass of harmonic univalent functions defined by a generalized Ruscheweyh derivatives operator, Appl. Math. Sci. (Ruse) 4 (2010) 327–336.
[29] S. Yalçin, M. Öztürk, Harmonic functions starlike of the complex order, Mat. Vesnik 58 (2006) 7–11.
[30] S. Yalçin, M. Öztürk, M. Yamankaradeniz, On the subclass of Salagean-type harmonic univalent functions, JIPAM. J. Inequal. Pure Appl. Math. 8 (2007) Article 54 9 pp.