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New Extensions of Polygroups by Polygroups

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Abstract. Extensions of polygroups such as direct hyper product and wreath product of polygroups have been introduced and studied earlier by Comer. In this paper, we define and study some other extensions. We first define regularly normal subpolygroups and prove that they induce quotient polygroup extensions. In addition, we prove that the kernel of every strong homomorphism is a regularly normal subpolygroup. This leads to new versions of the isomorphism theorems with respect to such subpolygroups. The main objective of the paper is to present a new extension K of a polygroup L by a polygroup H via the quotient polygroup H/I where I is regularly normal in H. This extension is a generalization of both the direct hyper product and the wreath product of polygroups.

1. Introduction and Preliminaries

The idea of constructing extensions of polygroups via factor polygroups comes from an extension that De Salvo introduced in [13] which is called (H, G)-hypergroups. Basically, given a hypergroup $(H, +)$ and mutually disjoint sets $\{A_i\}_{i\in G}$ where G is a given group such that $A_0 = H$. Set $K = \bigcup_{i\in G} A_i$ and define a hyper operation $\oplus$ on K as follows: For all $x, y \in H$, $x \oplus y = x + y$. For all $x \in A_i$ and $y \in A_j$ such that $A_i \times A_j \neq H \times H$, $x \oplus y = A_k$ where $i + j = k$. This extension of G by H represents a hypergroup. The wreath product $H[G]$ introduced in [2] can be obtained by De Salvo's construction when H and G are polygroups, $A_0 = H$ and $A_i = \{i\}$ for $i \neq 0$.

In our construction, we consider two polygroups H and K. We restrict the cardinalities of sets A_i, $i \neq 0$ to be equal to the cardinality of some factor polygroup H/I and the cardinality of A_0 equals to that of H. The hyper operation on $K = \bigcup_{i\in L} A_i$ is based on the hyper operations on the factor polygroup H/I and the polygroup L. In principle, the element zero of L is enlarged by the polygroup H and the rest of the elements of L are enlarged by isomorphic copies of the factor polygroup H/I.

This construction yields a polygroup in the case when the subpolygroup I is normal. However, the kernel of a strong homomorphism is not necessarily normal, [10]. Therefore, by weakening the condition of normality, we obtain the utmost possible extension. Indeed, we define and study regularly normal polygroups. After introducing the isomorphism theorems subject to these polygroups, we are able to present our new extension via factor polygroups.

Now, we recall some definitions and basic results for the development of our paper.

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A hypergroupoid $(H, +)$ is a non-empty set H together with a hyper operation $+$ defined on H, i.e., a mapping $+ : H \times H \to \wp^*(H)$ where $\wp^*(H)$ is the set of all non-empty subsets of H. The hyper operation $+$ can be extended to subsets of H in a normal way, so that $A + B$ is defined by $A + B = \bigcup \{a + b : a \in A, b \in B\}$. In particular, for $x \in H$, we have $x + A = \{x\} + A$.

A hypergroupoid $(H, +)$ is called a hypergroup if for all $a, b, c \in H$, we have $a + H = H + a = H$ and $a + (b + c) = (a + b) + c$. The last equality means that

$$\bigcup_{u \in b+c} (a + u) = \bigcup_{v \in a+b} (v + c).$$

A non empty subset K of a hypergroup $(H, +)$ is called a subhypergroup if it is a hypergroup under the hyper operation of H.

For a hypergroup $(H, +)$ and an integer $n > 1$, consider the relation β_n defined on H as $a\beta_n b \iff \exists\,(x_1, x_2 \ldots.., x_n) \in H^n$ such that $\{a, b\} \subseteq \sum_{i=1}^{n} x_n$. Moreover, let $\beta_1 = \{(x, x) : x \in H\}$ and $\beta = \bigcup_{n \geq 1} \beta_n$. Then β is an equivalence relation on H and the set of equivalence classes of β on H is denoted by H/β. If we define a hyper operation $\oplus$ on H/β as $\bar{x} \oplus \bar{y} = \{\bar{z} : z \in x + y\}$, then $(H/\beta, \oplus)$ is a group,[11]. In fact, β is the smallest equivalence relation on H such that H/β is a group. The kernel $\{x \in H : \Psi_H(x) = \bar{0}\}$ of the canonical projection $\Psi_H : H \longrightarrow H/\beta$ is called the heart of the hypergroup H and is denoted by ω_H.

Definition 1.1. *[12]Let A be a non empty subset of the hypergroup H. Then A is called a complete part of H if for any $n \in N$ and for all $a_1, a_2, ...a_n \in H$, the following implication holds: $A \cap \sum_{i=1}^{n} a_i \neq \emptyset \Longrightarrow \sum_{i=1}^{n} a_i \subseteq A$.*

If A is a subset of a hypergroup H, then the smallest complete part of H containing A is called the complete closure of A and is denoted by $C(A)$. The class of all complete parts subhypergroups of H is denoted by $CPS(H)$. In the following proposition, we show how the heart of a hypergroup H is related to complete parts of H.

Proposition 1.2. *[12]Let $(H, +)$ be a hypergroup and A be a non-empty subset of H. Then*

1. $\omega_H + A = A + \omega_H = C(A)$.
2. ω_H is a complete part of H.
3. $\omega_H = \bigcap_{K \in CPS(H)} K$.

Definition 1.3. *[11]A hypergroup H is called a polygroup if the following conditions are satisfied.*

1. There exists an element $0 \in P$ such that $0 + x = x + 0 = x$ for all $x \in P$.
2. Every element $x \in H$ has an inverse $-x \in H$ which means that $0 \in x + (-x) \cap -x + x$.
3. For $x, y, z \in H$, we have $z \in x + y$ implies $x \in z + (-y)$ and $y \in (-x) + z$.

A non empty subset K of a polygroup $(P, +)$ is called a subpolygroup of P if for all $a, b \in K$, $a + b \subseteq K$ and $-a \in K$.

Definition 1.4. *[12]A subpolygroup N of a polygroup P is called normal in P if $a + N - a \subseteq N$ for all $a \in P$.*

If N is a normal subpolygroup of a polygroup P, then we define on P a relation N_P where xN_Py if and only if $(x - y) \cap N \neq \emptyset$ for $x, y \in P$.

Proposition 1.5. *[12]Let N be a normal subpolygroup of a polygroup P. Then*

1. The relation N_P is an equivalence relation on P.
2. $N_P(x) = N + x$ for all $x \in P$.
3. For all $x, y \in P$, $N + (x + y) = N + z$ for all $z \in x + y$.
4. $P/N = \{N + x : x \in P\}$ is a polygroup under the hyper operation $\oplus$ defined as $(N + x) \oplus (N + y) = \{N + z : z \in x + y\}$.

Definition 1.6. *[12]Let $(P_1,+)$ and $(P_2,\oplus)$ be polygroups with identities e_1 and e_2 respectively. Let $\varphi : P_1 \to P_2$ be a mapping such that $\varphi(e_1) = e_2$. Then φ is called a strong homomorphism if $\varphi(x+y) = \varphi(x) \oplus \varphi(y)$ for all $x, y \in P_1$.*

The kernel of a strong homomorphism $\varphi : P_1 \to P_2$ is defined as $Ker(\varphi) = \{x \in P_1 : \varphi(x) = e_2\}$ which is a subpolygroup of P_1 but not normal in general. In [4], Comer determined that the heart of a polygroup P is a subpolygroup of P generated by all $x - x$ where $x \in P$.

In the case of normal subpolygroups, Davvas, [10] presented the following isomorphism theorems.

Theorem 1.7. *[10](First isomorphism theorem) Let P_1 and P_2 be polygroups and let $\varphi : P_1 \to P_2$ be a strong homomorphism with kernel K such that K is a normal in P_1. Then $P_1/K \cong lm\varphi$.*

Theorem 1.8. *[10](Second isomorphism theorem) If K and N are subpolygroups of a polygroup P where N is normal in P, then $K/(N \cap K) \cong (N+K)/N$.*

Theorem 1.9. *[10](Third isomorphism theorem) If K and N are normal subpolygroups of a polygroup P such that $N \subseteq K$, then K/N is a normal subpolygroup of P/N and $(P/N)/(K/N) \cong P/K$.*

Definition 1.10. *[12]Let $(P,+)$ be a polygroup and Ω be a non empty set. A map $f : \Omega \times P \to \wp^*(\Omega)$ is called an action of P on Ω if the following axioms hold.*

1. $f(\omega, e) = \{\omega\} = \omega$ for all $\omega \in \Omega$.
2. $f(f(\omega, g), h) = \bigcup_{\alpha \in g+h} f(\omega, \alpha)$ for all $g, h \in P$ and $\omega \in \Omega$.
3. $\bigcup_{\omega \in \Omega} f(\omega, g) = \Omega$ for all $g \in P$.
4. $\forall g \in P$, we have $\alpha \in f(\beta, g) \Longrightarrow \beta \in f(\alpha, -g)$.

For $\omega \in \Omega$, we write $\omega^g = f(\omega, g)$. The kernel of the action is defined as $K = \{g \in P : \omega^g = \{\omega\} \ \textit{for all}\ \omega \in \Omega\}$.

Extensions of polygroups by polygroups in a notion of product can be done as the direct hyper product of two polygroups and as the wreath product introduced in [2]. The direct hyper product of two polygroups $(P_1,+)$ and $(P_2,\oplus)$ is $(P_1 \times P_2, \otimes)$ where for $(x_1,y_1),(x_2,y_2) \in P_1 \times P_2$, $(x_1,y_1) \otimes (x_2,y_2) = \{(x,y) : x \in x_1 + x_2,\ y \in y_1 \oplus y_2\}$.

Suppose that $(A,+)$ and $(B,+)$ are two polygroups whose elements have been renamed so that $A \cap B = \{e\}$. An extension $(M,\oplus)$ of A by B (denoted by $A[B]$) is formed in the following way: We set $M = A \cup B$ and let $e \oplus x = x \oplus e = x$ for all $x \in M$ and for all $x, y \in M \backslash \{e\}$,

$$x \oplus y = \begin{cases} x + y & if\ x, y \in A \\ x & if\ x \in B, y \in A \\ y & if\ x \in A, y \in B \\ x + y & if\ x, y \in B, y \neq -x \\ (x+y) \cup A & if\ x, y \in B, y = -x \end{cases}$$

In this case, $A[B]$ is a polygroup which is called the wreath product of A by B, [2].

2. Regularly normal subpolygroups and isomorphism theorems

In this section, we define regularly normal subpolygroups by weakening the condition of normality. We prove that kernel of a polygroup strong homomorphism is always regularly normal and then give new versions of isomorphism theorems for such subpolygroups.

In [9], Davas has proved that if I is a normal subpolygroup of a polygroup H, then I induces an equivalence relation φ on H defined as $x\varphi y$ if and only if $(x-y) \cap I \neq \emptyset$. In the following lemma, we prove that this is in fact true for any subpolygroup of H.

Lemma 2.1. *Let $(H,+)$ be a polygroup and I be a subpolygroup of H. Then I induces an equivalence relation ρ on H defined as $x\rho y$ if and only if $(x-y) \cap I \neq \emptyset$.*

Proof. Since $0 \in x - x$ for all $x \in H$ and $0 \in I$, then $(x - x) \cap I \neq \emptyset$. So, $x\rho x$ and ρ is reflexive on H. Let $x, y \in H$ with $x\rho y$ and let $a \in (x - y) \cap I$. Then $a \in x - y$ and $a \in I$ which imply that $-a \in -(x - y)$ and $-a \in I$. Hence, $-a \in (y - x) \cap I$ and $y\rho x$. Thus, ρ is symmetric on H. Finally, we check the transitivity of ρ on H. Let $x, y, z \in H$ such that $x\rho y$ and $y\rho z$. Then $(x - y) \cap I \neq \emptyset$ and $(y - z) \cap I \neq \emptyset$. Let $a \in (x - y) \cap I$ and $b \in (y - z) \cap I$. Then $a \in x - y$, $a \in I$, $b \in y - z$ and $b \in I$. By the reversibility in H, we have $x \in a + y$ and $y \in b + z$. Thus, $x \in a + y \subseteq a + (b + z) = (a + b) + z$ and $a + b \subseteq I$. So, there is $u \in a + b$ such that $x \in u + z$ and then $u \in x - z$ and $u \in I$. It follows that $(x - z) \cap I \neq \emptyset$ and $x\rho z$. □

If I is a subpolygroup of a polygroup H, then $I + x = \rho(x)$ for all $x \in H$. Indeed, for $y \in I + x$, there exists $i \in I$ such that $y \in i + x$ which implies that $i \in y - x$. So, $(y - x) \cap I \neq \emptyset$ and $y\rho x$. Thus $y \in \rho(x)$ and $I + x \subseteq \rho(x)$. Similarly, we can justify the other inclusion.

Definition 2.2. *Let H be a polygroup and I be a subpolygroup of H. Then I is called regularly normal in H if $x + I = I + x$ for all $x \in H$.*

If a subpolygroup I is normal in a polygroup H, then it is regularly normal. Indeed, suppose $x + I - x \subseteq I$ for all $x \in H$. Then $x + I - x + x \subseteq I + x$ and so $x + I \subseteq I + x$ as $-x + x \subseteq I$ since I is normal. Similarly, we can verify the other inclusion. However, the converse is not always true as we demonstrate in the following example.

Example 2.3. *Let $H = \{0, 1, 2, 3, a, b\}$ with a hyperoperation $+$ defined as follow.*

+	0	1	2	3	a	b
0	0	1	2	3	a	b
1	1	1	0123	3	ab	b
2	2	012	2	23	a	ab
3	3	13	3	0123	ab	ab
a	a	a	ab	ab	$0123ab$	$23ab$
b	b	ab	b	ab	$13ab$	$0123ab$

Then $I = \{0, 1, 2, 3\}$ is a subpolygroup of H. It is clear that $I + a = \{a, b\} = a + I$ and $I + b = \{a, b\} = b + I$. But $a + I - a = \{a, b\} - a = \{0, 1, 2, 3, a, b\}$ and so I is not normal in H.

Proposition 2.4. *Let $(H, +)$ be a polygroup and I be a regularly normal subpolygroup of H. Then $H/I = \{I+x : x \in H\}$ is a polygroup under the hyperoperation $\oplus$ defined as $(I + x) \oplus (I + y) = \{I + t : t \in x + y\}$.*

Proof. First, we check that the hyper operation is well defined. Let $I + a = I + x$, $I + b = I + y$ and $I + t \in (I + a) \oplus (I + b)$. Then $t \in a + b \subseteq I + a + b = I + I + a + b = I + a + I + b = I + x + I + y = I + x + y$. Hence, $t \in I + u$ for some $u \in x + y$ and so $I + t \subseteq I + u \subseteq (I + x) \oplus (I + y)$. Thus, $(I + a) \oplus (I + b) \subseteq (I + x) \oplus (I + y)$. The other inclusion can be proved similarly. For the associativity, we let $I + x, I + y, I + z \in H/I$. Then

$$\begin{aligned}(I + x) \oplus [(I + y) \oplus (I + z)] &= (I + x) \oplus \{I + t \ : t \in y + z\} = \{I + h : h \in x + (y + z)\} \\ &= \{I + h : h \in (x + y) + z\} = [(I + x) \oplus (I + y)] \oplus (I + z).\end{aligned}$$

The identity element in H/I is $I + 0 = I$ where 0 is the identity element of H. Indeed, for every $I + a \in H/I$, $(I + 0) \oplus (I \oplus a) = \{I + t | t \in 0 + a\} = I + a = (I + a) \oplus (I + 0)$. Next, for $I + x \in H/I$, there exists a unique $-x \in H$ such that $0 \in x - x$. Hence, $I + 0 \subseteq \{I + t : t \in x - x\} = (I + x) \oplus (I + (-x))$. Therefore, $I + (-x)$ is the inverse of $I + x$. Finally, for $x, y, z \in H$, let $(I + x) \in (I + y) \oplus (I + z)$. Then $x \in y + z$ and so $y \in x - z$ and $z \in -y + x$. Thus, $I + y \in ((I + x) \oplus (I + (-z))$ and $I + z \in (I + (-y)) \oplus (I + x)$. □

Lemma 2.5. *Let I and J be subpolygroups of a polygroup H with I regularly normal in H. Then*

1. $I + J = J + I$ is a subpolygroup of H.
2. I is regularly normal in $I + J$.

Proof. (1) Since I is regularly normal then, $I+J=\{I+x: x\in J\}=\{x+I: x\in J\}=J+I$. Let $a,b\in I+J$. Then there exist $i_1,i_2\in I$ and $j_1.j_2\in J$ such that $a\in i_1+j_1$ and $b\in i_2+j_2$. Hence, $a+b\in(i_1+j_1)+(i_2+j_2)\subseteq i_1+J+I+j_2=i_1+I+J+j_2=I+J$ since I is regularly normal. Moreover, $-a\in(-j)+(-i)\subseteq J+I=I+J$.

(2) It is clear since $I\subseteq I+J$ and I is regularly normal in H. □

Proposition 2.6. *Let H_1 and H_2 be polygroups, $f: H_1\to H_2$ be a strong homomorphism and $K=ker\ (f)$. Then K is a regularly normal subpolygroup of H_1.*

Proof. It is easy to see that K is a subpolygroup of H_1. Let $x\in a+K$ for some $a\in H_1$. Then $f(x)=f(a)$ and so $f(x)-f(a)=f(a)-f(a)\supseteq e_2=f(e_1)$. Thus, $f(x-a)\supseteq f(e_1)$ and $(x-a)\cap K\neq\varnothing$. It follows that $x\rho a$ and $x\in K+a$. by Lemma 2.1. Conversely in the same manner, let $x\in K+a$ for some $a\in H_1$. Then there exists $k\in K$ such that $x\in k+a$. Since f is a strong homomorphism and $k\in K$, then $f(x)=f(a)$. By the uniqueness of inverses in polygroups, $-f(x)=-f(a)$ and so $-f(x)=f(-a)$. Thus, $e_2\in -f(x)+f(x)=f(-a)+f(x)=f(-a+x)$. Hence, there exist $t\in -a+x$ such that$f(t)=e_2$ and so $t\in K$. By reversibility, $x\in a+t\in a+K$. Therefore, $x+K=K+x$ and K is regularly normal. □

Now, by considering regularly normal subpolygroups, we present the following new versions of isomorphism theorems.

Theorem 2.7. *(First Isomorphism Theorem). Let $\varphi: H_1\to H_2$ be a polygroup strong homomorphism with kernel K. Then $H_1/K\cong Im\varphi$.*

Proof. By Proposition 2.6, K is a regularly normal subpolygroup. Define a mapping $\Psi: H_1/K\to Im\varphi$ by $\Psi(K+x)=\varphi(x)$. For $K+x=K+y$, we have $\rho(x)=\rho(y)$ and so $(x-y)\cap K\neq\varnothing$. Hence, clearly $\varphi(x)=\varphi(y)$ and $\Psi(K+x)=\Psi(K+y)$. Thus, Ψ is well defined. Now, for $K+x, K+y\in H_1/K$,

$$\begin{aligned}\Psi[(K+x)\oplus(K+y)] &= \Psi[\{K+t: t\in x+y\}]=\{\varphi(t): t\in x+y\}\\ &= \varphi(x+y)=\varphi(x)+\varphi(y)=\Psi(K+x)+\Psi(K+y).\end{aligned}$$

Hence, Ψ is a strong homomorphism. Let $\Psi(K+x)=\Psi(K+y)$ for $x,y\in H_1$. Then $\varphi(x)=\varphi(y)$ and $0\in\varphi(y)-\varphi(x)=\varphi(y-x)$. Thus, $\varphi(z)=0$ for some $z\in y-x$ and $z\in K$. By reversibility, $y\in z+x\subseteq K+x$. Thus, $K+y\subseteq K+x$. Similarly, $K+x\subseteq K+y$ and Ψ is one to one. Clearly,Ψ is onto and so Ψ is a strong isomorphism. □

Theorem 2.8. *(Second Isomorphism Theorem). If I and J are subpolygroups of a polygroup H where I is regularly normal in H, then $J/(I\cap J)\cong(I+J)/I$.*

Proof. Since I is regularly normal in H and by Lemma 2.5, $I+J$ is a subpolygroup of H. Now, $I\subseteq I+J$ is a regularly normal subpolygroup of $I+J$, so we can consider the polygroup $(I+J)/I$. Define $\Psi: J\to(I+J)/I$ by $\Psi(j)=I+j$ and let $x,y\in J$. Then

$$\Psi(x+y)=\Psi[\{t: t\in x+y\}]=\{I+t: t\in x+y\}=(I+x)\oplus(I+y)=\Psi(x)\oplus\Psi(y)$$

and so Ψ is a strong homomorphism. Let $I+a\in(I+J)/I$ where $a\in I+J$. Then $a\in i+j$ for some $i\in I$ and $j\in J$ and so $j\in -i+a\subseteq I+a$. Thus, $I+j=I+a$ and $\Psi(j)=I+j=I+a$. Hence, Ψ is onto. Moreover, $Ker\ (\Psi)=\{j\in J:\Psi(j)=I\}=\{j\in J: I+j=I\}=I\cap J$. It follows by Theorem 2.7 that $J/(I\cap J)\cong(I+J)/I$. □

Theorem 2.9. *(Third Isomorphism Theorem). Let I and J be regularly normal subpolygroups of a polygroup H such that $I\subseteq J$. Then J/I is a regularly normal subpolygroup of H/I and $(H/I)/(J/I)\cong H/J$.*

Proof. Since I,J are regularly normal in H and $I\subseteq J$, then I is regularly normal in J. Let $\Psi: H/I\to H/J$ be defined by $\Psi(I+a)=J+a$. Then Ψ is well defined since if $I+a=I+b$, then a and b are in the same equivalence class of the relation ρ in Lemma 2.1. Hence, $I\cap(a-b)\neq\varnothing$ and since $I\subseteq J$, then $J\cap(a-b)\neq\varnothing$. Therefore, $J+a=J+b$ and $\Psi(I+a)=\Psi(I+b)$. Now, for $I+a, I+b\in H/I$, we have

$$\begin{aligned}\Psi((I+a)+(I+b)) &= \Psi(\{I+k: k\in a+b\})=\{J+k: k\in a+b\}\\ &= (J+a)+(J+b)=\Psi(I+a)+\Psi(I+b).\end{aligned}$$

So, Ψ is a strong homomorphism with $Ker\ (\Psi) = \{I + a : \Psi(I + a) = J\} = \{I + a : J + a = J\} = \{I + a : a \in J\} = J/I$. By Proposition 2.6, J/I is regularly normal in H/I. Since clearly Ψ is onto, then $(H/I)/(J/I) \cong H/J$ by Theorem 2.7. $\square$

If a subpolygroup I of a polygroup H is normal, then $x + I - x \subseteq I$ for all $x \in H$. Hence, we have $x - x \subseteq I$ and so $\omega_H = \langle x - x \rangle \subseteq I$. Since $\omega_H + I = I$ and by Proposition 1.2, we see that I is a complete part in H and so absorbs any non-empty intersection with hyper sums in H. Hence, the equivalence relation ρ of Lemma 2.1 takes the form $x\rho y \Leftrightarrow x - y \subseteq I$. Therefore, when a subpolygroup I is normal, then it induces a stronger equivalence relation on H.

Proposition 2.10. *Let I be a regularly normal subpolygroup of a polygroup H. Then I is normal in H if and only if $(H/I, \oplus)$ is a group where for $I + x, I + y \in H/I$, $(I + x) \oplus (I + y) = \{I + t : t \in x + y\}$.*

Proof. Since I is regularly normal in H, then $(H/I, \oplus)$ is a polygroup by Proposition 2.4. Let I be a normal subpolygroup of H. Then I is a complete part in H and the equivalence relation ρ takes the form $x\rho y \Leftrightarrow x - y \subseteq I$ for all $x, y \in H$. Let $a, b \in (I+x)\oplus(I+y)$. Since $(I+x)+(I+y) = I+I+x+y = \{I+z : z \in x+y\} = (I+x)\oplus(I+y)$, then there exist $u_1, u_2 \in I + x$ and $v_1, v_2 \in I + y$ such that $a \in u_1 + v_1$ and $b \in u_2 + v_2$. Now, $u_1\rho u_2$ and $v_1\rho v_2$ and so $u_1 - u_2 \subseteq I$ and $v_1 - v_2 \subseteq I$. Thus, $a - b \subseteq (u_1 + v_1) - (u_2 + v_2) = u_1 + v_1 - v_2 - u_2 \subseteq u_1 + I - u_2 \subseteq I$ and then $a\rho b$. This implies that any two elements from $(I + x) \oplus (I + y)$ are in the same equivalence class. In conclusion, $\left|(I + x) \oplus (I + y)\right| = 1$ and H/I is a group. Conversely, suppose $(H/I, \oplus)$ is a group. Define a mapping $f : H \to H/I$ by $f(x) = I + x$. Then clearly, f is a strong homomorphism with $Ker(f) = I$. For $x \in H$, we have $f(x + I - x) = f(x) - f(x) = I$ since I is the zero of H/I. The last equality holds since $(H/I, \oplus)$ is a group, hence $x + I - x \subseteq I$ and I is normal in H. $\square$

3. Construction of The Extension

Let $(H, +)$ and $(L, +)$ be two polygroups and I be a regularly normal subpolygroup of H. Let $\{A_i\}_{i\in L}$ be a family of pair wise disjoint sets such that $A_0 = H$ and for all $i \neq 0$, $|A_i| = |H/I|$. For all $0 \neq i \in L$, we may write $A_i = \left\{a_i^{\overline{h}} : \overline{h} \in H/I\right\}$ and with no loss of generality we have $a_0^{\overline{0}} = I$ and $a_0^{\overline{h}} = \overline{h}$. For all $i \in L$, let $f_i : H \to A_i$ be a mapping such that f_0 is the identity mapping and for $i \neq 0$, $f_i(x) = a_i^{\overline{h}}$ for $x \in \overline{h}$.
Let $K = \bigcup\limits_{i\in L} A_i$ and define a hyper operation $\oplus$ on K as follows: For $k_1 \in A_i \subseteq K$ and $k_2 \in A_j \subseteq K$,

$$k_1 \oplus k_2 = \bigcup_{x\in i+j} f_x(f_i^{-1}(k_1) + f_j^{-1}(k_2)).$$

As special cases, we have:

1. For $k_1, k_2 \in A_0 = H$,

$$k_1 \oplus k_2 = \bigcup_{x\in 0+0} f_x(f_0^{-1}(k_1) + f_0^{-1}(k_2)) = k_1 + k_2.$$

2. For $i,j \neq 0$, $k_1 = a_i^{\overline{h}} \in A_i$ and $k_2 = a_j^{\overline{k}} \in A_j$,

$$\begin{aligned} k_1 \oplus k_2 &= \bigcup_{x\in i+j} f_x(f_i^{-1}(a_i^{\overline{h}}) + f_j^{-1}(a_j^{\overline{k}})) = \bigcup_{x\in i+j} f_x(\overline{h} + \overline{k}) \\ &= \bigcup_{x\in i+j} f_x(\{\overline{t} : t \in h + k\}) = \left\{a_p^{\overline{q}} : p \in i + j, \overline{q} \in \overline{h} + \overline{k}\right\}. \end{aligned}$$

3. For $k_1 \in A_0 = H$ and $k_2 = a_i^{\overline{h}} \in A_i$, $i \neq 0$,

$$\begin{aligned} k_1 \oplus k_2 &= \bigcup_{x\in 0+i} f_x(f_0^{-1}(k_1) + f_i^{-1}(a_i^{\overline{h}})) = \bigcup_{x\in i} f_x(k_1 + \overline{h}) = f_i(\overline{k_1} + \overline{h}) \\ &= f_i(\{\overline{t} : t \in k_1 + h\}) = \{a_i^{\overline{q}} : \overline{q} \in \overline{k_1} + \overline{h}\}. \end{aligned}$$

Proposition 3.1. *Let H, L, I, K and $\oplus$ be defined as above. Then $(K, \oplus)$ is a polygroup.*

Proof. First, we check the associativity. For $i, j, k \neq 0$, let $a_i^{\overline{n}} \in A_i$, $a_j^{\overline{m}} \in A_j$ and $a_k^{\overline{h}} \in A_k$. Then

$$\begin{aligned} a_i^{\overline{n}} \oplus \left(a_j^{\overline{m}} \oplus a_k^{\overline{h}}\right) &= a_i^{\overline{n}} \oplus \left\{a_x^{\overline{y}} : x \in j+k, \overline{y} \in \overline{m} + \overline{h}\right\} = \left\{a_p^{\overline{q}} : p \in i + (j+k), \overline{q} \in \overline{n} + (\overline{m} + \overline{h})\right\} \\ &= \left\{a_p^{\overline{q}} : p \in (i+j)+k, \overline{q} \in (\overline{n} + \overline{m}) + \overline{h}\right\} = (a_i^{\overline{n}} \oplus a_j^{\overline{m}}) \oplus a_k^{\overline{h}}. \end{aligned}$$

For $h \in H$, we have,

$$\begin{aligned} h \oplus (a_i^{\overline{n}} \oplus a_j^{\overline{m}}) &= h \oplus \left\{a_z^{\overline{y}} : z \in i+j, \overline{y} \in \overline{n} + \overline{m}\right\} = \left\{a_p^{\overline{q}} : p \in i+j, \overline{q} \in \overline{h} + (\overline{n} + \overline{m})\right\} \\ &= \left\{a_p^{\overline{q}} : p \in i+j, \overline{q} \in (\overline{h} + \overline{n}) + \overline{m}\right\} = (h \oplus a_i^{\overline{n}}) \oplus a_j^{\overline{m}}. \end{aligned}$$

For $h, k \in H$, we have,

$$\begin{aligned} h \oplus (k \oplus a_i^{\overline{n}}) &= h \oplus \left\{a_i^{\overline{z}} : \overline{z} \in \overline{k} + \overline{n}\right\} = \left\{a_i^{\overline{q}} : \overline{q} \in \overline{h} + (\overline{k} + \overline{n})\right\} \\ &= \left\{a_i^{\overline{q}} : \overline{q} \in (\overline{h} + \overline{k}) + \overline{n}\right\} = (h \oplus k) \oplus a_i^{\overline{n}}. \end{aligned}$$

The identity element 0 of H is also the identity element of $(K, \oplus)$. Indeed, for $a_i^{\overline{n}} \in K$, we have $0 \oplus a_i^{\overline{n}} = \left\{a_i^{\overline{q}} : \overline{q} \in 0 + \overline{n}\right\} = a_i^{\overline{n}} = a_i^{\overline{n}} \oplus 0$. Now, let $a_i^{\overline{k}} \in A_i$ and consider the elements $-i \in L$ and $-k \in H$ such that $0 \in i - i$ and $0 \in k - k$. Then $a_{-i}^{\overline{-k}} \in A_{-i}$ and $0 \in I = a_0^{\overline{0}} \subseteq \left\{a_p^{\overline{q}} : p \in i - i, \overline{q} \in \overline{k} - \overline{k}\right\} = a_i^{\overline{k}} \oplus a_{-i}^{\overline{-k}}$ and so $-a_i^{\overline{k}} = a_{-i}^{\overline{-k}}$. Finally, for $a_i^{\overline{n}}, a_j^{\overline{m}}, a_k^{\overline{h}} \in K$, let $a_i^{\overline{n}} \in a_j^{\overline{m}} \oplus a_k^{\overline{h}} = \left\{a_x^{\overline{y}} : x \in j+k, \overline{y} \in \overline{m} + \overline{h}\right\}$. Then $i \in j + k$ and $\overline{n} \in \overline{m} + \overline{h}$. By the reversibility in the polygroups L and H/I, we have $j \in i - k$, $k \in -j + i$, $\overline{m} \in \overline{n} - \overline{h}$ and $\overline{h} \in -\overline{m} + \overline{n}$. Hence, $a_j^{\overline{m}} \in \left\{a_p^{\overline{q}} : p \in i - k, \overline{q} \in \overline{n} - \overline{h}\right\} = a_i^{\overline{n}} \oplus a_{-k}^{\overline{-h}} = a_i^{\overline{n}} \oplus (-a_k^{\overline{h}})$ and $a_k^{\overline{h}} \in \left\{a_p^{\overline{q}} : p \in -j + i, \overline{q} \in -\overline{m} + \overline{n}\right\} = a_{-j}^{\overline{-m}} \oplus a_i^{\overline{n}} = -a_j^{\overline{m}} \oplus a_i^{\overline{n}}$. If $h \in H$ with $h \in a_j^{\overline{m}} \oplus a_i^{\overline{n}} = \left\{a_p^{\overline{q}} : p \in j + i, \overline{q} \in \overline{m} + \overline{n}\right\}$, then $i = -j$ since $h \in \overline{h} = a_0^{\overline{h}}$. Hence, $\overline{h} \in \overline{m} + \overline{n}$ and so $\overline{m} \in \overline{h} - \overline{n}$. Now, $h \oplus (-a_i^{\overline{n}}) = h \oplus a_{-i}^{\overline{-n}} = \left\{a_{-i}^{\overline{q}} : \overline{q} \in \overline{x} - \overline{n}\right\} \supseteq a_{-i}^{\overline{m}} = a_j^{\overline{m}}$. Similarly, we can see that $a_i^{\overline{n}} \in -a_j^{\overline{m}} + h$. If $a_i^{\overline{m}} \in h \oplus a_i^{\overline{k}}$ for $h \in \overline{h} = a_0^{\overline{h}}$, then we have $a_i^{\overline{m}} \in \left\{a_i^{\overline{q}} : \overline{q} \in \overline{h} + \overline{k}\right\}$ and so $\overline{m} \in \overline{h} + \overline{k}$. Thus, $\overline{h} \in \overline{m} - \overline{k}$ and $h \in \overline{h} = a_0^{\overline{h}} \subseteq \left\{a_p^{\overline{q}} : p \in i - i, \overline{q} \in \overline{m} - \overline{k}\right\} = a_i^{\overline{m}} \oplus a_{-i}^{\overline{-k}} = a_i^{\overline{m}} \oplus (-a_i^{\overline{k}})$. □

Proposition 3.2. *Let H, L and K be defined as in Proposition 3.1. Then $K/H \simeq L$.*

Proof. Let $\pi : K \to L$ be defined as $\pi(x) = \left\{ \begin{array}{ll} 0 & if\ x \in H \\ i & if\ x = a_i^{\overline{k}},\ i \neq 0 \end{array} \right\}$. Then π is a strong homomorphism. Indeed, if $a_i^{\overline{h}}, a_j^{\overline{k}} \in K$ with $i, j, i + j \neq 0$, then $\pi\left(a_i^{\overline{h}} \oplus a_j^{\overline{k}}\right) = \pi\left\{a_p^{\overline{q}} : p \in i + j, \overline{q} \in \overline{h} + \overline{k}\right\} = i + j = \pi(a_i^{\overline{h}}) + \pi(a_i^{\overline{k}})$. Similarly, we can check that π is a strong homomorphism in all other cases for any pair of elements in K. Moreover, clearly π is surjective and $Ker(\pi) = \{x \in K : \pi(x) = 0\} = H = Im(f)$. By Theorem 2.7, we conclude that $K/H \simeq L$. □

Remark 3.3. *We will denote the above extension K of L via H/I by $L \times_{H/I} H$.*

In the following two corollaries, we justify that the new extension $K = L \times_{H/I} H$ is in fact a generalization of both the direct hyper product $L \times H$ and the wreath product $H[L]$.

Corollary 3.4. *Let H and L be polygroups and let $I = \{0\} \subseteq H$. If $K = L \times_{H/I} H$, then $(K, \oplus)$ is isomorphic to the direct hyper product $L \times H$ of L and H.*

Proof. For $I = \{0\}$, the factor polygroup $H/I \simeq H$ which implies that $|A_i| = |H|$ for all i. With no loss of generality, we can rename the elements of H so that each $h \in H$ can be written as a_0^h and the elements of A_i as a_i^h, $h \in H$. For all $i \in L$, the mapping $f_i : H \to A_i$ is defined by $f_i\left(a_0^h\right) = a_i^h$, $h \in H$. Now, $K = \bigcup_{i\in L} A_i = \left\{a_p^q : p \in L, q \in H\right\}$. Let $\varphi : K \to L \times H$ be defined by $\varphi(a_p^q) = (p, q)$. Then it is straightforward to prove that φ is a bijection. Moreover,

$$\begin{aligned}\varphi\left(a_j^k \oplus a_i^h\right) &= \varphi\left\{a_p^q : p \in j + i, q \in k + h\right\} = \{(p, q) : p \in j + i, q \in k + h\} \\ &= (j, k) + (i, h) = \varphi(a_j^k) + \varphi(a_i^h).\end{aligned}$$

So, φ is a strong homomorphism and $K = L \times_{H/I} H$ is isomorphic to $L \times H$. □

Corollary 3.5. *Let H and L be polygroups and let $I = H$. If $K = L \times_{H/I} H$, then $(K, \oplus)$ is isomorphic to the wreath product $H[L]$.*

Proof. For $I = H$, we have $H/H \simeq \{0\}$ which implies that $|A_i| = 1$ for all $i \in L\setminus\{0\}$. Set $A_i = \{i\}$ for $i \in L\setminus\{0\}$ and $A_0 = H$. Then $K = H \cup L$ and by following the definition of the hyper operation $\oplus$ on K, we have the following cases:

For $x, y \in H$,

$$x \oplus y = x + y.$$

For $i, j \in L\setminus\{0\}$ and $i \neq -j$, we have

$$i \oplus j = \bigcup_{x\in i+j} f_x(f_i^{-1}(i) + f_j^{-1}(j)) = \bigcup_{x\in i+j} f_x(H + H) = i + j.$$

For $i, j \in L\setminus\{0\}$ and $i = -j$,

$$i \oplus j = \bigcup_{x\in i+j} f_x(H) = f_0(H) \cup \left\{\bigcup_{x\in i+j\setminus\{0\}} f_x(H)\right\} = H \cup (i + j).$$

For $x \in H, i \in L$,

$$x \oplus i = \bigcup_{x\in 0+i} f_x(f_0^{-1}(x) + f_i^{-1}(i)) = f_i(x + H) = f_i(H) = i.$$

Similarly, we can see that $i \oplus x = i$ for $i \in L$ and $x \in H$. Thus, for $x, y \in K\setminus\{0\}$, we have

$$x \oplus y = \left\{\begin{array}{cc} x + y & if\ x, y \in H \\ x & if\ x \in L, y \in H \\ y & if\ x \in H, y \in L \\ x + y & if\ x, y \in L, y \neq -x \\ x + y \cup H & if\ x, y \in L, y = -x \end{array}\right\}.$$

This is exactly the hyper operation on $H[L]$. □

Corollary 3.6. *Let H and L be polygroups and $K = L \times_{H/I} H$. Then $J = I \cup \{a_i^{\overline{0}} : i \in L\setminus\{0\}\}$ is a regularly normal subpolygroup of K. Moreover, $K/J \simeq H/I$.*

Proof. For $h_1, h_2 \in I$, $h_1 - h_2 \subseteq I \subseteq J$. For $a_i^{\overline{0}}, a_j^{\overline{0}} \in J$ where $i, j \in L/\{0\}$, we have $-a_j^{\overline{0}} = a_{-j}^{\overline{0}} \in J$. Hence, $a_i^{\overline{0}} \oplus -a_j^{\overline{0}} = a_i^{\overline{0}} \oplus a_{-j}^{\overline{0}} = \{a_x^{\overline{0}} : x \in i - j\} \subseteq J$. For $h \in I$ and $a_i^{\overline{0}} \in J$, $i \neq 0$, we have $h \oplus (-a_i^{\overline{0}}) = -a_i^{\overline{0}} \in J$. Therefore, J is a subpolygroup in K. Now, let $h \in I \subseteq K$. Then $h + J = J = J + h$. For $a_j^{\overline{k}} \in K$ and since $I = a_0^{\overline{0}}$ we have

$$\begin{aligned} a_j^{\overline{k}} \oplus J &= \{a_j^{\overline{k}} \oplus a_i^{\overline{0}} : i \in L\} = \{a_x^{\overline{y}} : x \in j + i \text{ for all } i \in L \text{ and } \overline{y} \in \overline{k} + \overline{0}\} = \{a_x^{\overline{k}} : x \in L\} \\ &= \{a_x^{\overline{y}} : x \in i + j \text{ for all } i \in L \text{ and } \overline{y} \in \overline{0} + \overline{k}\} = \{a_i^{\overline{0}} \oplus a_j^{\overline{k}} : i \in L\} = J \oplus a_j^{\overline{k}}. \end{aligned}$$

Therefore, J is regularly normal. Now, let $\Psi : K \to H/I$ be defined by $\Psi(a_i^{\overline{h}}) = \overline{h}$ for all $i \in L$. Then Ψ is clearly well defined and onto. Let $a_i^{\overline{h}}, a_j^{\overline{k}} \in K$ where $i, j, i + j \neq 0$. Then

$$\Psi(a_i^{\overline{h}} \oplus a_j^{\overline{k}}) = \Psi(\{a_x^{\overline{y}} : x \in i + j, \overline{y} \in \overline{h} + \overline{k}\}) = \{\overline{y} : \overline{y} \in \overline{h} + \overline{k}\} = \overline{h} + \overline{k} = \Psi(a_i^{\overline{h}}) + \Psi(a_j^{\overline{k}}).$$

The other cases for pairs of elements of K can be checked similarly. Thus, Ψ is a strong homomorphism. Also,

$$ker\,(\Psi) = \{a_i^{\overline{h}} : \Psi(a_i^{\overline{h}}) = I\} = \{a_i^{\overline{h}} : \overline{h} = I\} = \{a_i^{\overline{h}} : \overline{h} = \overline{0}\} = I \cup \{a_i^{\overline{0}} : i \in L\} = J.$$

Based on the first isomorphism Theorem 2.7, we have $K/J \cong H/I$. □

Corollary 3.7. *Let H and L be polygroups, $K = L \times_{H/I} H$ and $J = I \cup \{a_i^{\overline{0}} | i \in L\backslash\{0\}\}$. Then $J \simeq L \times_{I/I} I \simeq I[L]$.*

Proof. Based on Corollary 3.5, $L \times_{I/I} I = I \cup L$ where $I \cap L = \{0\}$ and we can define a hyper operation $\oplus'$ on $L \times_{I/I} I$ as: $x \oplus' y = \left\{ \begin{array}{cc} x + y & if\ x, y \in I \\ x & if\ x \in L, y \in I \\ y & if\ x \in I, y \in L \\ x + y & if\ x, y \in L, y \neq -x \\ (x + y) \cup I & if\ x, y \in L, y = -x \end{array} \right\}$. . . . . (*)

Define $\varphi : J \to L \times_{I/I} I$ by $\varphi(x) = \left\{ \begin{array}{cc} x & if\ x \in I \\ i & if\ x = a_i^{\overline{0}} \in J\backslash I \end{array} \right\}$. Then clearly φ is bijective. Now, let $a_i^{\overline{0}}, a_j^{\overline{0}} \in J$ with $a_i^{\overline{0}} \neq -a_j^{\overline{0}}$. Then $i \neq -j$ and from (*) we have that $i \oplus' j = i + j$. So,

$$\varphi(a_i^{\overline{0}} \oplus a_j^{\overline{0}}) = \varphi\left\{a_x^{\overline{0}} : x \in i + j\right\} = i + j = i \oplus' j = \varphi(a_i^{\overline{0}}) \oplus' \varphi(a_j^{\overline{0}}).$$

For $a_i^{\overline{0}}, a_j^{\overline{0}} \in J$ with $a_i^{\overline{0}} = -a_j^{\overline{0}}$, we have $i = -j$ and by (*) we have $i \oplus' j = (i + j) \cup I$. So,

$$\begin{aligned} \varphi(a_i^{\overline{0}} \oplus a_j^{\overline{0}}) &= \varphi\left\{a_x^{\overline{0}} : x \in i + j\right\} = \varphi(I) \cup \varphi\left\{a_x^{\overline{0}} : x \in (i + j)\backslash\{0\}\right\} \\ &= I \cup ((i + j)\backslash\{0\}) = I \cup (i + j) = i \oplus' j = \varphi(a_i^{\overline{0}}) \oplus' \varphi(a_j^{\overline{0}}). \end{aligned}$$

For $h, k \in I$, we have $h + k = h \oplus' k$. So,

$$\varphi(h + k) = \varphi(\{t : t \in h + k\}) = \{t : t \in h + k\} = h \oplus' k = \varphi(h) \oplus' \varphi(k).$$

Finally, for $h \in I$ and $a_i^{\overline{0}} \in J\backslash I$, we have

$$\varphi(h \oplus a_i^{\overline{0}}) = \varphi(a_i^{\overline{0}}) = i = h \oplus' i = \varphi(h) \oplus' \varphi(a_i^{\overline{0}}).$$

Therefore, $J \simeq L \times_{I/I} I \simeq I[L]$. □

Corollary 3.8. *Let H, L be polygroups and $K = L \times_{H/I} H$. Then $K/I \cong (H/I) \times L$.*

Proof. Let $\varphi : K \to (H/I) \times L$ be defined by $\varphi(k) = \left\{ \begin{array}{ll} (\bar{h}, i) & if\ k = a_i^{\bar{h}}, i \in L\backslash\{0\} \\ (\bar{h}, 0) & if\ k \in I + h \end{array} \right\}$.Then φ is obviously well defined and onto. Let $a_i^{\bar{h}}, a_j^{\bar{k}} \in K$ where $i, j, i + j \neq 0$. Then

$$\begin{aligned} \varphi(a_i^{\bar{h}} \oplus a_j^{\bar{k}}) &= \varphi(\{a_x^{\bar{y}} : x \in i + j, \bar{y} \in \bar{h} + \bar{k}\}) \\ &= \{(\bar{y}, x) : x \in i + j, \bar{y} \in \bar{h} + \bar{k}\} = (\bar{h}, i) + (\bar{k}, j) = \varphi(a_i^{\bar{h}}) + \varphi(a_j^{\bar{k}}). \end{aligned}$$

After considering the other cases, we conclude that φ is a strong homomorphism. Moreover, $Ker\ (\varphi) = \{a_i^{\bar{h}} : \varphi(a_i^{\bar{h}}) = (\bar{0}, 0)\} = \{a_i^{\bar{h}} : (\bar{h}, i) = (\bar{0}, 0)\} = \{a_0^{\bar{0}}\} = I$. According to Theorem 2.7, we get $K/I \cong (H/I) \times L$. □

Corollary 3.9. *Let I and I_1 be regularly normal subpolygroup of polygroup H such that $I \subseteq I_1$ and J be a regularly normal subpolygroup of a polygroup L. Then, $J \times_{I_1/I} I_1$ is a regularly normal subpolygroup of $L \times_{H/I} H$.*

Proof. Since $I \subseteq I_1$ and I is regularly normal in H, then I is regularly normal in I_1 and so $J \times_{I_1/I} I_1$ is a polygroup. Since $I_1 \subseteq H$, $J \subseteq L$ and $I_1/I \subseteq H/I$, then $J \times_{I_1/I} I_1 \subseteq L \times_{H/I} H$. Let $x, y \in J \times_{I_1/I} I_1$ where $x = a_j^{\bar{k}}$ and $y = a_l^{\bar{h}}$ for $0 \neq j, l \in J$ and $\bar{k}, \bar{h} \in I_1/I$. Then $x \oplus y = \left\{a_d^{\bar{t}} : d \in j + l, \bar{t} \in \bar{k} + \bar{h}\right\} \in J \times_{I_1/I} I_1$ since I_1/I is a subpolygroup of H/I. The other cases of elements of $J \times_{I_1/I} I_1$ can be checked similarly. Also, if $x = a_j^{\bar{k}} \in J \times_{I_1/I} I_1$, then $-\bar{k} \in I_1/I$ and $-j \in J$. Hence, $-x = a_{-j}^{-\bar{k}} \in J \times_{I_1/I} I_1$. Therefore, $J \times_{I_1/I} I_1$ is a subpolygroup of $L \times_{H/I} H$. By Theorem 2.9, I_1/I is a regularly normal subpolygroup of H/I. Thus, for $a_i^{\bar{h}} \in L \times_{H/I} H$, $\bar{h} \in H/I$ and $i \in L$, we have

$$\begin{aligned} &a_i^{\bar{h}} \oplus (J \times_{I_1/I} I_1) = a_i^{\bar{h}} \oplus \{I_1 \cup \left\{a_x^{\bar{y}} : x \in J\backslash\{0\}, \bar{y} \in I_1/I\right\}\} = a_i^{\bar{h}} \oplus \left\{a_x^{\bar{y}} : x \in J, \bar{y} \in I_1/I\right\} \\ &= \left\{a_e^{\bar{f}} : e \in i + J, \bar{f} \in \bar{h} + (I_1/I)\right\} = \left\{a_e^{\bar{f}} : e \in J + i, \bar{f} \in (I_1/I) + \bar{h}\right\} = (J \times_{I_1/I} I_1) \oplus a_i^{\bar{h}}. \end{aligned}$$

It follows that, $J \times_{I_1/I} I_1$ is regularly normal in $L \times_{H/I} H$. □

Corollary 3.10. *Let $L \times_{H/I} H$ and $J \times_{I_1/I} I_1$ be as in Corollary 2.9. Then $(L \times_{H/I} H)/(J \times_{I_1/I} I_1) \cong (L/J) \times (H/I_1)$.*

Proof. Define a mapping $\Psi : (L \times_{H/I} H)/(J \times_{I_1/I} I_1) \to (L/J) \times (H/I_1)$ by $\Psi((J \times_{I_1/I} I_1) \oplus a_i^{\bar{h}}) = (J + i, I_1 + h)$ where $a_i^{\bar{h}} \in L \times_{H/I} H$. Then Ψ is well-defined and one to one. Indeed, for $a_i^{\bar{h}}, a_j^{\bar{k}} \in L \times_{H/I} H$, we have

$$\begin{aligned} (J \times_{I_1/I} I_1) \oplus a_i^{\bar{h}} &= (J \times_{I_1/I} I_1) \oplus a_j^{\bar{k}} \\ &\Leftrightarrow (a_i^{\bar{h}} \oplus a_{-j}^{-\bar{k}}) \cap (J \times_{I_1/I} I_1) \neq \emptyset \\ &\Leftrightarrow (i - j) \cap J \neq \emptyset \text{ and } (h - k) \cap I_1 \neq \emptyset \\ &\Leftrightarrow (J + i, I_1 + h) = (J + j, I_1 + k) \\ &\Leftrightarrow \Psi((J \times_{I_1/I} I_1) \oplus a_i^{\bar{h}}) = \Psi((J \times_{I_1/I} I_1) \oplus a_j^{\bar{k}}). \end{aligned}$$

Now, let $(J \times_{I_1/I} I_1) \oplus a_i^{\bar{h}}, (J \times_{I_1/I} I_1) \oplus a_j^{\bar{k}} \in (L \times_{H/I} H)/(J \times_{I_1/I} I_1)$. Then

$$\begin{aligned} \Psi(((J \times_{I_1/I} I_1) \oplus a_i^{\bar{h}}) \oplus ((J \times_{I_1/I} I_1) \oplus a_j^{\bar{k}})) &= \Psi(\left\{(J \times_{I_1/I} I_1) \oplus a_x^{\bar{y}} : x \in i + j, \bar{y} \in \bar{h} + \bar{k}\right\}) \\ &= \{(J + x, I_1 + y) : x \in i + j, y \in h + k\} = (J + i, I_1 + h) \times (J + j, I_1 + k) \\ &= \Psi((J \times_{I_1/I} I_1) \oplus a_i^{\bar{h}}) \times \Psi((J \times_{I_1/I} I_1) \oplus a_j^{\bar{k}}). \end{aligned}$$

Hence, Ψ is a strong homomorphism. Finally, it is straightforward that Ψ is onto and the result follows. □

Corollary 3.11. *Let M, L and H be polygroups and let $J \subseteq L$, $I \subseteq H$ be regularly normal subpolygroups in L and H respectively. Then $(M \underset{L/J}{\times} L) \underset{H/I}{\times} H \cong M \underset{(L \underset{H/I}{\times} H)/(J \underset{I/I}{\times} I)}{\times} (L \underset{H/I}{\times} H)$.*

Proof. By Corollary 3.10, $(L \times_{H/I} H)/(J \times_{I_1/I} I_1) \cong (L/J) \times (H/I_1)$. Thus, the elements of $M \underset{(L \underset{H/I}{\times} H)/(J \underset{I/I}{\times} I)}{\times} (L \underset{H/I}{\times} H)$ can be denoted by $c_i^{(\bar{h}_J, \bar{k}_I)}$ where $\bar{h}_J = J + h \in L/J$, $\bar{k}_I = I + k \in H/I$ and $i \in M$. In this case, the hyper operation $\oplus$ can be expressed as

$$c_i^{(\bar{h}_J, \bar{k}_I)} \oplus c_j^{(\bar{p}_J, \bar{q}_I)} = \left\{ c_x^{(y_J, z_I)} : x \in i + j,\ \bar{y}_J \in \bar{h}_J + \bar{p}_J,\ \bar{z}_I \in \bar{k}_I + \bar{q}_I \right\}.$$

On the other hand, we may denote the elements of $(M \underset{L/J}{\times} L) \underset{H/I}{\times} H$ by $b^{\bar{k}_I}_{a_i^{\bar{h}_J}}$ where $a_i^{\bar{h}_J} \in M \underset{L/J}{\times} L$ and $\bar{k}_I = I + k \in H/I$. The hyper operation $\oplus$ can be written as $b^{k_I}_{a_i^{\bar{h}_J}} \oplus b^{\bar{q}_I}_{a_j^{\bar{p}_J}} = \left\{ b^{\bar{z}_I}_{a_x^{\bar{y}_J}} : x \in i + j,\ \bar{y}_J \in \bar{h}_J + \bar{p}_J,\ \bar{z}_I \in \bar{k}_I + \bar{q}_I \right\}$. Therefore, it is easy to check that the mapping $\Psi : (M \underset{L/J}{\times} L) \underset{H/I}{\times} H \to M \underset{(L \underset{H/I}{\times} H)/(J \underset{I/I}{\times} I)}{\times} (L \underset{H/I}{\times} H)$ defined by $\Psi(b^{\bar{k}_I}_{a_i^{\bar{h}_J}}) = c_i^{(\bar{h}_J, \bar{k}_I)}$ is a strong isomorphism. $\square$

Example 3.12. *Consider the two polygroups $H = \{0, 1, 2, 3\}$and $L = \{0, x\}$with the following tables*

H	0	1	2	3
0	0	1	2	3
1	1	01	23	23
2	2	23	013	123
3	3	23	123	012

L	0	x
0	0	x
x	x	$0x$

Then clearly, $I = \{0, 1\}$ is a subpolygroup of H with $I + 2 = \{2, 3\} = 2 + I$. We now present the extensions $K = L \times_{H/I} H$ of L by H via H/I when $I = \{0\}$, $I = H$ and $I = \{0, 1\}$.

Case 1: $I = H$. In this case, K is isomorphic to $H[L]$ with the following table.

K	0	1	2	3	x
0	0	1	2	3	x
1	1	01	23	23	x
2	2	23	013	123	x
3	3	23	123	012	x
x	x	x	x	x	$0123x$

This is the extension of L by H via H/H and it is the minimal extension since $|K| = |H| + |L \backslash \{0\}| = 4 + 1 = 5$. *From the table, we can see that K is obtained by enlarging the element* 0 *in L by H.*

Case 2: $I = \{0, 1\}$. In this case, K is represented as follows.

K	0	1	2	3	a_x^0	a_x^2
0	0	1	2	3	a_x^0	a_x^2
1	1	01	23	23	a_x^0	a_x^2
2	2	23	013	123	a_x^2	$a_x^0 a_x^2$
3	3	23	123	012	a_x^2	$a_x^0 a_x^2$
a_x^0	a_x^0	a_x^0	a_x^2	a_x^2	$01a_x^0$	$23a_x^2$
a_x^2	a_x^2	a_x^2	$a_x^0 a_x^2$	$a_x^0 a_x^2$	$23a_x^2$	$0123a_x^0 a_x^2$

and $|K| = |H| + |H/I||L \backslash \{0\}| = 4 + 2 = 6$.

Case 3: $I = \{0\}$.

K	0	1	2	3	a_x^0	a_x^1	a_x^2	a_x^3
0	0	1	2	3	a_x^0	a_x^1	a_x^2	a_x^3
1	1	01	23	23	a_x^1	$a_x^0a_x^1$	$a_x^2a_x^3$	$a_x^2a_x^3$
2	2	23	013	123	a_x^2	$a_x^2a_x^3$	$a_x^0a_x^1a_x^3$	$a_x^1a_x^2a_x^3$
3	3	23	123	012	a_x^3	$a_x^2a_x^3$	$a_x^1a_x^2a_x^3$	$a_x^0a_x^1a_x^3$
a_x^0	a_x^0	a_x^1	a_x^2	a_x^3	$0a_x^0$	$1a_x^1$	$2a_x^2$	$3a_x^3$
a_x^1	a_x^1	$a_x^0a_x^1$	$a_x^2a_x^3$	$a_x^2a_x^3$	$1a_x^1$	$01a_x^0a_x^1$	$23a_x^2a_x^3$	$23a_x^2a_x^3$
a_x^2	a_x^2	$a_x^2a_x^3$	$a_x^0a_x^1a_x^3$	$a_x^1a_x^2a_x^3$	$2a_x^2$	$23a_x^2a_x^3$	$013a_x^0a_x^1a_x^3$	$123a_x^1a_x^2a_x^3$
a_x^3	a_x^3	$a_x^2a_x^3$	$a_x^1a_x^2a_x^3$	$a_x^0a_x^1a_x^3$	$3a_x^3$	$23a_x^2a_x^3$	$123a_x^1a_x^2a_x^3$	$012a_x^1a_x^2a_x^3$

This is the extension of L by H via $H/\{0\}$ and it is isomorphic to $L \times H$. It is the maximal extension of L by H since $|K| = |H| + |H/\{0\}||L\backslash\{0\}| = |H| + |H|.|L\backslash\{0\}| = |H|.|L| = (4)(2) = 8$. *From the table, we can see that K is obtained by enlarging the elements of L by copies of the polygroup H.*

4. Properties of the construction

An extension of a polygroup L by a polygroup H can always be done as their direct hyper product $L \times H$ or wreath product $H[L]$. The possibility of other extensions depends on the existence of regularly normal subpolygroups I of H. In that case, we can collapse the polygroup H into the factor polygroup H/I and use the canonical homomorphism $\varphi : H \to H/I$ as the base for the extension. Every non zero element of L is enlarging by isomorphic copies of H/I and the zero element of L is enlarging by H. Indeed, for $A_i = \left\{a_i^{\bar{h}} : \bar{h} \in H/I\right\}, 0 \neq i \in L$, the functions $f_i : H \to A_i$ are defined as copies of the canonical homomorphism $\varphi : H \to H/I$. For all $i \in L$, we define a hyper operation $\oplus_i$ on the set A_i as $a_i^{\bar{h}} \oplus_i a_i^{\bar{k}} = \left\{a_i^{\bar{x}} : \bar{x} \in \bar{h} + \bar{k}\right\}$. Then one can easily see that $(A_i, \oplus_i)$ is a polygroup and f_i is a homomorphism for all i.

Proposition 4.1. *Consider the extension $K = L \times_{H/I} H$ defined in proposition 3.1. Then for all $i \in L\backslash\{0\}$, H acts on A_i by the action $g_i : A_i \times H \to \rho(A_i)$ defined as $g_i\left(a_i^{\bar{k}}, h\right) = a_i^{\bar{k}} \oplus h = \{a_i^{\bar{x}} : \bar{x} \in \bar{k} + \bar{h}\}$.*

Proof. We check the axioms of Definition *1.10*.

1) For all $a_i^{\bar{k}} \in A_i$, $g_i\left(a_i^{\bar{k}}, 0\right) = a_i^{\bar{k}} \oplus 0 = a_i^{\bar{k}}$.

2) For all $h, h' \in H$ and $a_i^{\bar{k}} \in A_i$, we have

$$\begin{aligned} g_i\left(g_i\left(a_i^{\bar{k}}, h\right), h'\right) &= g_i\left(a_i^{\bar{k}} \oplus h, h'\right) = g_i\left(\{(t, h') \,|t \in a_i^{\bar{k}} \oplus h\}\right) \\ &= \{t \oplus h' | t \in a_i^{\bar{k}} \oplus h\} = \left(a_i^{\bar{k}} \oplus h\right) \oplus h' \\ &= a_i^{\bar{k}} \oplus (h \oplus h') = \{a_i^{\bar{k}} \oplus t : t \in h + h'\} = \bigcup_{t \in h + h'} g_i\left(a_i^{\bar{k}}, t\right). \end{aligned}$$

3) For all $h \in H$,

$$\begin{aligned} \bigcup_{a_i^{\bar{k}} \in A_i} g_i\left(a_i^{\bar{k}}, h\right) &= \left\{a_i^{\bar{k}} \oplus h : a_i^{\bar{k}} \in A_i\right\} = \left\{a_i^{\bar{x}} : \bar{x} \in \bar{k} + \bar{h}, \bar{k} \in H/I\right\} \\ &= \left\{a_i^{\bar{x}} : \bar{x} \in (H/I) + \bar{h}\right\} = \left\{a_i^{\bar{x}} : \bar{x} \in H/I\right\} = A_i \end{aligned}$$

4) Let $h \in H$ and let $a_i^{\bar{k}} \in g_i\left(a_i^{\bar{n}}, h\right)$. Then $a_i^{\bar{k}} \in a_i^{\bar{n}} \oplus h = \{a_i^{\bar{x}} | \bar{x} \in \bar{n} + \bar{h}\}$. Hence, $\bar{k} \in \bar{n} + \bar{h}$ and so $\bar{n} \in \bar{k} - \bar{h}$. It follows that $a_i^{\bar{n}} \in \{a_i^{\bar{x}} | \bar{x} \in \bar{k} - \bar{h}\} = a_i^{\bar{k}} \oplus (-h) = g_i\left(a_i^{\bar{k}}, -h\right)$. □

Remark 4.2. *In the previous Proposition, we consider the right action of H on A_i . Similarly, we can consider the left action defined as $g_i\left(h, a_i^{\bar{k}}\right) = h \oplus a_i^{\bar{k}}$. For simplicity, we will denote the right action $g_i\left(a_i^{\bar{k}}, h\right)$ of H on A_i by $\left(a_i^{\bar{k}}\right)^h$*

Proposition 4.3. *The kernel of the action of H on A_i defined by $g_i\left(a_i^{\bar{k}}, h\right) = a_i^{\bar{k}} \oplus h$ is the subpolygroup I of H .*

Proof. For all $i \in L$, $Ker(g_i) = \{h \in H : g_i\left(a_i^{\bar{k}}, h\right) = a_i^{\bar{k}}$ for all $k \in H/I\}$. Let $x \in Ker(g_i) \subseteq H$. Then $a_i^{\bar{k}} \oplus x = a_i^{\bar{k}}$ for all $a_i^{\bar{k}} \in A_i$ and so $\{a_i^{\bar{y}} | \bar{y} \in \bar{k} + \bar{x}\} = a_i^{\bar{k}}$. Hence, $\bar{k} + \bar{x} = \bar{k}$ for all $\bar{k} \in H/I$ and then $\bar{x} = \bar{0} = I$ since H/I is a polygroup. Therefore, $x \in I$ and $Ker(g_i) \subseteq I$. Conversely, for $x \in I$, $a_i^{\bar{k}} \oplus x = a_i^{\bar{k}}$ for all $a_i^{\bar{k}} \in A_i$. It follows that $g_i\left(a_i^{\bar{k}}, x\right) = a_i^{\bar{k}}$ for all $a_i^{\bar{k}} \in A_i$ and so $x \in Ker(g_i)$. Thus, $I \subseteq Ker(g_i)$ and the result follows. □

Remark 4.4. *For polygroups L and H, the direct product $L \times H$ is an extension of L by H via $H/\{0\}$. Thus, the kernel of the action g_i of H on A_i is $Ker(g_i) = \{0\}$. On the other hand, the Wreath product $H[L]$ is an extension of L by H via H/H and in this case, $Ker(g_i) = H$.*

To visualize the table of a polygroup extension of L by H via H/I and also simplify the construction, we consider some notations. By $B(A_i)$, we denote the block of A_i that represents the table of hyper sums of $(A_i, \oplus_i)$.

$$B(A_i) = \begin{array}{|c|c|c|} \hline \sum^i_{\bar{0}\bar{0}} & \sum^i_{\bar{0}\bar{h}} & \cdots \\ \hline \sum^i_{\bar{h}\bar{0}} & \sum^i_{\bar{h}\bar{h}} & \cdots \\ \hline \vdots & \vdots & \ddots \\ \hline \end{array} \quad \text{where } \sum_{\bar{x}\bar{y}}^{i} = a_i^{\bar{x}} \oplus_i a_i^{\bar{y}}$$

By the definition of the hyperoperation $\oplus$, we can write $a_i^{\bar{h}} \oplus a_j^{\bar{k}} = \left\{a_p^{\bar{q}} : p \in i+j, \bar{q} \in \bar{h} + \bar{k}\right\} = \bigcup\limits_{p \in i+j} \left\{a_p^{\bar{h}} \oplus_p a_p^{\bar{k}}\right\}$. Also, we have, $A_i \oplus A_j = \{A_x : x \in i+j\}$. Thus, we denote the block of hyper sums in $A_i \oplus A_j$ with $B(A_i \oplus A_j) = \bigcup\limits_{x \in i+j} B(A_x)$. where

$$\bigcup_{x \in i+j} B(A_x) = \begin{array}{|c|c|c|} \hline \bigcup\limits_{x \in i+j} \sum^x_{\bar{0}\bar{0}} & \bigcup\limits_{x \in i+j} \sum^x_{\bar{0}\bar{h}} & \cdots \\ \hline \bigcup\limits_{x \in i+j} \sum^x_{\bar{h}\bar{0}} & \bigcup\limits_{x \in i+j} \sum^x_{\bar{h}\bar{h}} & \cdots \\ \hline \vdots & \vdots & \ddots \\ \hline \end{array}$$

Whenever $0 \in i+j$, then $B(A_0) \in \bigcup\limits_{k \in i+j} B(A_k)$. Although$|B(A_0)| \geq |B(A_k)|$ for $k \in (i+j)\backslash\{0\}$, but the hyperoperation $\oplus$ defined on K fits $B(A_0)$ in the union by collapsing $A_0 = H$ into factor polygroup H/I. Indeed,

$$\begin{aligned} a_i^{\bar{h}} \oplus a_j^{\bar{k}} &= \left\{a_p^{\bar{q}} : p \in i+j, \bar{q} \in \bar{h} + \bar{k}\right\} = \left\{a_0^{\bar{q}} :\ \bar{q} \in \bar{h} + \bar{k}\right\} \cup \left\{a_p^{\bar{q}} : p \in (i+j)\backslash\{0\}, \bar{q} \in \bar{h} + \bar{k}\right\} \\ &= \left\{\bar{q} :\ \bar{q} \in \bar{h} + \bar{k}\right\} \cup \Big(\bigcup_{p \in (i+j)\backslash\{0\}} (a_p^{\bar{h}} \oplus_p a_p^{\bar{k}})\Big) \end{aligned}$$

Therefore, the table for $(K, \oplus)$ has the form

$\oplus$	H	A_i	A_j	$\cdots$
H	$B(H)$	$B(A_i^H)_L$	$B(A_j^H)_L$	$\cdots$
A_i	$B(A_i^H)_R$	$\bigcup\limits_{k \in i+i} B(A_k)$	$\bigcup\limits_{k \in i+j} B(A_k)$	$\cdots$
A_j	$B(A_i^H)_R$	$\bigcup\limits_{k \in j+i} B(A_k)$	$\bigcup\limits_{k \in j+j} B(A_k)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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