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Positive Periodic Solutions for Second-Order Neutral Differential Equations with Time-Dependent Deviating Arguments

Zhibo Cheng[a,b], **Feifan Li**[a], **Shaowen Yao**[a]

[a] *School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, Henan 454000, PR China*
[b] *Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, PR China*

Abstract. In this paper, we consider a kind of second-order neutral differential equation with time-dependent deviating arguments. By applications of Krasnoselskii's fixed point theorem, sufficient conditions for the existence of positive periodic solutions are established.

1. Introduction

In this paper, we consider the following second-order neutral differential equation with time-dependent deviating arguments

$$(x(t) - cx(t-\tau))'' + a(t)x(t) = f(t, x(t-\delta(t))), \tag{1}$$

where c, τ are constants with $|c| \neq 1$ and $0 < \tau < \omega$, $a(t) \in C(\mathbb{R}, (0, +\infty))$, $\delta(t) \in C(\mathbb{R}, \mathbb{R})$, $a(t)$ and $\delta(t)$ are ω-periodic functions with t, $f(t, x) \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $f(t+\omega, \cdot) \equiv f(t, \cdot)$, here ω is a positive constant.

Neutral differential equations manifest themselves in many fields. There has been a rapid growth of interest in neutral differential equations which appeared in the control models, blood cell production models and population models [3, 7]. For example, Bai and Xu [3] discussed the two-phase size-structured population model with infinite states-at-birth and distributed delay in birth process. In recent years, some good deal of works have been performed on the existence of periodic solutions of first-order and second-order neutral differential equations (see [4]-[6],[8]-[14]). In 2007, Wu and Wang [11] discussed a kind of second-order neutral delay differential equation

$$(x(t) - cx(t-\tau))'' + a(t)x(t) = \lambda b(t) f(x(t-\delta(t))), \tag{2}$$

which is a particular case of equation (1). By applications of a fixed point theorem in cones, some sufficient conditions of existence, multiplicity and nonexistence of positive periodic solutions were established with $c \in (-1, 0)$. Afterwards, Cheung, Ren and Han [4] investigated a kind of neutral differential equation

$$(x(t) - cx(t-\delta(t)))'' + a(t)x(t) = f(t, x(t-\delta(t))), \tag{3}$$

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Email addresses: czb_1982@126.com (Zhibo Cheng), lffwy126@126.com (Feifan Li), yaoshaowen@hpu.edu.com (Shaowen Yao)

where $|c| < 1$. Differing from equation (1), the deviating argument of equation (3) is the same as the time delay term. By means of Krasnoselskii's fixed point theorem, they obtained sufficient conditions for the existence of periodic solutions to equation (3).

In the above papers, the authors investigated second-order neutral differential equations only in the case that $|c| < 1$. And to our knowledge, the case $|c| > 1$ has not been investigated until now. In this paper, we try to fill this gap and establish the existence of positive periodic solutions of equation (1) in the cases that $|c| < 1$ and $|c| > 1$ by employing the property of the neutral operator $(Ax)(t) := x(t) - cx(t-\tau)$ and applying Krasnoselskii's fixed point theorem. The techniques used are quite different from that in [4, 11] and our results are more general than those in [4, 11] in two aspects. Firstly, by using the property of the neutral operator $(Ax)(t)$, we give $f(t,x)$ condition which is weaker than the $\tilde{F}(t,x) := f(t, x(t-\delta(t))) - ca(t)x(t-\delta(t))$ condition in [4]. Secondly, we establish the existence of positive periodic solutions of equation (1) in the cases that $|c| < 1$ and $|c| > 1$.

The paper is organized as follows. In Section 2, firstly, the Green's function is given and some useful properties for the Green's function are obtained. Afterwards, we analyze qualitative properties of the neutral operator $(Ax)(t)$ which is helpful for further studies of differential equations. In Section 3, we get existence results of positive periodic solutions for equation (1) in the case that $c \in (-\frac{m}{m+M}, \frac{m}{m+M})$, here $M := \max\{a(t) : t \in [0,\omega]\}$ and $m := \min\{a(t) : t \in [0,\omega]\}$. We prove the existence criteria of periodic solutions for equation (1) through a basic application of Krasnoselskii's fixed point theorem. In Section 4, we investigate the existence of positive periodic solutions for equation (1) in the case that $c \in (-\infty, -1) \cup (1, +\infty)$. Our results extend and improve some corresponding results in [4, 11].

2. Preparation

Let

$$C_\omega := \{x \in C(\mathbb{R}, \mathbb{R}) : x(t+\omega) = x(t)\}$$

with norm $\|x\| := \max\limits_{t\in[0,\omega]} |x(t)|$. Clearly, $(C_\omega, \|\cdot\|)$ is a Banach space.

Define

$$C_\omega^+ := \{x \in C(\mathbb{R}, (0,+\infty)) : x(t+\omega) = x(t)\},$$

$$\beta := \sqrt{M}, \quad \kappa := \frac{2M + m - \sqrt{(2M+m)^2 - 4Mm}}{2M}.$$

Next, we show the following main theorem and lemmas which we need.

Theorem 2.1. *(see [2])(**Krasnoselskii's fixed point theorem**) Let C_ω be a Banach space. Assume that Ω is a bounded closed convex subset of C_ω. If $Q, S : \Omega \to C_\omega$ satisfy*

(1) $Qx + Sy \in \Omega$, $\forall\, x, y \in \Omega$,

(2) S is a contractive operator and Q is a completely continuous operator.

Then $Q + S$ has a fixed point in Ω.

Lemma 2.1. *If $M < \left(\frac{\pi}{\omega}\right)^2$, then the equation*

$$y''(t) + My(t) = h(t), \quad h(t) \in C_\omega^+, \tag{4}$$

has a unique ω-periodic solution

$$y(t) = \int_0^\omega G(t,s)h(s)ds,$$

where

$$G(t,s)=\begin{cases}\dfrac{\cos\beta(t-s-\frac{\omega}{2})}{2\beta\sin\frac{\beta\omega}{2}}, & 0\le s\le t\le\omega,\\ \dfrac{\cos\beta(t-s+\frac{\omega}{2})}{2\beta\sin\frac{\beta\omega}{2}}, & 0\le t<s\le\omega.\end{cases}$$

Proof. This lemma had been proved in [4], for convenience of readers, we present the proof as follows. Applying the method of variation of constant, we can get the general solution of equation (4), which has the following form $y(t)=c_1(t)\cos\beta t+c_2(t)\sin\beta t$. Therefore, we can get

$$c_1'(t)=\frac{-\sin\beta t}{\beta}h(t),\ \ c_2'(t)=\frac{\cos\beta t}{\beta}h(t).$$

Since $y(t),\ y'(t)$ are periodic functions, we have

$$\begin{aligned}c_1(t)&=\int_0^t\frac{\cos\beta(s+\frac{\omega}{2})}{2\beta\sin\frac{\beta\omega}{2}}h(s)ds+\int_t^\omega\frac{\cos\beta(s-\frac{\omega}{2})}{2\beta\sin\frac{\beta\omega}{2}}h(s)ds,\\ c_2(t)&=\int_0^t\frac{\sin\beta(s+\frac{\omega}{2})}{2\beta\sin\frac{\beta\omega}{2}}h(s)ds+\int_t^\omega\frac{\sin\beta(s-\frac{\omega}{2})}{2\beta\sin\frac{\beta\omega}{2}}h(s)ds.\end{aligned}$$

Thus, we can get the solution of equation (4),

$$\begin{aligned}y(t)=&c_1(t)\cos\beta t+c_2(t)\sin\beta t\\ =&\int_0^t\frac{\cos\beta(s+\frac{\omega}{2})\cos\beta t}{2\beta\sin\frac{\beta\omega}{2}}h(s)ds+\int_t^\omega\frac{\cos\beta(s-\frac{\omega}{2})\cos\beta t}{2\beta\sin\frac{\beta\omega}{2}}h(s)ds\\ &+\int_0^t\frac{\sin\beta(s+\frac{\omega}{2})\sin\beta t}{2\beta\sin\frac{\beta\omega}{2}}h(s)ds+\int_t^\omega\frac{\sin\beta(s-\frac{\omega}{2})\sin\beta t}{2\beta\sin\frac{\beta\omega}{2}}h(s)ds\\ =&\int_0^t\frac{\cos\beta(t-s-\frac{\omega}{2})}{2\beta\sin\frac{\beta\omega}{2}}h(s)ds+\int_t^\omega\frac{\cos\beta(t-s+\frac{\omega}{2})}{2\beta\sin\frac{\beta\omega}{2}}h(s)ds.\end{aligned}$$

One can observe that $y(t+\omega)=y(t)$, and we can get the Green's function, i.e.

$$G(t,s)=\begin{cases}\dfrac{\cos\beta(t-s-\frac{\omega}{2})}{2\beta\sin\frac{\beta\omega}{2}}, & 0\le s\le t\le\omega,\\ \dfrac{\cos\beta(t-s+\frac{\omega}{2})}{2\beta\sin\frac{\beta\omega}{2}}, & 0\le t<s\le\omega.\end{cases}$$

□

Lemma 2.2. *(see [4])* $\int_0^\omega G(t,s)ds=\frac{1}{M}$. *And if* $M<(\frac{\pi}{\omega})^2$, *then* $G(t,s)>0$ *for all* $t\in[0,\omega]$ *and* $s\in[0,\omega]$. *Furthermore,* $G(t,s)$ *is a differentiable function with* t.

Lemma 2.3. *(see [14]) If* $|c|\neq 1$, *then the operator* A *has a continuous inverse* A^{-1} *on* C_ω, *satisfying*

$$\left|\left(A^{-1}x\right)(t)\right|\le\frac{||x||}{|1-|c||},\quad\forall\, x\in C_\omega.$$

3. Positive periodic solutions of equation (1) in the case that $c \in (-\frac{m}{M+m}, \frac{m}{M+m})$

Let $y(t) = (Ax)(t)$, from Lemma 2.3, we have $x(t) = (A^{-1}y)(t)$. Hence, equation (1) can be transformed into

$$y''(t) + a(t)y(t) - a(t)H(y(t)) = f(t, x(t - \delta(t))), \tag{5}$$

where $H(y(t)) = -c(A^{-1}y)(t - \tau) = -cx(t - \tau)$. We consider

$$y''(t) + a(t)y(t) - a(t)H(y(t)) = h(t), \quad h \in C_\omega^+. \tag{6}$$

Define the operators $T, N : C_\omega \to C_\omega$ by

$$(Th)(t) = \int_0^\omega G(t,s)h(s)ds, \quad (Ny)(t) = (M - a(t))y(t) + a(t)H(y(t)). \tag{7}$$

Clearly, $(Th)(t) > 0$, for $h(t) > 0$ and $M < (\frac{\pi}{\omega})^2$ (see Lemma 2.2). From equality (7), the solution for equation (6) can be written as

$$y(t) = (Th)(t) + (TNy)(t). \tag{8}$$

In view of $c \in (-\frac{m}{M+m}, \frac{m}{M+m})$, we have

$$\|TN\| \le \|T\|\|N\| \le \frac{1}{M}\left(M - m + \frac{M|c|}{1-|c|}\right) \le \frac{M - m(1-|c|)}{M(1-|c|)} < 1, \tag{9}$$

since $\|T\| \le \frac{1}{M}$ (see Lemma 2.2). Hence, we get

$$y(t) = (I - TN)^{-1}(Th)(t). \tag{10}$$

Define an operator $P : C_\omega \to C_\omega$ by

$$(Ph)(t) = (I - TN)^{-1}(Th)(t). \tag{11}$$

Obviously, if $M < (\frac{\pi}{\omega})^2$, for any $h \in C_\omega^+$, $y(t) = (Ph)(t)$ is the unique positive ω-periodic solution of equation (6). Define $\sigma := \frac{l}{L}$, where l, L are the maximum and minimum of $G(t,s)$ on $\mathbb{R} \times \mathbb{R}$, we can get the following lemmas.

Lemma 3.1. *Assume that $c \in (-\frac{m}{M+m}, 0)$, $|c| \le \sigma$ and $M < (\frac{\pi}{\omega})^2$ hold. Then*

$$(Th)(t) \le (Ph)(t) \le \frac{M(1-|c|)}{m - (M+m)|c|}\|Th\|, \quad \textit{for all } h \in C_\omega^+.$$

Proof. From equality (11) and $\|TN\| < 1$, we have

$$\begin{aligned} P =&(I - TN)^{-1}T \\ =&(I + TN + (TN)^2 + (TN)^3 + \cdots)T \\ =&T + TNT + (TN)^2T + (TN)^3T + \cdots . \end{aligned} \tag{12}$$

For all $h(t) \in C_\omega^+$, from inequality (9), we can get

$$(Ph)(t) = (I - TN)^{-1}(Th)(t) \le \frac{\|Th\|}{I - \|TN\|} \le \frac{M(1-|c|)}{m - (M+m)|c|}\|Th\|.$$

Since $c \in (-\frac{m}{M+m}, 0)$, $|c| \le \sigma$ and $M < (\frac{\pi}{\omega})^2$, from equality (7), it is easy to verify that $(TNTh)(t) \ge 0$ if $h \in C_\omega^+$. then we have from equality (12) that $(Th)(t) \le (Ph)(t)$. □

Lemma 3.2. *Assume that $c \in (0, \frac{m}{M+m})$ and $M < (\frac{\pi}{\omega})^2$ hold. Then*

$$\frac{m-(M+m)c}{M(1-c)}(Th)(t) \leq (Ph)(t) \leq \frac{M(1-c)}{m-(M+m)c}\|Th\|, \quad \textit{for all } h \in C_\omega^+.$$

Proof. Since $\|TN\| < 1$, similarly as the proof of Lemma 3.1, we can get that $(Ph)(t) \leq \frac{M(1-c)}{m-(M+m)c}\|Th\|$.

Since $c \in (0, \frac{m}{M+m})$, we can not get $(TNTh)(t) \geq 0$ for all $h \in C_\omega^+$. From equality (12), we have

$$\begin{aligned} P &= (I + TN + (TN)^2 + (TN)^3 + \cdots)T \\ &= (I + (TN)^2 + (TN)^4 + \cdots)T + (TN + (TN)^3 + (TN)^5 + \cdots)T \\ &= (I + (TN)^2 + (TN)^4 + \cdots)T + (I + (TN)^2 + (TN)^4 + \cdots)TNT \\ &= (I + (TN)^2 + (TN)^4 + \cdots)(I + TN)T. \end{aligned}$$

Then, we can get

$$(Ph)(t) \geq (I + TN)(Th)(t) \geq (I - \|TN\|)(Th)(t) \geq \frac{m-(m+M)c}{M(1-c)}(Th)(t) > 0, \quad \text{for all } h \in C_\omega^+.$$

□

We consider the existence of periodic solutions for equation (1). Define operators $Q, S : C_\omega \to C_\omega$ by

$$(Qx)(t) = P(f(t, x(t-\delta(t)))), \quad (Sx)(t) = cx(t-\tau). \tag{13}$$

In view of equation (6) and the above analysis, the existence of periodic solutions for equation (1) is equivalent to the existence of periodic solutions for the operator equation

$$Qx + Sx = x \tag{14}$$

in C_ω.

Now, we present our results of equation (1) in the case that $c \in (-\frac{m}{M+m}, \frac{m}{M+m})$.

Theorem 3.1. *Suppose that $c \in (0, \frac{m}{M+m})$ and $M < (\frac{\pi}{\omega})^2$ hold. Furthermore, assume that the following condition is satisfied:*

(F_1) There exist two non-negative constants r and R such that

$$\left(\frac{M(1-c)}{m-(M+m)c}\right)^2 r < R$$

and

$$\frac{M^2(1-c)^2}{m-(M+m)c} r \leq f(t,x) \leq (m-(M+m)c)R,$$

for all $t \in [0, \omega]$ and $x \in [r, R]$.

Then equation (1) has at least one ω-periodic solution $x(t)$ with $r \leq x(t) \leq R$.

Proof. Let

$$\Omega = \{x \in C_\omega : \ r \leq x \leq R, \ \text{for } t \in \mathbb{R}\}.$$

Obviously, Ω is a bounded closed convex set in C_ω.

For any $x \in \Omega$, $t \in \mathbb{R}$, from equality (13), we have

$$(Qx)(t+\omega) = P(f(t+\omega, x(t+\omega-\delta(t+\omega)))) = P(f(t, x(t-\delta(t)))) = (Qx)(t),$$

and

$$(Sx)(t+\omega) = cx(t+\omega-\tau) = cx(t-\tau) = (Sx)(t),$$

which show that $(Qx)(t)$ and $(Sx)(t)$ are ω-periodic. Thus, we get $Q(\Omega) \subset C_\omega$ and $S(\Omega) \subset C_\omega$.

For all $x, y \in \Omega$, from Lemma 3.2 and condition (F_1), we arrive at

$$\begin{aligned}(Qx)(t) + (Sy)(t) =& P(f(t, x(t-\delta(t)))) + cy(t-\tau)\\ \leq& \frac{M(1-c)}{m-(M+m)c}\|Tf\| + cy(t-\tau)\\ \leq& \frac{M(1-c)}{m-(M+m)c}\max_{t\in[0,\omega]}\int_0^\omega G(t,s)f(s, x(s-\delta(s)))ds + cy(t-\tau)\\ \leq& \frac{M(1-c)}{m-(M+m)c}\cdot(m-(M+m)c)R\cdot\frac{1}{M} + cR\\ =& R,\end{aligned}$$

since $\int_0^\omega G(t,s)ds = \frac{1}{M}$. On the other hand, from Lemma 3.2 and condition (F_1), it is clear that

$$\begin{aligned}(Qx)(t) + (Sy)(t) =& P(f(t, x(t-\delta(t)))) + cy(t-\tau)\\ \geq& \frac{m-(M+m)c}{M(1-c)}\int_0^\omega G(t,s)f(s, x(s-\delta(s)))ds + cy(t-\tau)\\ \geq& \frac{m-(M+m)c}{M(1-c)}\cdot\frac{M^2(1-c)^2 r}{m-(M+m)c}\cdot\frac{1}{M} + cr\\ =& r.\end{aligned}$$

Therefore, we obtain that $Qx + Sy \in \Omega$.

For all $x_1,\ x_2 \in \Omega$, we obtain

$$|(Sx_1)(t) - (Sx_2)(t)| = |cx_1(t-\tau) - cx_2(t-\tau)| \leq |c|\|x_1 - x_2\|.$$

By taking the norm of both sides, we see that

$$\|Sx_1 - Sx_2\| \leq |c|\|x_1 - x_2\|.$$

Thus, we have from $0 < c < \frac{m}{M+m}$ that S is contractive.

Next, we show that Q is completely continuous. According to equalities (11), (12) and (13), we shall prove that T is completely continuous and N is a continuous bounded operator. Firstly, we show that T is completely continuous.

Let $\{h_k\} \in \Omega$ be a convergent sequence of functions, such that $h_k(t) \to h(t)$ as $k \to \infty$. Since Ω is closed, for $h \in \Omega$ and $t \in [0,\omega]$, we deduce

$$\begin{aligned}|(Th_k)(t) - (Th)(t)| =& \left|\int_0^\omega G(t,s)h_k(s)ds - \int_0^\omega G(t,s)h(s)ds\right|\\ \leq& \int_0^\omega G(t,s)|h_k(s) - h(s)|ds.\end{aligned}$$

Since $|h_k(t) - h(t)| \to 0$ as $k \to \infty$, we obtain

$$\lim_{k\to\infty}\|(Th_k)(t) - (Th)(t)\| = 0. \tag{15}$$

Therefore, T is continuous. On the other hand, we have

$$\begin{aligned}|(Th)(t)| =& \left|\int_0^\omega G(t,s)h(s)ds\right|\\ \leq& \|h\| \int_0^\omega G(t,s)ds\\ \leq& \frac{\|h\|}{M},\end{aligned}$$

where $\|h\| := \max_{t\in[0,\omega]} |h(t)|$. Moreover, from Lemma 2.2, we get

$$\begin{aligned}|(T'h)(t)| =& \left|\int_0^\omega \frac{\partial G(t,s)}{\partial t} h(s)ds\right|\\ \leq& \|h\| \int_0^\omega \left|\frac{\partial G(t,s)}{\partial t}\right| ds\\ \leq& \|h\| G_0 \omega,\end{aligned}$$

where $G_0 := \max_{s,t\in[0,\omega]} \left|\frac{\partial G(t,s)}{\partial t}\right|$. From the above two inequalities, we conclude that $\{Th : h \in \Omega\}$ is uniformly bounded and equicontinuous on $t \in [0, \omega]$. Therefore, $T(\Omega)$ is relatively compact, i.e., T is a compact operator. In conclusion, T is completely continuous.

Secondly, we show that N is a continuous bounded operator. By using a similar argument, it is clearly that N is continuous. From $H(y(t)) = -c(A^{-1}y)(t - \tau)$ and Lemma 2.3, we obtain

$$\begin{aligned}|(Ny)(t)| =& |(M - a(t))y(t) + a(t)H(y(t))|\\ \leq& |(M - a(t))||y(t)| + |c||a(t)||(A^{-1}y)(t - \tau)|\\ \leq& (M - m)|y(t)| + |c|M|(A^{-1}y)(t - \tau)|\\ \leq& \left(M - m + M\frac{|c|}{1 - |c|}\right)\|y\|,\end{aligned}$$

where $\|y\| := \max_{t\in[0,\omega]} |y(t)|$. Therefore, N is a bounded operator.

From above analysis, we conclude that TN is completely continuous. From equalities (12) and (13), we have Q is completely continuous. Then, from Theorem 2.1, we can get that equation (1) has at least one ω-periodic solution $x(t)$ with $r \leq x(t) \leq R$. □

If $r = \frac{c}{M}$ and $R = \frac{1}{m}$, condition (F_1) can be rewritten as

(F_1^*) $\quad \frac{M(1-c)^2c}{m-(M+m)c} \leq f(t, x) \leq 1 - \frac{M+m}{m}c,$ for all $t \in [0, \omega]$ and $x \in [\frac{c}{M}, \frac{1}{m}]$.

Then, we can get the following corollary.

Corollary 3.1. *Suppose that $c \in (0, \frac{m}{M+m})$, $M < (\frac{\pi}{\omega})^2$ and condition (F_1^*) hold. Then equation (1) has at least one positive ω-periodic solution $x(t)$ with $\frac{c}{M} \leq x(t) \leq \frac{1}{m}$.*

Remark 3.1. *Corollary 3.1 extends and improves the Theorem 2.1 in [4].*

Next, we consider the existence of periodic solutions for equation (1) in the case that $c \in (-\frac{m}{M+m}, 0)$. Firstly, we consider the following equation

$$Mc^2 - (2M + m)c + m = 0. \tag{16}$$

We can get equation (16) has a solution $\kappa := \frac{2M+m-\sqrt{(2M+m)^2-4Mm}}{2M}$ and $0 < \kappa < \frac{m}{M+m}$. If $c < \kappa$, we can get $Mc^2 - (2M + m)c + m > 0$.

On the other hand, for any $c > 0$, we have $(M+m)c^2 - (M+m)c + M > 0$. Therefore, if $R > \frac{(M+m)|c|^2-(M+m)|c|+M}{M|c|^2-(2M+m)|c|+m} r > 0$, we have $M(r+|c|R) < \frac{m-(M+m)|c|}{1-|c|}(R+|c|r)$.

Then, we can obtain the following theorem.

Theorem 3.2. *Suppose that $c < 0$, $|c| < \min\{\sigma, \kappa\}$ and $M < (\frac{\pi}{\omega})^2$ hold. Furthermore, assume that the following condition is satisfied:*

(F_2) There exist two non-negative constants r, R such that

$$\frac{(M+m)|c|^2-(M+m)|c|+M}{M|c|^2-(2M+m)|c|+m} r < R$$

and

$$M(r+|c|R) \le f(t,x) \le \frac{m-(M+m)|c|}{1-|c|}(R+|c|r),$$

for all $t \in [0,\omega]$ and $x \in [r,R]$.

Then equation (1) has at least one ω-periodic solution $x(t)$ with $r \le x(t) \le R$.

Proof. We follow the same notations and use a similar argument as in the proof of Theorem 3.1. It can be easily shown that $(Qx)(t)$ and $(Sx)(t)$ are ω-periodic with t. One can observe that Q is completely continuous and S is contractive. Next, we claim that $Qx + Sy \in \Omega$, for all $x, y \in \Omega$. From Lemma 3.1 and condition (F_2), we have

$$\begin{aligned}
(Qx)(t)+(Sy)(t) =& P(f(t,x(t-\delta(t)))) + cy(t-\tau)\\
\le& \frac{M(1-|c|)}{m-(M+m)|c|}\|Tf\| + cy(t-\tau)\\
\le& \frac{M(1-|c|)}{m-(M+m)|c|}\max_{t\in[0,\omega]}\int_0^\omega G(t,s)f(s,x(s-\delta(s)))ds + cy(t-\tau)\\
\le& \frac{M(1-|c|)}{m-(M+m)|c|}\cdot\frac{m-(M+m)|c|}{1-|c|}(R+|c|r)\cdot\frac{1}{M} - |c|r\\
=& R,
\end{aligned}$$

since $\int_0^\omega G(t,s)ds = \frac{1}{M}$. On the other hand, from Lemma 3.1 and condition (F_2), we get

$$\begin{aligned}
(Qx)(t)+(Sy)(t) =& P(f(t,x(t-\delta(t)))) + cy(t-\tau)\\
\ge& \int_0^\omega G(t,s)f(s,x(s-\delta(s)))ds + cy(t-\tau)\\
\ge& M(r+|c|R)\cdot\frac{1}{M} - |c|R\\
=& r.
\end{aligned}$$

From the above two inequalities, we obtain that $Qx + Sy \in \Omega$. Then, from Theorem 2.1, equation (1) has at least one ω-periodic solution $x(t)$ with $r \le x(t) \le R$. □

If $r = 0$ and $R = 1$, condition (F_2) can be rewritten as

(F_2^*) $\quad |c|M < f(t,x) \le \frac{m-(M+m)|c|}{1-|c|}$, for all $t \in [0,\omega]$ and $x \in [0,1]$.

Then, we can get the following corollary.

Corollary 3.2. *Suppose that $c < 0, |c| < \min\{\sigma,\kappa\}, M < (\frac{\pi}{\omega})^2$ and condition (F_2^*) hold. Then equation (1) has at least one positive ω-periodic solution $x(t)$ with $0 < x(t) \le 1$.*

Remark 3.2. *Corollary 3.2 extends and improves the Theorem 2.3 in [4].*

Remark 3.3. *If $|c| > 1$, from Lemma 2.3 and equations (5)–(8), we have $\|TN\| \le 1 - \frac{m}{M} + \frac{|c|}{|c|-1}$. Since $1 - \frac{m}{M} + \frac{|c|}{|c|-1} > 1$, we can not get $(I-TN)^{-1}$. Therefore, the above method does not apply to the case that $|c| > 1$. Next, we have to find another way to get over this problem.*

4. Positive periodic solutions of equation (1) in the case that $|c| > 1$

According to Remark 3.3, we consider the positive periodic solutions of equation (1) in the case that $|c| > 1$. Firstly, we consider the following equation

$$y''(t) + a(t)y(t) = h(t), \quad h \in C_\omega^+. \tag{17}$$

And define $\tilde{N} : C_\omega \to C_\omega$ by

$$(\tilde{N}y)(t) = (M - a(t))y(t). \tag{18}$$

From Lemma 2.1 and $(Th)(t) = \int_0^\omega G(t,s)h(s)ds$, the solution of equation (17) can be written as

$$y(t) = (Th)(t) + (T\tilde{N}y)(t). \tag{19}$$

Since $\|\tilde{N}\| \leq M - m$ and $\|T\| \leq \frac{1}{M}$, we have

$$\|T\tilde{N}\| \leq \|T\|\|\tilde{N}\| \leq 1 - \frac{m}{M} < 1. \tag{20}$$

Therefore, we have

$$y(t) = (I - T\tilde{N})^{-1}(Th)(t). \tag{21}$$

We define an operator $\tilde{P} : C_\omega \to C_\omega$ by

$$(\tilde{P}h)(t) = (I - T\tilde{N})^{-1}(Th)(t). \tag{22}$$

Clearly, $y(t) = (\tilde{P}h)(t)$ is the unique positive ω-periodic solution of equation (17), for any $h \in C_\omega^+$ and $M < (\frac{\pi}{\omega})^2$. And we can get the following lemma which is similar to Lemma 3.1.

Lemma 4.1. *Assume that $M < (\frac{\pi}{\omega})^2$ holds. Then, we have*

$$(Th)(t) \leq (\tilde{P}h)(t) \leq \frac{M}{m}\|Th\|, \quad \textit{for all } h \in C_\omega^+.$$

Now, we consider the existence of periodic solution for equation (1). If $|c| > 1$, equation (1) can be written as

$$-c\Big(x(t-\tau) - \frac{1}{c}x(t)\Big)'' - ca(t)\Big(x(t-\tau) - \frac{1}{c}x(t)\Big) = f(t, x(t-\delta(t))) - ca(t)x(t-\tau), \tag{23}$$

which can be further written as

$$\Big(x(t-\tau) - \frac{1}{c}x(t)\Big)'' + a(t)\Big(x(t-\delta(t)) - \frac{1}{c}x(t)\Big) = a(t)x(t-\tau) - \frac{f(t, x(t-\delta(t)))}{c}. \tag{24}$$

Taking $y(t) = x(t-\tau) - \frac{1}{c}x(t)$, then equation (24) can be transformed into

$$y''(t) + a(t)y(t) = a(t)x(t-\tau) - \frac{f(t, x(t-\delta(t)))}{c}. \tag{25}$$

For convenience, let

$$F(t, x(t)) := a(t)x(t-\tau) - \frac{f(t, x(t-\delta(t)))}{c}.$$

Furthermore, we have

$$x(t) = y(t+\tau) + \frac{1}{c}x(t+\tau) = \tilde{P}(F(t+\tau, x(t+\tau))) + \frac{1}{c}x(t+\tau). \tag{26}$$

Define operators $\tilde{Q}, \tilde{S} : C_\omega \to C_\omega$ by

$$(\tilde{Q}x)(t) = \tilde{P}(F(t+\tau, x(t+\tau))), \ \ (\tilde{S}x)(t) = \frac{1}{c}x(t+\tau). \tag{27}$$

In view of equations (25), (26) and above analysis, the existence of periodic solutions of equation (1) is equivalent to the existence of solutions for the operator equation

$$\tilde{Q}x + \tilde{S}x = x \tag{28}$$

in C_ω.

Next, we present our main results about the existence of periodic solutions for equation (1).

Theorem 4.1. *Suppose that $c > 1$ and $M < (\frac{\pi}{\omega})^2$ hold. Furthermore, assume that the following condition is satisfied:*
(F_3) There exist two constants r, R, such that $0 < \frac{M}{m}r < R$ and

$$M\left(1 - \frac{1}{c}\right)r \le F(t, x(t)) \le m\left(1 - \frac{1}{c}\right)R,$$

for all $t \in [0, \omega]$ and $x \in [r, R]$.

Then equation (1) has at least one ω-periodic solution $x(t)$ with $r \le x(t) \le R$.

Proof. We define Ω as in the proof of Theorem 3.1. For any $x \in \Omega$, $t \in \mathbb{R}$, from equality (27), we have

$$\begin{aligned}
(\tilde{Q}x)(t+\omega) =& \tilde{P}(F(t+\omega+\tau, x(t+\omega+\tau))) \\
=& \tilde{P}\left(a(t+\omega+\tau)x(t+\omega) - \frac{f(t+\omega+\tau, x(t+\omega+\tau+\delta(t+\omega+\tau)))}{c}\right) \\
=& \tilde{P}\left(a(t+\tau)x(t) - \frac{f(t+\tau, x(t+\tau+\delta(t+\tau)))}{c}\right) \\
=& \tilde{P}(F(t+\tau, x(t+\tau))) = (\tilde{Q}x)(t),
\end{aligned}$$

and

$$(\tilde{S}x)(t+\omega) = \frac{1}{c}x(t+\omega+\tau) = \frac{1}{c}x(t+\tau) = (\tilde{S}x)(t),$$

which show that $(\tilde{Q}x)(t)$ and $(\tilde{S}x)(t)$ are ω-periodic. Thus, we have $\tilde{Q}(\Omega) \subset C_\omega$ and $\tilde{S}(\Omega) \subset C_\omega$.

For any $x, y \in \Omega$, from Lemma 4.1 and condition (F_3), we get

$$\begin{aligned}
(\tilde{Q}x)(t) + (\tilde{S}y)(t) =& \tilde{P}(F(t+\tau, x(t+\tau))) + \frac{1}{c}y(t+\tau) \\
\le& \frac{M}{m}\|TF\| + \frac{1}{c}y(t+\tau) \\
\le& \frac{M}{m}\max_{t\in[0,\omega]}\int_0^\omega G(t+\tau, s)\left(a(s)x(s-\tau) - \frac{f(s, x(s-\delta(s)))}{c}\right)ds + \frac{1}{c}y(t+\tau) \\
\le& \frac{M}{m}\cdot m\left(1 - \frac{1}{c}\right)R \cdot \frac{1}{M} + \frac{1}{c}R \\
=& R,
\end{aligned}$$

since $\int_0^\omega G(t,s)ds = \frac{1}{M}$. On the other hand, from Lemma 4.1 and condition (F_3), we arrive at

$$\begin{aligned}(\tilde{Q}x)(t) + (\tilde{S}y)(t) =& \tilde{P}(F(t+\tau, x(t+\tau))) + \frac{1}{c}y(t+\tau)\\ \geq& \int_0^\omega G(t+\tau,s)\left(a(s)x(s-\tau) - \frac{f(s,x(s-\delta(s)))}{c}\right)ds + \frac{1}{c}y(t+\tau)\\ \geq& M(1-\frac{1}{c})r\cdot\frac{1}{M} + \frac{1}{c}r\\ =& r.\end{aligned}$$

Combing with the above two inequalities, we obtain $\tilde{Q}x + \tilde{S}y \in \Omega$.

For all $x_1,\ x_2 \in \Omega$, we see that

$$|(\tilde{S}x_1)(t) - (\tilde{S}x_2)(t)| = \left|\frac{1}{c}x_1(t+\tau) - \frac{1}{c}x_2(t+\tau)\right| \leq \frac{1}{|c|}\|x_1 - x_2\|.$$

By taking the norm of both sides, we can get

$$\|\tilde{S}x_1 - \tilde{S}x_2\| \leq \frac{1}{|c|}\|x_1 - x_2\|.$$

Thus, we have from $|c| > 1$ that $\tilde{S}$ is contractive.

By using a similar argument as in the proof of Theorem 3.1, we can observe that $T\tilde{N}$ is completely continuous. Then, from Lemma 4.1 and equation (27), we have $\tilde{Q}$ is completely continuous. And from Theorem 2.1, we get that equation (1) has at least one ω-periodic solution $x(t)$ with $r \leq x(t) \leq R$. □

For $c < -1$, we can get the following theorem.

Theorem 4.2. *Suppose that $c < -\frac{M}{m}$ and $M < (\frac{\pi}{\omega})^2$ hold. Furthermore, assume that the following condition is satisfied:*

(F_4) There exist two constants r, R such that $0 < \frac{M|c|-m}{m|c|-M}r < R$ and

$$M\left(r + \frac{1}{|c|}R\right) \leq F(t, x(t)) \leq m\left(R + \frac{1}{|c|}r\right),$$

for all $t \in [0,\omega]$ and $x \in [r, R]$.

Then equation (1) has at least one ω-periodic solution $x(t)$ with $r \leq x(t) \leq R$.

Proof. We define Ω as in the proof of Theorem 3.1 and use a similar argument as in the proof of Theorem 4.1. It can be easily shown that $(\tilde{Q}x)(t)$ and $(\tilde{S}x)(t)$ are ω-periodic with t. One can observe that $\tilde{Q}$ is completely continuous and $\tilde{S}$ is contractive. Next, we show that $\tilde{Q}x + \tilde{S}y \in \Omega$, for all $x, y \in \Omega$. From Lemma 4.1 and condition (F_4), we deduce

$$\begin{aligned}(\tilde{Q}x)(t) + (\tilde{S}y)(t) =& \tilde{P}(F(t+\tau, x(t+\tau))) + \frac{1}{c}y(t+\tau)\\ \leq& \frac{M}{m}\|TF\| + \frac{1}{c}y(t+\tau)\\ \leq& \frac{M}{m}\max_{t\in[0,\omega]}\int_0^\omega G(t+\tau,s)\left(a(s)x(s-\tau) - \frac{f(s,x(s-\delta(s)))}{c}\right)ds + \frac{1}{c}y(t+\tau)\\ \leq& \frac{M}{m}\cdot m\left(R + \frac{1}{|c|}r\right)\cdot\frac{1}{M} - \frac{1}{|c|}r\\ =& R.\end{aligned}$$

On the other hand, from Lemma 4.1 and condition (F_4), we have

$$\begin{aligned}(\tilde{Q}x)(t)+(\tilde{S}y)(t)=&\tilde{P}(F(t+\tau,x(t+\tau)))+\frac{1}{c}y(t+\tau)\\ \geq&\int_0^{\omega}G(t+\tau,s)\left(a(s)x(s-\tau)-\frac{f(s,x(s-\delta(s)))}{c}\right)ds+\frac{1}{c}y(t+\tau)\\ \geq&M\left(r+\frac{1}{|c|}R\right)\cdot\frac{1}{M}-\frac{1}{|c|}R\\ =&r.\end{aligned}$$

Combing the above two inequalities, we obtain that $\tilde{Q}x+\tilde{S}y\in\Omega$. Then, from Theorem 2.1, equation (1) has at least one ω-periodic solution $x(t)$ with $r\leq x(t)\leq R$. □

Finally, we present an example to illustrate our results.

Example 4.1. *Consider the following equation*

$$\left(x(t)-\frac{1}{4}x\left(t-\frac{\pi}{4}\right)\right)''+(3-\cos 6t)x(t)=3e^{\sin 6t}+\frac{1}{8}\cos^2 x\left(t-\frac{1}{2}\cos 6t\right). \tag{29}$$

Comparing equation (29) *to equation* (1), *we have* $\omega=\frac{\pi}{3},\tau=\frac{\pi}{4}$, $c=\frac{1}{4}$, $\delta(t)=\frac{1}{2}\cos 6t$, $a(t)=3-\cos 6t$, $(\frac{\pi}{\omega})^2=9$, $f(t,x(t-\delta(t)))=3e^{\sin 6t}+\frac{1}{8}\cos^2 x(t-\frac{1}{2}\cos 6t)$, *and we can easily get* $M=\max\{a(t):t\in[0,\frac{\pi}{3}]\}=4<9$, $m=\min\{a(t):t\in[0,\frac{\pi}{3}]\}=2$, $1\leq f(t,x)\leq 9$. *Let* $r=\frac{1}{18}$ *and* $R=18$, *we can verify that the condition* (F_1) *is satisfied. From Theorem 3.1, equation* (29) *has at least one* $\frac{\pi}{3}$*-periodic solution with* $\frac{1}{18}\leq x(t)\leq 18$.

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