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The Product-Type Operators from Logarithmic Bloch Spaces to Zygmund-Type Spaces

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Abstract. The boundedness and compactness of a product-type operator, recently introduced by S. Stević, A. Sharma and R. Krishan,

$$T^n_{\psi_1,\psi_2,\varphi}f(z) = \psi_1(z)f^{(n)}(\varphi(z)) + \psi_2(z)f^{(n+1)}(\varphi(z)),\ f \in H(\mathbb{D}),$$

from the logarithmic Bloch spaces to Zygmund-type spaces are characterized, where $\psi_1, \psi_2 \in H(\mathbb{D})$, φ is an analytic self-map of $\mathbb{D}$ and n a positive integer.

1. Introduction

Firstly, we introduce the notations used in this paper. Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. Let μ be a weight, that is, μ is a positive continuous function on $\mathbb{D}$.

The logarithmic Bloch space and Zygmund-type space is defined as follows, respectively:

$$\mathcal{B}_{\log} = \left\{ f \in H(\mathbb{D}) : \|f\| = \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)\left(\log \frac{2}{1-|z|}\right)|f'(z)| < \infty \right\},$$

and

$$\mathcal{Z}_\mu = \left\{ f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}} \mu(z)|f''(z)| < \infty \right\}.$$

The quantity appearing in the definition of the logarithmic Bloch space appears in [1], in characterizing multipliers of the Bloch functions. The space itself has been defined later. The space $\mathcal{B}_{\log}$ is a Banach space under the norm $\|f\|_{\mathcal{B}_{\log}} = |f(0)| + \|f\|$. With the norm $\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f''(z)|$, the

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Zygmund-type space $\mathcal{Z}_\mu$ is also a Banach space. When $\mu(z) = 1 - |z|^2$, it is the Zygmund space $\mathcal{Z}$, which was essentially introduced in [22]. For the case of the unit ball see, e.g., [40]. Let

$$\mathcal{B}_{\log,0} = \left\{ f \in \mathcal{B}_{\log} : \lim_{|z|\to 1} \left(1 - |z|^2\right) \left(\log \frac{2}{1 - |z|}\right) |f'(z)| = 0 \right\}.$$

Having studied composition operators (see, e.g., [5] and the references therein) and integral operators on spaces of analytic functions on various domains, some experts started studying operator-theoretic properties of their product-type operators (for the case of the unit disc see [3, 12, 13, 17, 19–21, 33, 53, 54], while for the case of the unit ball see [10, 24, 26–28, 34–43, 49, 55–57]). After the publication of [7], some experts started studying product-type operators involving the differentiation operator (see, e.g., [14–16, 18, 44]). Some of these papers study the operators from or to Bloch-type and/or Zygmund-type spaces ([2, 4, 6, 8, 9, 17, 20, 29, 50]).

Motivated by the study of weighted differentiation composition operators (see [32, 45–47]), quite recently, S. Stević, A. Sharma and R. Krishan in [48] introduced the operator

$$T^n_{\psi_1,\psi_2,\varphi} f(z) = \psi_1(z) f^{(n)}(\varphi(z)) + \psi_2(z) f^{(n+1)}(\varphi(z)), \ f \in H(\mathbb{D}),$$

where $\psi_1, \psi_2 \in H(\mathbb{D})$, φ is an analytic self-map of $\mathbb{D}$ and n a positive integer. The boundedness and compactness of the product-type operator $T^n_{\psi_1,\psi_2,\varphi} : F(p, q, s)$ (or $F_0(p, q, s)$) $\to \mathcal{B}_\mu$ have been studied by them. Note that, for $\psi_2 \equiv 0$, we obtain the weighted differentiation composition operator. For some later results on the weighted differentiation composition operator on various spaces of analytic functions see, e.g., [11, 23, 31, 51, 58–60].

Inspired by the results [25, 30, 32, 45, 47], our aim is to consider the boundedness and compactness of the operators $T^n_{\psi_1,\psi_2,\varphi} : \mathcal{B}_{\log}$ (or $\mathcal{B}_{\log,0}$) $\to \mathcal{Z}_\mu$.

2. Auxiliary results

Here we quote three lemmas which will be used in the proofs of the main results in this paper.

Lemma 2.1 ([30]) *Suppose $f \in \mathcal{B}_{\log}$, there exists a constant C such that*

$$|f^{(n)}(z)| \le \frac{C\|f\|_{\mathcal{B}_{\log}}}{(1 - |z|^2)^n \log \frac{2}{1-|z|}},$$

for every $z \in \mathbb{D}$, and all positive integer $n = 1, 2, \cdots$.

The following lemma was essentially introduced in [52], we will sketch the details of the proof to maintain completeness.

Lemma 2.2 *Let*

$$g_t(z) = \frac{(1 - |z|) \log \frac{2}{1-|z|}}{(1 - |tz|) \log \frac{2}{1-|tz|}}, \ t \in [0, 1], \ z \in \mathbb{D},$$

then for all $t \in [0, 1]$,

$$|g_t(z)| < 2, \ for\ every\ z \in \mathbb{D}.$$

Proof. Let $f(x) = (1 - x) \log \frac{2}{1-x}, x \in [0, 1)$. By the Product Rule $f'(x) = -\log \frac{2}{1-x} + 1$. Set $x_0 = 1 - \frac{2}{e}$, clearly f increases on $[0, x_0]$ and f decreases on $[x_0, 1)$. Noting that $0 < \frac{4}{3}x_0 = \frac{4e-8}{3e} < 1$, we have

(1) If $1 \ge t > \frac{3}{4}$ and $\frac{4}{3}x_0 < x < 1$, then $1 > x \ge tx > x_0$, so $f(x) \le f(tx)$, thus $|g_t(z)| = \frac{f(|z|)}{f(t|z|)} \le 1$, if $t \in (3/4, 1]$ and $\frac{4}{3}x_0 < |z| < 1$.

(2) If $1 \geq t > \frac{3}{4}$ and $0 \leq x \leq \frac{4}{3}x_0$, then $0 \leq tx \leq \frac{4}{3}x_0$, so $f(x) \leq f(x_0)$ and $f(tx) \geq \min\left\{f(0), f(\frac{4}{3}x_0)\right\}$. Since $1 - \frac{4}{3}x_0 = \frac{8-e}{3e} > 5/8$, $\frac{2}{1-\frac{4}{3}x_0} = \frac{6e}{8-e} > \frac{307}{100}$ and $(\frac{307}{100})^5 > 2^8$, thus $f(\frac{4}{3}x_0) > \frac{5}{8}\log\frac{307}{100} > \log 2$, so

$$|g_t(z)| = \frac{f(|z|)}{f(t|z|)} \leq \frac{f(x_0)}{\min\{f(0), f(\frac{4}{3}x_0)\}} = \frac{2/e}{\log 2} < 2, \text{ if } t \in (3/4, 1] \text{ and } 0 \leq |z| \leq \frac{4}{3}x_0.$$

(3) If $0 \leq t \leq \frac{3}{4}$, then for $x \in [0, 1)$, $f(x) \leq f(x_0) = 2/e$. Since $tx \in [0, 3/4]$, we have

$$f(tx) \geq \min\left\{f(0), f\left(\frac{3}{4}\right)\right\} = \min\left\{\log 2, \frac{3}{4}\log 2\right\} = \frac{3}{4}\log 2,$$

thus

$$|g_t(z)| = \frac{f(|z|)}{f(t|z|)} \leq \frac{8}{3e\log 2} < 2, \text{ if } t \in [0, 3/4] \text{ and } z \in \mathbb{D}.$$

So the proof is complete.

The following compactness criterion follows from standard arguments, for example, those in [5, Proposition 3.11].

Lemma 2.3 *Let $\psi_1, \psi_2 \in H(\mathbb{D})$, φ be an analytic self-map of $\mathbb{D}$, n a positive integer and μ a weight. Then $T^n_{\psi_1,\psi_2,\varphi} : \mathcal{B}_{\log}$ (or $\mathcal{B}_{\log,0}$) $\to \mathcal{Z}_\mu$ is compact if and only if $T^n_{\psi_1,\psi_2,\varphi} : \mathcal{B}_{\log}$ (or $\mathcal{B}_{\log,0}$) $\to \mathcal{Z}_\mu$ is bounded and for any bounded sequence $\{f_k\}$ in $\mathcal{B}_{\log}$ (or $\mathcal{B}_{\log,0}$) which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, we have $\|T^n_{\psi_1,\psi_2,\varphi} f_k\|_{\mathcal{Z}_\mu} \to 0$ as $k \to \infty$.*

3. Boundedness and compactness of $T^n_{\psi_1,\psi_2,\varphi}$ from $\mathcal{B}_{\log}$ (or $\mathcal{B}_{\log,0}$) to $\mathcal{Z}_\mu$ spaces

In this section, we prove our main results.

Theorem 3.1. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$, φ be an analytic self-map of $\mathbb{D}$, n a positive integer, and μ a weight. Then the following statements are equivalent.*

(1) $T^n_{\psi_1,\psi_2,\varphi} : \mathcal{B}_{\log} \to \mathcal{Z}_\mu$ *is bounded;*

(2) $T^n_{\psi_1,\psi_2,\varphi} : \mathcal{B}_{\log,0} \to \mathcal{Z}_\mu$ *is bounded;*

(3)

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi_1''(z)|}{\left(1 - |\varphi(z)|^2\right)^n \log\frac{2}{1-|\varphi(z)|}} < \infty, \tag{1}$$

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi_1(z)\varphi''(z) + 2\psi_1'(z)\varphi'(z) + \psi_2''(z)|}{\left(1 - |\varphi(z)|^2\right)^{n+1} \log\frac{2}{1-|\varphi(z)|}} < \infty, \tag{2}$$

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi_1(z)(\varphi'(z))^2 + 2\psi_2'(z)\varphi'(z) + \psi_2(z)\varphi''(z)|}{\left(1 - |\varphi(z)|^2\right)^{n+2} \log\frac{2}{1-|\varphi(z)|}} < \infty, \tag{3}$$

and

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi_2(z)||\varphi'(z)|^2}{\left(1 - |\varphi(z)|^2\right)^{n+3} \log\frac{2}{1-|\varphi(z)|}} < \infty. \tag{4}$$

Proof. (3) ⇒ (1). Since

$$\begin{aligned}\left(T^n_{\psi_1,\psi_2,\varphi}f\right)'(z) &= \psi_1'(z)f^{(n)}(\varphi(z))\\ &+\left(\psi_1(z)\varphi'(z)+\psi_2'(z)\right)f^{(n+1)}(\varphi(z))+\psi_2(z)\varphi'(z)f^{(n+2)}(\varphi(z)),\end{aligned}$$

and

$$\begin{aligned}\left(T^n_{\psi_1,\psi_2,\varphi}f\right)''(z) &= \psi_1''(z)f^{(n)}(\varphi(z))\\ &+\left(\psi_1(z)\varphi''(z)+2\psi_1'(z)\varphi'(z)+\psi_2''(z)\right)f^{(n+1)}(\varphi(z))\\ &+\left(\psi_1(z)(\varphi'(z))^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z)\right)f^{(n+2)}(\varphi(z))\\ &+\psi_2(z)(\varphi'(z))^2f^{(n+3)}(\varphi(z)),\end{aligned}$$

thus for every $z\in\mathbb{D}$ and $f\in\mathcal{B}_{\log}$, by Lemma 2.1 and the hypothesis we obtain that

$$\begin{aligned}&\mu(z)\left|\left(T^n_{\psi_1,\psi_2,\varphi}f\right)''(z)\right|\\ &\le \mu(z)|\psi_1''(z)|\left|f^{(n)}(\varphi(z))\right|\\ &+\mu(z)|\psi_1(z)\varphi''(z)+2\psi_1'(z)\varphi'(z)+\psi_2''(z)|\left|f^{(n+1)}(\varphi(z))\right|\\ &+\mu(z)\left|\psi_1(z)(\varphi'(z))^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z)\right|\left|f^{(n+2)}(\varphi(z))\right|\\ &+\mu(z)\left|\psi_2(z)(\varphi'(z))^2\right|\left|f^{(n+3)}(\varphi(z))\right|\\ &\le C\|f\|_{\mathcal{B}_{\log}}\frac{\mu(z)|\psi_1''(z)|}{\left(1-\left|\varphi(z)\right|^2\right)^n\log\frac{2}{1-|\varphi(z)|}}\\ &+C\|f\|_{\mathcal{B}_{\log}}\frac{\mu(z)|\psi_1(z)\varphi''(z)+2\psi_1'(z)\varphi'(z)+\psi_2''(z)|}{\left(1-\left|\varphi(z)\right|^2\right)^{n+1}\log\frac{2}{1-|\varphi(z)|}}\\ &+C\|f\|_{\mathcal{B}_{\log}}\frac{\mu(z)|\psi_1(z)(\varphi'(z))^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z)|}{\left(1-\left|\varphi(z)\right|^2\right)^{n+2}\log\frac{2}{1-|\varphi(z)|}}\\ &+C\|f\|_{\mathcal{B}_{\log}}\frac{\mu(z)|\psi_2(z)||\varphi'(z)|^2}{\left(1-\left|\varphi(z)\right|^2\right)^{n+3}\log\frac{2}{1-|\varphi(z)|}}\\ &\le C\|f\|_{\mathcal{B}_{\log}}.\end{aligned}\tag{5}$$

On the other hand, we have

$$\begin{aligned}\left|\left(T^n_{\psi_1,\psi_2,\varphi}f\right)(0)\right| &= \left|\psi_1(0)f^{(n)}(\varphi(0))+\psi_2(0)f^{(n+1)}(\varphi(0))\right|\\ &\le C\frac{|\psi_1(0)|+|\psi_2(0)|}{\left(1-\left|\varphi(0)\right|^2\right)^{n+1}\log\frac{2}{1-|\varphi(0)|}}\|f\|_{\mathcal{B}_{\log}},\end{aligned}\tag{6}$$

and

$$\begin{aligned}&\left|\left(T^n_{\psi_1,\psi_2,\varphi}f\right)'(0)\right|\\ &=\left|\psi_1'(0)f^{(n)}(\varphi(0))+(\psi_1(0)\varphi'(0)+\psi_2'(0))f^{(n+1)}(\varphi(0))+\psi_2(0)\varphi'(0)f^{(n+2)}(\varphi(0))\right|\\ &\le C\frac{|\psi_1'(0)|+|\psi_1(0)\varphi'(0)+\psi_2'(0)|+|\psi_2(0)\varphi'(0)|}{\left(1-\left|\varphi(0)\right|^2\right)^{n+2}\log\frac{2}{1-|\varphi(0)|}}\|f\|_{\mathcal{B}_{\log}}.\end{aligned}\tag{7}$$

It follows from (5), (6) and (7) that $T^n_{\psi_1,\psi_2,\varphi}:\mathcal{B}_{\log}\to\mathcal{Z}_\mu$ is bounded.

(1) ⇒ (2). It is obvious.

(2) ⇒ (3). Assume that $T^n_{\psi_1,\psi_2,\varphi}:\mathcal{B}_{\log,0}\to\mathcal{Z}_\mu$ is bounded, that is, there exists a constant C such that

$$\|T^n_{\psi_1,\psi_2,\varphi}f\|_{\mathcal{Z}_\mu}\le C\|f\|_{\mathcal{B}_{\log}},\ \text{for every } f\in\mathcal{B}_{\log,0}. \tag{8}$$

For $f(z)=\frac{z^n}{n!}\in\mathcal{B}_{\log,0}$ in (8), we have that

$$K_1:=\sup_{z\in\mathbb{D}}\mu(z)|\psi_1''(z)|<\infty. \tag{9}$$

Taking $f(z)=\frac{z^{n+1}}{(n+1)!}\in\mathcal{B}_{\log,0}$ in (8), we obtain that

$$\sup_{z\in\mathbb{D}}\mu(z)|\psi_1''(z)\varphi(z)+\psi_1(z)\varphi''(z)+2\psi_1'(z)\varphi'(z)+\psi_2''(z)|<\infty. \tag{10}$$

From (9) and (10), and since the function φ is bounded on $\mathbb{D}$, it follows that,

$$K_2:=\sup_{z\in\mathbb{D}}\mu(z)|\psi_1(z)\varphi''(z)+2\psi_1'(z)\varphi'(z)+\psi_2''(z)|<\infty. \tag{11}$$

Taking $f(z)=\frac{z^{n+2}}{(n+2)!}\in\mathcal{B}_{\log,0}$ in (8), we have that

$$\begin{aligned}\sup_{z\in\mathbb{D}}\mu(z)|&\frac{1}{2}\psi_1''(z)(\varphi(z))^2+(\psi_1(z)\varphi''(z)+2\psi_1'(z)\varphi'(z)+\psi_2''(z))\varphi(z)\\&+(\psi_1(z)(\varphi'(z))^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z))|<\infty.\end{aligned} \tag{12}$$

By (9), (11), (12) and the boundedness of φ, we have that

$$K_3:=\sup_{z\in\mathbb{D}}\mu(z)\left|\psi_1(z)(\varphi'(z))^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z)\right|<\infty. \tag{13}$$

Taking $f(z)=\frac{z^{n+3}}{(n+3)!}\in\mathcal{B}_{\log,0}$ in (8), we also get

$$\begin{aligned}\sup_{z\in\mathbb{D}}\mu(z)|&\frac{1}{6}\psi_1''(z)(\varphi(z))^3+\frac{1}{2}\left(\psi_1(z)\varphi''(z)+2\psi_1'(z)\varphi'(z)+\psi_2''(z)\right)(\varphi(z))^2\\&+\left(\psi_1(z)(\varphi'(z))^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z)\right)\varphi(z)+\psi_2(z)(\varphi'(z))^2|<\infty.\end{aligned} \tag{14}$$

By (9), (11), (13), (14), and the boundedness of φ, we have that

$$K_4:=\sup_{z\in\mathbb{D}}\mu(z)|\psi_2(z)||\varphi'(z)|^2<\infty. \tag{15}$$

For a fixed $\omega\in\mathbb{D}$ and constants a,b,c, set

$$\begin{aligned}f_\omega(z)\ &=a\frac{1-|\varphi(\omega)|^2}{\left(1-z\overline{\varphi(\omega)}\right)\log\frac{2}{1-|\varphi(\omega)|}}+b\frac{(1-|\varphi(\omega)|^2)^2}{\left(1-z\overline{\varphi(\omega)}\right)^2\log\frac{2}{1-|\varphi(\omega)|}}\\&+c\frac{(1-|\varphi(\omega)|^2)^3}{\left(1-z\overline{\varphi(\omega)}\right)^3\log\frac{2}{1-|\varphi(\omega)|}}+\frac{(1-|\varphi(\omega)|^2)^4}{\left(1-z\overline{\varphi(\omega)}\right)^4\log\frac{2}{1-|\varphi(\omega)|}}.\end{aligned} \tag{16}$$

It is easy to check that

$$\begin{aligned}f_\omega^{(n)}(z)\ &=an!\frac{(1-|\varphi(\omega)|^2)\left(\overline{\varphi(\omega)}\right)^n}{\left(1-z\overline{\varphi(\omega)}\right)^{n+1}\log\frac{2}{1-|\varphi(\omega)|}}+b(n+1)!\frac{(1-|\varphi(\omega)|^2)^2\left(\overline{\varphi(\omega)}\right)^n}{\left(1-z\overline{\varphi(\omega)}\right)^{n+2}\log\frac{2}{1-|\varphi(\omega)|}}\\&+c\frac{(n+2)!}{2}\frac{(1-|\varphi(\omega)|^2)^3\left(\overline{\varphi(\omega)}\right)^n}{\left(1-z\overline{\varphi(\omega)}\right)^{n+3}\log\frac{2}{1-|\varphi(\omega)|}}+\frac{(n+3)!}{6}\frac{(1-|\varphi(\omega)|^2)^4\left(\overline{\varphi(\omega)}\right)^n}{\left(1-z\overline{\varphi(\omega)}\right)^{n+4}\log\frac{2}{1-|\varphi(\omega)|}};\end{aligned} \tag{17}$$

$$\begin{aligned} f_\omega^{(n+1)}(z) &= a(n+1)!\frac{(1-|\varphi(\omega)|^2)\left(\overline{\varphi(\omega)}\right)^{n+1}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+2}\log\frac{2}{1-|\varphi(\omega)|}} + b(n+2)!\frac{(1-|\varphi(\omega)|^2)^2\left(\overline{\varphi(\omega)}\right)^{n+1}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+3}\log\frac{2}{1-|\varphi(\omega)|}} \\ &+c\frac{(n+3)!}{2}\frac{(1-|\varphi(\omega)|^2)^3\left(\overline{\varphi(\omega)}\right)^{n+1}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+4}\log\frac{2}{1-|\varphi(\omega)|}} + \frac{(n+4)!}{6}\frac{(1-|\varphi(\omega)|^2)^4\left(\overline{\varphi(\omega)}\right)^{n+1}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+5}\log\frac{2}{1-|\varphi(\omega)|}}; \end{aligned} \tag{18}$$

$$\begin{aligned} f_\omega^{(n+2)}(z) &= a(n+2)!\frac{(1-|\varphi(\omega)|^2)\left(\overline{\varphi(\omega)}\right)^{n+2}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+3}\log\frac{2}{1-|\varphi(\omega)|}} + b(n+3)!\frac{(1-|\varphi(\omega)|^2)^2\left(\overline{\varphi(\omega)}\right)^{n+2}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+4}\log\frac{2}{1-|\varphi(\omega)|}} \\ &+c\frac{(n+4)!}{2}\frac{(1-|\varphi(\omega)|^2)^3\left(\overline{\varphi(\omega)}\right)^{n+2}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+5}\log\frac{2}{1-|\varphi(\omega)|}} + \frac{(n+5)!}{6}\frac{(1-|\varphi(\omega)|^2)^4\left(\overline{\varphi(\omega)}\right)^{n+2}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+6}\log\frac{2}{1-|\varphi(\omega)|}}; \end{aligned} \tag{19}$$

$$\begin{aligned} f_\omega^{(n+3)}(z) &= a(n+3)!\frac{(1-|\varphi(\omega)|^2)\left(\overline{\varphi(\omega)}\right)^{n+3}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+4}\log\frac{2}{1-|\varphi(\omega)|}} + b(n+4)!\frac{(1-|\varphi(\omega)|^2)^2\left(\overline{\varphi(\omega)}\right)^{n+3}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+5}\log\frac{2}{1-|\varphi(\omega)|}} \\ &+c\frac{(n+5)!}{2}\frac{(1-|\varphi(\omega)|^2)^3\left(\overline{\varphi(\omega)}\right)^{n+3}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+6}\log\frac{2}{1-|\varphi(\omega)|}} + \frac{(n+6)!}{6}\frac{(1-|\varphi(\omega)|^2)^4\left(\overline{\varphi(\omega)}\right)^{n+3}}{\left(1-z\overline{\varphi(\omega)}\right)^{n+7}\log\frac{2}{1-|\varphi(\omega)|}}; \end{aligned} \tag{20}$$

By Lemma 2.2 we have

$$\begin{aligned} &\sup_{z\in\mathbb{D}}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|f_\omega'(z)| \\ &\leq C\sup_{z\in\mathbb{D}}(1-|z|)\left(\log\frac{2}{1-|z|}\right)\frac{(1-|\varphi(\omega)|)}{\left(1-\left|\overline{\varphi(\omega)}\right|\right)\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\log\frac{2}{1-|\varphi(\omega)|}} \\ &+C\sup_{z\in\mathbb{D}}(1-|z|)\left(\log\frac{2}{1-|z|}\right)\frac{(1-|\varphi(\omega)|)^2}{\left(1-\left|\overline{\varphi(\omega)}\right|\right)^2\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\log\frac{2}{1-|\varphi(\omega)|}} \\ &+C\sup_{z\in\mathbb{D}}(1-|z|)\left(\log\frac{2}{1-|z|}\right)\frac{(1-|\varphi(\omega)|)^3}{\left(1-\left|\overline{\varphi(\omega)}\right|\right)^3\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\log\frac{2}{1-|\varphi(\omega)|}} \\ &+C\sup_{z\in\mathbb{D}}(1-|z|)\left(\log\frac{2}{1-|z|}\right)\frac{(1-|\varphi(\omega)|)^4}{\left(1-\left|\overline{\varphi(\omega)}\right|\right)^4\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\log\frac{2}{1-|\varphi(\omega)|}} \\ &\leq C\sup_{z\in\mathbb{D}}\frac{(1-|z|)\log\frac{2}{1-|z|}}{\left(1-\left|z\overline{\varphi(\omega)}\right|\right)\log\frac{2}{1-\left|z\overline{\varphi(\omega)}\right|}} \\ &\leq 2C, \end{aligned} \tag{21}$$

hence $f_\omega \in \mathcal{B}_{\log}$ and $\sup_{\omega\in\mathbb{D}}\|f_\omega\|_{\mathcal{B}_{\log}} \leq 2C$.

On the other hand, for each fix $\omega \in \mathbb{D}$, by (21) we obtain that

$$(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|f_\omega'(z)| \leq C\frac{(1-|z|)\log\frac{2}{1-|z|}}{(1-|\varphi(\omega)|)\log 2} \to 0 \text{ (as } |z|\to 1), \tag{22}$$

it follows that $f_\omega \in \mathcal{B}_{\log,0}$ for each fix $\omega \in \mathbb{D}$.

For a system of linear equations

$$\begin{aligned} a + (n+2)b + (n+2)(n+3)c/2 &= -(n+2)(n+3)(n+4)/6, \\ a + (n+3)b + (n+3)(n+4)c/2 &= -(n+3)(n+4)(n+5)/6, \\ a + (n+4)b + (n+4)(n+5)c/2 &= -(n+4)(n+5)(n+6)/6, \end{aligned} \tag{23}$$

since

$$\begin{vmatrix} 1 & n+2 & (n+2)(n+3)/2 \\ 1 & n+3 & (n+3)(n+4)/2 \\ 1 & n+4 & (n+4)(n+5)/2 \end{vmatrix} = 1 \neq 0,$$

the system (23) by Cramer's Rule has non-zero solution. From (17), (18), (19) and (20), there are constants a, b, c, such that $f_\omega^{(n+1)}(\varphi(\omega)) = f_\omega^{(n+2)}(\varphi(\omega)) = f_\omega^{(n+3)}(\varphi(\omega)) = 0$ and

$$f_\omega^{(n)}(\varphi(\omega)) = C_1(a,b,c,n)\frac{\left(\overline{\varphi(\omega)}\right)^n}{(1-|\varphi(\omega)|^2)^n \log \frac{2}{1-|\varphi(\omega)|}},$$

where $C_1(a,b,c,n) = an! + b(n+1)! + \frac{c(n+2)!}{2} + \frac{(n+3)!}{6} \neq 0$. Hence for the test functions f_ω, where $\omega \in \mathbb{D}$ and $\varphi(\omega) \neq 0$, we get

$$\begin{aligned} C &\geq \|T^n_{\psi_1,\psi_2,\varphi} f_\omega\|_{\mathcal{Z}_\mu} \\ &\geq \mu(\omega)\left|\psi_1''(\omega) f_\omega^{(n)}(\varphi(\omega))\right| \\ &= |C_1(a,b,c,n)| \frac{\mu(\omega)|\psi_1''(\omega)|\left|\overline{\varphi(\omega)}\right|^n}{\left(1-\left|\varphi(\omega)\right|^2\right)^n \log\frac{2}{1-|\varphi(\omega)|}}. \end{aligned} \tag{24}$$

By (24), we obtain that

$$\begin{aligned} &\sup_{\frac{1}{2}<|\varphi(\omega)|<1} \frac{\mu(\omega)|\psi_1''(\omega)|}{\left(1-\left|\varphi(\omega)\right|^2\right)^n \log\frac{2}{1-|\varphi(\omega)|}} \\ &\leq 2^n \sup_{\frac{1}{2}<|\varphi(\omega)|<1} \frac{\mu(\omega)|\psi_1''(\omega)|\left|\overline{\varphi(\omega)}\right|^n}{\left(1-\left|\varphi(\omega)\right|^2\right)^n \log\frac{2}{1-|\varphi(\omega)|}} \\ &\leq 2^n \sup_{\omega\in\mathbb{D}} \frac{\mu(\omega)|\psi_1''(\omega)|\left|\overline{\varphi(\omega)}\right|^n}{\left(1-\left|\varphi(\omega)\right|^2\right)^n \log\frac{2}{1-|\varphi(\omega)|}} \\ &\leq C < \infty. \end{aligned} \tag{25}$$

And from (9), we have

$$\begin{aligned} &\sup_{|\varphi(\omega)|\leq\frac{1}{2}} \frac{\mu(\omega)|\psi_1''(\omega)|}{\left(1-\left|\varphi(\omega)\right|^2\right)^n \log\frac{2}{1-|\varphi(\omega)|}} \\ &\leq \sup_{|\varphi(\omega)|\leq\frac{1}{2}} \frac{\mu(\omega)|\psi_1''(\omega)|}{\left(1-\left|\varphi(\omega)\right|^2\right)^n \log 2} \\ &\leq \left(\frac{4}{3}\right)^n \frac{1}{\log 2} \sup_{|\varphi(\omega)|\leq\frac{1}{2}} \mu(\omega)|\psi_1''(\omega)| \\ &\leq \left(\frac{4}{3}\right)^n \frac{K_1}{\log 2} < \infty. \end{aligned} \tag{26}$$

Thus from (26) with (25) we see that (1) holds.

Since

$$\begin{vmatrix} 1 & n+1 & (n+1)(n+2)/2 \\ 1 & n+3 & (n+3)(n+4)/2 \\ 1 & n+4 & (n+4)(n+5)/2 \end{vmatrix} = 3 \neq 0,$$

from (17), (18), (19) and (20), there are constants a, b, c in (16) such that $g_\omega^{(n)}(\varphi(\omega)) = g_\omega^{(n+2)}(\varphi(\omega)) = g_\omega^{(n+3)}(\varphi(\omega)) = 0$ and

$$g_\omega^{(n+1)}(\varphi(\omega)) = C_2(a,b,c,n)\frac{\left(\overline{\varphi(\omega)}\right)^{n+1}}{(1-|\varphi(\omega)|^2)^{n+1}\log\frac{2}{1-|\varphi(\omega)|}},$$

where $C_2(a,b,c,n) = a(n+1)! + b(n+2)! + \frac{c(n+3)!}{2} + \frac{(n+4)!}{6} \neq 0$ and g_ω denotes the corresponding function. Therefore, for g_ω, where $\omega \in \mathbb{D}$ and $\varphi(\omega) \neq 0$, we get

$$\begin{aligned} C &\geq \|T^n_{\psi_1,\psi_2,\varphi} g_\omega\|_{\mathcal{Z}_\mu} \\ &\geq \mu(\omega)\left|\psi_1(\omega)\varphi''(\omega) + 2\psi_1'(\omega)\varphi'(\omega) + \psi_2''(\omega)\right|\left|g_\omega^{(n+1)}(\varphi(\omega))\right| \\ &= |C_2(a,b,c,n)|\frac{\mu(\omega)\left|\psi_1(\omega)\varphi''(\omega) + 2\psi_1'(\omega)\varphi'(\omega) + \psi_2''(\omega)\right|\left|\overline{\varphi(\omega)}\right|^{n+1}}{\left(1-\left|\varphi(\omega)\right|^2\right)^{n+1}\log\frac{2}{1-|\varphi(\omega)|}}. \end{aligned} \tag{27}$$

From (27), we obtain

$$\begin{aligned} &\sup_{\frac{1}{2}<|\varphi(\omega)|<1}\frac{\mu(\omega)|\psi_1(\omega)\varphi''(\omega) + 2\psi_1'(\omega)\varphi'(\omega) + \psi_2''(\omega)|}{\left(1-\left|\varphi(\omega)\right|^2\right)^{n+1}\log\frac{2}{1-|\varphi(\omega)|}} \\ &\leq 2^{n+1}\sup_{\frac{1}{2}<|\varphi(\omega)|<1}\frac{\mu(\omega)|\psi_1(\omega)\varphi''(\omega) + 2\psi_1'(\omega)\varphi'(\omega) + \psi_2''(\omega)|\left|\overline{\varphi(\omega)}\right|^{n+1}}{\left(1-\left|\varphi(\omega)\right|^2\right)^{n+1}\log\frac{2}{1-|\varphi(\omega)|}} \\ &\leq 2^{n+1}\sup_{\omega\in\mathbb{D}}\frac{\mu(\omega)|\psi_1(\omega)\varphi''(\omega) + 2\psi_1'(\omega)\varphi'(\omega) + \psi_2''(\omega)|\left|\overline{\varphi(\omega)}\right|^{n+1}}{\left(1-\left|\varphi(\omega)\right|^2\right)^{n+1}\log\frac{2}{1-|\varphi(\omega)|}} \\ &\leq C < \infty. \end{aligned} \tag{28}$$

By (11), we see that

$$\begin{aligned} &\sup_{|\varphi(\omega)|\leq\frac{1}{2}}\frac{\mu(\omega)|\psi_1(\omega)\varphi''(\omega) + 2\psi_1'(\omega)\varphi'(\omega) + \psi_2''(\omega)|}{\left(1-\left|\varphi(\omega)\right|^2\right)^{n+1}\log\frac{2}{1-|\varphi(\omega)|}} \\ &\leq \sup_{|\varphi(\omega)|\leq\frac{1}{2}}\frac{\mu(\omega)|\psi_1(\omega)\varphi''(\omega) + 2\psi_1'(\omega)\varphi'(\omega) + \psi_2''(\omega)|}{\left(1-\left|\varphi(\omega)\right|^2\right)^{n+1}\log 2} \\ &\leq \left(\frac{4}{3}\right)^{n+1}\frac{1}{\log 2}\sup_{|\varphi(\omega)|\leq\frac{1}{2}}\mu(\omega)|\psi_1(\omega)\varphi''(\omega) + 2\psi_1'(\omega)\varphi'(\omega) + \psi_2''(\omega)| \\ &\leq \left(\frac{4}{3}\right)^{n+1}\frac{K_2}{\log 2} < \infty. \end{aligned} \tag{29}$$

Thus combining (28) with (29) we get the condition (2).

Next, we prove that (3). Since

$$\begin{vmatrix} 1 & n+1 & (n+1)(n+2)/2 \\ 1 & n+2 & (n+2)(n+3)/2 \\ 1 & n+4 & (n+4)(n+5)/2 \end{vmatrix} = 3 \neq 0,$$

from (17), (18), (19) and (20), there are constants a, b, c in (16) such that $h_\omega^{(n)}(\varphi(\omega)) = h_\omega^{(n+1)}(\varphi(\omega)) = h_\omega^{(n+3)}(\varphi(\omega)) = 0$ and

$$h_\omega^{(n+2)}(\varphi(\omega)) = C_3(a,b,c,n)\frac{\left(\overline{\varphi(\omega)}\right)^{n+2}}{(1-|\varphi(\omega)|^2)^{n+2}\log\frac{2}{1-|\varphi(\omega)|}},$$

where $C_3(a,b,c,n) = a(n+2)! + b(n+3)! + \frac{c(n+4)!}{2} + \frac{(n+5)!}{6} \neq 0$ and h_ω denotes the corresponding function. Hence for h_ω, where $\omega \in \mathbb{D}$ and $\varphi(\omega) \neq 0$, we get

$$\begin{aligned} C &\geq \|T^n_{\psi_1,\psi_2,\varphi} h_\omega\|_{\mathcal{Z}_\mu} \\ &\geq \mu(\omega)\left|\psi_1(\omega)(\varphi'(\omega))^2 + 2\psi_2'(\omega)\varphi'(\omega) + \psi_2(\omega)\varphi''(\omega)\right|\left|h_\omega^{(n+2)}(\varphi(\omega))\right| \\ &= |C_3(a,b,c,n)|\frac{\mu(\omega)\left|\psi_1(\omega)(\varphi'(\omega))^2 + 2\psi_2'(\omega)\varphi'(\omega) + \psi_2(\omega)\varphi''(\omega)\right|\left|\overline{\varphi(\omega)}\right|^{n+2}}{\left(1-\left|\varphi(\omega)\right|^2\right)^{n+2}\log\frac{2}{1-|\varphi(\omega)|}}. \end{aligned} \tag{30}$$

From (30) it follows that

$$\begin{aligned} &\sup_{\frac{1}{2}<|\varphi(\omega)|<1}\frac{\mu(\omega)\left|\psi_1(\omega)(\varphi'(\omega))^2 + 2\psi_2'(\omega)\varphi'(\omega) + \psi_2(\omega)\varphi''(\omega)\right|}{\left(1-\left|\varphi(\omega)\right|^2\right)^{n+2}\log\frac{2}{1-|\varphi(\omega)|}} \\ &\leq 2^{n+2}\sup_{\frac{1}{2}<|\varphi(\omega)|<1}\frac{\mu(\omega)\left|\psi_1(\omega)(\varphi'(\omega))^2 + 2\psi_2'(\omega)\varphi'(\omega) + \psi_2(\omega)\varphi''(\omega)\right|\left|\overline{\varphi(\omega)}\right|^{n+2}}{\left(1-\left|\varphi(\omega)\right|^2\right)^{n+2}\log\frac{2}{1-|\varphi(\omega)|}} \\ &\leq 2^{n+2}\sup_{\omega\in\mathbb{D}}\frac{\mu(\omega)\left|\psi_1(\omega)(\varphi'(\omega))^2 + 2\psi_2'(\omega)\varphi'(\omega) + \psi_2(\omega)\varphi''(\omega)\right|\left|\overline{\varphi(\omega)}\right|^{n+2}}{\left(1-\left|\varphi(\omega)\right|^2\right)^{n+2}\log\frac{2}{1-|\varphi(\omega)|}} \\ &\leq C < \infty. \end{aligned} \tag{31}$$

Using (13), we have

$$\begin{aligned} &\sup_{|\varphi(\omega)|\leq\frac{1}{2}}\frac{\mu(\omega)|\psi_1(\omega)(\varphi'(\omega))^2 + 2\psi_2'(\omega)\varphi'(\omega) + \psi_2(\omega)\varphi''(\omega)|}{\left(1-\left|\varphi(\omega)\right|^2\right)^{n+2}\log\frac{2}{1-|\varphi(\omega)|}} \\ &\leq \sup_{|\varphi(\omega)|\leq\frac{1}{2}}\frac{\mu(\omega)|\psi_1(\omega)(\varphi'(\omega))^2 + 2\psi_2'(\omega)\varphi'(\omega) + \psi_2(\omega)\varphi''(\omega)|}{\left(1-\left|\varphi(\omega)\right|^2\right)^{n+2}\log 2} \\ &\leq \left(\frac{4}{3}\right)^{n+2}\frac{1}{\log 2}\sup_{|\varphi(\omega)|\leq\frac{1}{2}}\mu(\omega)\left|\psi_1(\omega)(\varphi'(\omega))^2 + 2\psi_2'(\omega)\varphi'(\omega) + \psi_2(\omega)\varphi''(\omega)\right| \\ &\leq \left(\frac{4}{3}\right)^{n+2}\frac{K_3}{\log 2} < \infty. \end{aligned} \tag{32}$$

From (31) and (32), condition (3) follows, as desired.

Finally, we prove that (4). Since

$$\begin{vmatrix} 1 & n+1 & (n+1)(n+2)/2 \\ 1 & n+2 & (n+2)(n+3)/2 \\ 1 & n+3 & (n+3)(n+4)/2 \end{vmatrix} = 1 \neq 0,$$

from (17), (18), (19) and (20), there are constants a, b, c in (16) such that $k_\omega^{(n)}(\varphi(\omega)) = k_\omega^{(n+1)}(\varphi(\omega)) = k_\omega^{(n+2)}(\varphi(\omega)) = 0$ and

$$k_\omega^{(n+3)}(\varphi(\omega)) = C_4(a, b, c, n)\frac{\left(\overline{\varphi(\omega)}\right)^{n+3}}{(1 - |\varphi(\omega)|^2)^{n+3}\log\frac{2}{1-|\varphi(\omega)|}},$$

where $C_4(a, b, c, n) = a(n+3)! + b(n+4)! + \frac{c(n+5)!}{2} + \frac{(n+6)!}{6} \neq 0$ and k_ω denotes the corresponding function. Hence for k_ω, where $\omega \in \mathbb{D}$ and $\varphi(\omega) \neq 0$, we get

$$\begin{aligned} C &\geq \|T^n_{\psi_1,\psi_2,\varphi}k_\omega\|_{\mathcal{Z}_\mu} \\ &\geq \mu(\omega)|\psi_2(\omega)||\varphi'(\omega)|^2 \left|h_\omega^{(n+3)}(\varphi(\omega))\right| \\ &= |C_4(a, b, c, n)|\frac{\mu(\omega)|\psi_2(\omega)||\varphi'(\omega)|^2 \left|\overline{\varphi(\omega)}\right|^{n+3}}{\left(1 - \left|\varphi(\omega)\right|^2\right)^{n+3}\log\frac{2}{1-|\varphi(\omega)|}}. \end{aligned} \tag{33}$$

By (33), we obtain that

$$\begin{aligned} &\sup_{\frac{1}{2}<|\varphi(\omega)|<1}\frac{\mu(\omega)|\psi_2(\omega)||\varphi'(\omega)|^2}{\left(1 - \left|\varphi(\omega)\right|^2\right)^{n+2}\log\frac{2}{1-|\varphi(\omega)|}} \\ &\leq 2^{n+3}\sup_{\frac{1}{2}<|\varphi(\omega)|<1}\frac{\mu(\omega)|\psi_2(\omega)||\varphi'(\omega)|^2 \left|\overline{\varphi(\omega)}\right|^{n+3}}{\left(1 - \left|\varphi(\omega)\right|^2\right)^{n+3}\log\frac{2}{1-|\varphi(\omega)|}} \\ &\leq 2^{n+3}\sup_{\omega\in\mathbb{D}}\frac{\mu(\omega)|\psi_2(\omega)||\varphi'(\omega)|^2 \left|\overline{\varphi(\omega)}\right|^{n+3}}{\left(1 - \left|\varphi(\omega)\right|^2\right)^{n+3}\log\frac{2}{1-|\varphi(\omega)|}} \\ &\leq C < \infty. \end{aligned} \tag{34}$$

On the other hand, by using (15), we have

$$\begin{aligned} &\sup_{|\varphi(\omega)|\leq\frac{1}{2}}\frac{\mu(\omega)|\psi_2(\omega)||\varphi'(\omega)|^2}{\left(1 - \left|\varphi(\omega)\right|^2\right)^{n+3}\log\frac{2}{1-|\varphi(\omega)|}} \\ &\leq \sup_{|\varphi(\omega)|\leq\frac{1}{2}}\frac{\mu(\omega)|\psi_2(\omega)||\varphi'(\omega)|^2}{\left(1 - \left|\varphi(\omega)\right|^2\right)^{n+3}\log 2} \\ &\leq \left(\frac{4}{3}\right)^{n+3}\frac{1}{\log 2}\sup_{|\varphi(\omega)|\leq\frac{1}{2}}\mu(\omega)|\psi_2(\omega)||\varphi'(\omega)|^2 \\ &\leq \left(\frac{4}{3}\right)^{n+3}\frac{K_4}{\log 2} < \infty. \end{aligned} \tag{35}$$

Hence, (34) and (35) imply (4), completing the proof of the theorem. Note that we have used the fact that the functions $g_\omega, h_\omega, k_\omega \in \mathcal{B}_{\log,0}$ for each fix $\omega \in \mathbb{D}$.

Theorem 3.2. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$, φ be an analytic self-map of $\mathbb{D}$, n a positive integer, and μ a weight. Then the following statements are equivalent.*

(1) $T^n_{\psi_1,\psi_2,\varphi} : \mathcal{B}_{\log} \to \mathcal{Z}_\mu$ *is compact;*

(2) $T^n_{\psi_1,\psi_2,\varphi} : \mathcal{B}_{\log,0} \to \mathcal{Z}_\mu$ *is compact;*

(3) $T^n_{\psi_1,\psi_2,\varphi} : \mathcal{B}_{\log} \to \mathcal{Z}_\mu$ *is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\psi_1''(z)|}{\left(1-\left|\varphi(z)\right|^2\right)^n \log\frac{2}{1-|\varphi(z)|}} = 0, \tag{36}$$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\psi_1(z)\varphi''(z)+2\psi_1'(z)\varphi'(z)+\psi_2''(z)|}{\left(1-\left|\varphi(z)\right|^2\right)^{n+1} \log\frac{2}{1-|\varphi(z)|}} = 0, \tag{37}$$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\psi_1(z)(\varphi'(z))^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z)|}{\left(1-\left|\varphi(z)\right|^2\right)^{n+2} \log\frac{2}{1-|\varphi(z)|}} = 0, \tag{38}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\psi_2(z)||\varphi'(z)|^2}{\left(1-\left|\varphi(z)\right|^2\right)^{n+3} \log\frac{2}{1-|\varphi(z)|}} = 0. \tag{39}$$

Proof. (3) $\Rightarrow$ (1). Assume that $T^n_{\psi_1,\psi_2,\varphi} : \mathcal{B}_{\log} \to \mathcal{Z}_\mu$ is bounded, and that conditions (36), (37) and (38) hold. For any bounded sequence $\{f_k\}$ in $\mathcal{B}_{\log}$ with $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$. By Lemma 2.3 we have to show that

$$\|T^n_{\psi_1,\psi_2,\varphi} f_k\|_{\mathcal{Z}_\mu} \to 0, \text{ if } k \to \infty.$$

We may assume that $\|f_k\|_{\mathcal{B}_{\log}} \le 1$ for every $k \in \mathbb{N}$. Let us fix $\varepsilon > 0$. From (36), (37), (38) and (39) there exists $\rho \in (0,1)$ such that

$$\frac{\mu(z)|\psi_1''(z)|}{\left(1-\left|\varphi(z)\right|^2\right)^n \log\frac{2}{1-|\varphi(z)|}} < \varepsilon, \tag{40}$$

$$\frac{\mu(z)|\psi_1(z)\varphi''(z)+2\psi_1'(z)\varphi'(z)+\psi_2''(z)|}{\left(1-\left|\varphi(z)\right|^2\right)^{n+1} \log\frac{2}{1-|\varphi(z)|}} < \varepsilon, \tag{41}$$

$$\frac{\mu(z)\left|\psi_1(z)(\varphi'(z))^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{n+2} \log\frac{2}{1-|\varphi(z)|}} < \varepsilon, \tag{42}$$

and

$$\frac{\mu(z)|\psi_2(z)||\varphi'(z)|^2}{\left(1-\left|\varphi(z)\right|^2\right)^{n+3} \log\frac{2}{1-|\varphi(z)|}} < \varepsilon, \tag{43}$$

if $\rho < |\varphi(z)| < 1$. Since $T^n_{\psi_1,\psi_2,\varphi} : \mathcal{B}_{\log} \to \mathcal{Z}_\mu$ is bounded, thus (9), (11), (13) and (15) hold by Theorem 3.1. Since $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, Cauchy's estimate implies that $f_k^{(n)}$, $f_k^{(n+1)}$, $f_k^{(n+2)}$ and $f_k^{(n+3)}$

converges to 0 uniformly on compact subsets of $\mathbb{D}$, there exists a $K_0 \in \mathbb{N}$ such that

$$\begin{aligned}
&\left|(T^n_{\psi_1,\psi_2,\varphi} f_k)(0)\right| + \left|(T^n_{\psi_1,\psi_2,\varphi} f_k)'(0)\right| + \sup_{|\varphi(z)|\le\rho} \mu(z)\left|(T^n_{\psi_1,\psi_2,\varphi} f_k)''(z)\right| \\
&\le |\psi_1(0)|\left|f_k^{(n)}(\varphi(0))\right| + \left|\psi_2(0) f_k^{(n+1)}(\varphi(0))\right| + |\psi_1'(0)|\left|f_k^{(n)}(\varphi(0))\right| \\
&+|\psi_1(0)\varphi'(0)|\left|f_k^{(n+1)}(\varphi(0))\right| + \left|\psi_2(0)\varphi'(0) f_k^{(n+2)}(\varphi(0))\right| \\
&+ \sup_{|\varphi(z)|\le\rho} \mu(z)|\psi_1''(z)|\left|f_k^{(n)}(\varphi(z))\right| \\
&+ \sup_{|\varphi(z)|\le\rho} \mu(z)|\psi_1(z)\varphi''(z) + 2\psi_1'(z)\varphi'(z) + \psi_2''(z)|\left|f_k^{(n+1)}(\varphi(z))\right| \\
&+ \sup_{|\varphi(z)|\le\rho} \mu(z)\left|\psi_1(z)(\varphi'(z))^2 + 2\psi_2'(z)\varphi'(z) + \psi_2(z)\varphi''(z)\right|\left|f_k^{(n+2)}(\varphi(z))\right| \\
&+ \sup_{|\varphi(z)|\le\rho} \mu(z)|\psi_2(z)||\varphi'(z)|^2\left|f_k^{(n+3)}(\varphi(z))\right| \\
&\le C\varepsilon + K_1 \sup_{|\varphi(z)|\le\rho}\left|f_k^{(n)}(\varphi(z))\right| + K_2 \sup_{|\varphi(z)|\le\rho}\left|f_k^{(n+1)}(\varphi(z))\right| \\
&+K_3 \sup_{|\varphi(z)|\le\rho}\left|f_k^{(n+2)}(\varphi(z))\right| + K_4 \sup_{|\varphi(z)|\le\rho}\left|f_k^{(n+3)}(\varphi(z))\right| \\
&< C\varepsilon, \qquad (44)
\end{aligned}$$

whenever $k > K_0$. From (40), (41), (42), (43), (44) and Lemma 2.1 we have

$$\begin{aligned}
&\|T^n_{\psi_1,\psi_2,\varphi} f_k\|_{\mathcal{Z}_\mu} \\
&= \left|\left(T^n_{\psi_1,\psi_2,\varphi} f_k\right)(0)\right| + \left|\left(T^n_{\psi_1,\psi_2,\varphi} f_k\right)'(0)\right| + \sup_{z\in\mathbb{D}} \mu(z)\left|\left(T^n_{\psi_1,\psi_2,\varphi} f_k\right)''(z)\right| \\
&\le \left|\left(T^n_{\psi_1,\psi_2,\varphi} f_k\right)(0)\right| + \left|\left(T^n_{\psi_1,\psi_2,\varphi} f_k\right)'(0)\right| \\
&+ \sup_{|\varphi(z)|\le\rho} \mu(z)\left|\left(T^n_{\psi_1,\psi_2,\varphi} f_k\right)''(z)\right| + \sup_{\rho<|\varphi(z)|<1} \mu(z)\left|\left(T^n_{\psi_1,\psi_2,\varphi} f_k\right)''(z)\right| \\
&< C\varepsilon + C \sup_{\rho<|\varphi(z)|<1} \frac{\mu(z)|\psi_1''(z)|}{\left(1-|\varphi(z)|^2\right)^n \log\frac{2}{1-|\varphi(z)|}}\|f\|_{\mathcal{B}_{\log}} \\
&+C \sup_{\rho<|\varphi(z)|<1} \frac{\mu(z)|\psi_1(z)\varphi''(z) + 2\psi_1'(z)\varphi'(z) + \psi_2''(z)|}{\left(1-|\varphi(z)|^2\right)^{n+1} \log\frac{2}{1-|\varphi(z)|}}\|f\|_{\mathcal{B}_{\log}} \\
&+C \sup_{\rho<|\varphi(z)|<1} \frac{\mu(z)|\psi_1(z)(\varphi'(z))^2 + 2\psi_2'(z)\varphi'(z) + \psi_2(z)\varphi''(z)|}{\left(1-|\varphi(z)|^2\right)^{n+2} \log\frac{2}{1-|\varphi(z)|}}\|f\|_{\mathcal{B}_{\log}} \\
&+C \sup_{\rho<|\varphi(z)|<1} \frac{\mu(z)|\psi_2(z)||\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^{n+3} \log\frac{2}{1-|\varphi(z)|}}\|f\|_{\mathcal{B}_{\log}} \\
&< 4C\varepsilon, \qquad (45)
\end{aligned}$$

whenever $k > K_0$. Hence $T^n_{\psi_1,\psi_2,\varphi} : \mathcal{B}_{\log} \to \mathcal{Z}_\mu$ is compact.

(1) ⇒ (2). It is obvious.

(2) ⇒ (3). Assume that $T^n_{\psi_1,\psi_2,\varphi} : \mathcal{B}_{\log,0} \to \mathcal{Z}_\mu$ is compact. Then it is clear that $T^n_{\psi_1,\psi_2,\varphi} : \mathcal{B}_{\log,0} \to \mathcal{Z}_\mu$ is bounded. By Theorem 3.1 we get that $T^n_{\psi_1,\psi_2,\varphi} : \mathcal{B}_{\log} \to \mathcal{Z}_\mu$ is bounded. Let $\{z_k\}$ be a sequence in $\mathbb{D}$ such

that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Set

$$f_k(z) = f_{z_k}(z) = a\frac{1-|\varphi(z_k)|^2}{\left(1-z\overline{\varphi(z_k)}\right)\log\frac{2}{1-|\varphi(z_k)|}} + b\frac{(1-|\varphi(z_k)|^2)^2}{\left(1-z\overline{\varphi(z_k)}\right)^2\log\frac{2}{1-|\varphi(z_k)|}}$$
$$+c\frac{(1-|\varphi(z_k)|^2)^3}{\left(1-z\overline{\varphi(z_k)}\right)^3\log\frac{2}{1-|\varphi(z_k)|}} + \frac{(1-|\varphi(z_k)|^2)^4}{\left(1-z\overline{\varphi(z_k)}\right)^4\log\frac{2}{1-|\varphi(z_k)|}}.$$

Note that

$$|f_k(z)| \le \left|a\frac{(1-|\varphi(z_k)|^2)}{\left(1-z\overline{\varphi(z_k)}\right)\log\frac{2}{1-|\varphi(z_k)|}}\right| + \left|b\frac{(1-|\varphi(z_k)|^2)^2}{\left(1-z\overline{\varphi(z_k)}\right)^2\log\frac{2}{1-|\varphi(z_k)|}}\right|$$
$$+\left|c\frac{(1-|\varphi(z_k)|^2)^3}{\left(1-z\overline{\varphi(z_k)}\right)^3\log\frac{2}{1-|\varphi(z_k)|}}\right| + \left|\frac{(1-|\varphi(z_k)|^2)^4}{\left(1-z\overline{\varphi(z_k)}\right)^4\log\frac{2}{1-|\varphi(z_k)|}}\right|$$
$$\le \frac{|a|(1+|\varphi(z_k)|)(1-|\varphi(z_k)|)}{(1-|\varphi(z_k)|)\log\frac{2}{1-|\varphi(z_k)|}} + \frac{|b|(1+|\varphi(z_k)|)^2(1-|\varphi(z_k)|)^2}{(1-|\varphi(z_k)|)^2\log\frac{2}{1-|\varphi(z_k)|}}$$
$$+\frac{|c|(1+|\varphi(z_k)|)^3(1-|\varphi(z_k)|)^3}{(1-|\varphi(z_k)|)^3\log\frac{2}{1-|\varphi(z_k)|}} + \frac{(1+|\varphi(z_k)|)^4(1-|\varphi(z_k)|)^4}{(1-|\varphi(z_k)|)^4\log\frac{2}{1-|\varphi(z_k)|}}$$
$$\le \frac{C}{\log\frac{2}{1-|\varphi(z_k)|}} \to 0\ (k\to\infty),$$

for $|z| < 1$. From which, (21) and (22), we see that f_k is a bounded sequence in $\mathcal{B}_{\log,0}$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$. By Lemma 2.3, we have

$$\lim_{k\to\infty} \|T^n_{\psi_1,\psi_2,\varphi} f_k\|_{\mathcal{Z}_\mu} = 0.$$

Note that

$$f_k^{(n+1)}(\varphi(z_k)) = f_k^{(n+2)}(\varphi(z_k)) = f_k^{(n+3)}(\varphi(z_k)) = 0,$$
$$f_k^{(n)}(\varphi(z_k)) = \frac{C_1(a,b,c,n)\left(\overline{\varphi(z_k)}\right)^n}{(1-|\varphi(z_k)|^2)^n\log\frac{2}{1-|\varphi(z_k)|}}.$$

From (24) and using the compactness of $T^n_{\psi_1,\psi_2,\varphi} : \mathcal{B}_{\log,0} \to \mathcal{Z}_\mu$ we obtain

$$|C_1(a,b,c,n)|\frac{\mu(z_k)|\psi_1''(z_k)|\left|\overline{\varphi(z_k)}\right|^n}{\left(1-\left|\varphi(z_k)\right|^2\right)^n\log\frac{2}{1-|\varphi(z_k)|}} \le \|T^n_{\psi_1,\psi_2,\varphi} f_k\|_{\mathcal{Z}_\mu} \to 0 \text{ as } k\to\infty. \tag{46}$$

From (46) and $|\varphi(z_k)| \to 1$, it follows that

$$\lim_{k\to\infty}\frac{\mu(z_k)|\psi_1''(z_k)|}{\left(1-\left|\varphi(z_k)\right|^2\right)^n\log\frac{2}{1-|\varphi(z_k)|}} = 0,$$

and consequently (36) holds. The idea and the process of the proof of (37), (38) and (39) is quite similar to that of (36) by using test functions $g_k(z) = g_{z_k}(z)$, $h_k(z) = h_{z_k}(z)$ and $k_k(z) = k_{z_k}(z)$, hence it will be omitted due to the space limitation. The details are left to interested readers.

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