



Some Characterizing Results for Hemi-Slant Warped Product Submanifolds of a Kaehler Manifold

Kamran Khan^a, Viqar Azam Khan^b

^aDepartment of Applied Mathematics, Aligarh Muslim University, Aligarh, India

^bDepartment of Mathematics, Aligarh Muslim University, Aligarh, India

Abstract. Many differential geometric properties of submanifolds of a Kaehler manifold are looked into via canonical structure tensors P and F on the submanifold. For instance, a CR -submanifold of a Kaehler manifold is a CR -product (i.e. locally a Riemannian product of a holomorphic and a totally real submanifold) if and only if the canonical tensor P is parallel on the submanifold. Since, warped product manifolds are generalized version of Riemannian product of manifolds, in this article, we consider the covariant derivatives of the structure tensors on a hemi-slant submanifold of a Kaehler manifold. Our investigations have led us to characterize hemi-slant warped product submanifolds.

1. Introduction

To construct a class of manifolds of negative sectional curvatures, R. L. Bishop and B. O'Neill [4] generalized the notion of product metric by homothetically warping the product metric of a product manifold on to the fibers. These manifolds appear in differential geometric studies in a natural way. Moreover, they provide an excellent setting to model space-time near black holes or bodies with high gravitational fields. For instance, the Schwarzschild space-time model is a warped product manifold. This makes the study of warped product manifolds more significant geometrically as well. Bishop and O'Neill [4] obtained some intrinsic geometric properties of these manifolds. The study of warped product manifolds with extrinsic geometric point of view was initiated by B.-Y. Chen [10–12] when he investigated CR -submanifolds of a Kaehler manifold as warped products. Later, warped products were explored in other ambient spaces as well. For instance, F. R. Al-Solamy et. al [1], B. Sahin [22], V. A. Khan et.al [17, 18], V. Bonanzinga and K. Matsumoto [5], S. Uddin et. al [24] explored the existence of various warped products in Kaehler, nearly Kaehler and locally conformal Kaehler (l.c.K.) manifolds and studied the geometric properties of the existent warped product submanifolds.

To explore the submanifolds of an almost Hermitian manifold, many of the extrinsic geometric features of the submanifold are looked into via the shape operator or the structure tensor fields P and F (defined in Section 2). For example, the characterizations of CR -products in Kaehler and nearly Kaehler manifolds (cf. [8, 16]). Moreover, along the years, there has been interests to find an analogous of classical deRahm's theorem to warped products. In the present note, we have considered hemi-slant (also named as pseudo-slant or anti-slant) submanifolds of a Kaehler manifold and have sought some characterizations under

2010 *Mathematics Subject Classification.* Primary 53C40; Secondary 53C42, 53B25

Keywords. Warped product, hemi-slant submanifold, Kaehler manifold

Received: 24 August 2017; Accepted: 10 June 2018

Communicated by Ljubica Velimirović

Email addresses: kamran35d@gmail.com (Kamran Khan), viqarster@gmail.com (Viqar Azam Khan)

which the submanifolds would become warped products. More explicitly, we have worked out formulae for $\bar{\nabla}P$ and $\bar{\nabla}F$ under which a hemi-slant-submanifold of a Kaehler manifold reduces to a warped product submanifold. To achieve the objective, we fix up the basic terminology and notations in Section 2, then obtain some initial results and relevant differential geometric properties of hemi-slant submanifolds of a Kaehler manifold in Section 3. These results are used in obtaining the said characterizations in Section 4. The obtained results extend and generalize many existing results in almost Hermitian setting.

2. Preliminaries

Let (\bar{M}, J, g) be an almost Hermitian manifold with an almost complex structure J and Hermitian metric g . That is, the (1,1) tensor field J satisfies the following relations.

$$J^2 = -I, \quad g(JU, JV) = g(U, V) \tag{1}$$

for any vector fields U, V on \bar{M} , where I denotes the identity transformation.

Let M be an n -dimensional Riemannian manifold isometrically immersed into an almost Hermitian manifold \bar{M} with tangent bundle TM and the normal bundle $T^\perp M$. Following B.-Y. Chen’s notation in [9], for any $x \in M$ and any $U \in T_x M$, we put

$$JU = PU + FU, \quad PU \in T_x M \text{ and } FU \in T_x^\perp M, \tag{2}$$

thus defining endomorphism $P : T_x M \rightarrow T_x M$ and a normal valued linear map $F : T_x M \rightarrow T_x^\perp M$. The (1,1) tensor field and the normal valued 1-form on M determined by P and F will be denoted by the same letters. Similarly, for $\xi \in T^\perp M$, we put

$$t\xi = \tan(J\xi) \text{ and } f\xi = \text{nor}(J\xi). \tag{3}$$

Let $\bar{\nabla}$ be the Levi-Civita connection on the ambient manifold \bar{M} and ∇ , the induced Riemannian connection on the submanifold M . The two connections are related by means of Gauss-Weingarten formulas as:

$$\bar{\nabla}_U V = \nabla_U V + h(U, V) \tag{4}$$

and

$$\bar{\nabla}_U \xi = -A_\xi U + \nabla_U^\perp \xi \tag{5}$$

for $U, V \in TM$ and $\xi \in T^\perp M$; where h is the second fundamental form, A_ξ , the shape operator (associated with the normal vector field ξ) of the immersion of M into \bar{M} and ∇^\perp is the connection in the normal bundle $T^\perp M$. A_ξ and h are related by

$$g(A_\xi U, V) = g(h(U, V), \xi),$$

where g denotes the Riemannian metric on \bar{M} as well as the one induced on M .

The covariant derivatives of the tensor fields P, F, t and f are defined as:

$$(\bar{\nabla}_U P)V = \nabla_U PV - P\nabla_U V, \tag{6}$$

$$(\bar{\nabla}_U F)V = \nabla_U^\perp FV - F\nabla_U V, \tag{7}$$

$$(\bar{\nabla}_U t)\xi = \nabla_U t\xi - t\nabla_U^\perp \xi, \tag{8}$$

$$(\bar{\nabla}_U f)\xi = \nabla_U^\perp f\xi - f\nabla_U^\perp \xi, \tag{9}$$

for $U, V \in TM$.

For a submanifold M of an almost Hermitian manifold \bar{M} , the action of the almost complex structure J on the tangent bundle TM gives rise to various distributions on M . For instance,

A distribution D on a submanifold M of an almost Hermitian manifold \bar{M} is called a *holomorphic* (or *complex*) distribution if $JD \subseteq D$.

A distribution D on M is called *totally real* (or *Lagrangian*) distribution if $JD \subseteq T^\perp M$.

A. Bejancu [3] introduced the notion of *CR-submanifolds* of an almost Hermitian manifold to provide a single setting to study holomorphic and totally real distributions. More specifically, a *CR-submanifold* of an almost Hermitian manifold is endowed with a pair of orthogonal complementary distributions D and D^\perp such that D is holomorphic and D^\perp is totally real. The notion of holomorphic and totally real distributions were further generalized when B.-Y. Chen [9] introduced the idea of slant immersions.

Let D be a distribution on a submanifold M of an almost Hermitian manifold (\bar{M}, J, g) . For any $x \in M$ and any non-zero vector $X \in D_x$, if the angle $\theta(X) \in [0, \pi/2]$ between JX and the vector space D_x does not depend on the choice of $x \in M$ and $X \in D_x$, D is said to be a *slant distribution* on M . The constant angle θ is called the *Wirtinger angle* of D in M [9].

Usually, a slant distribution with Wirtinger angle θ is denoted by D^θ . A submanifold M is called a *slant submanifold* if the tangent bundle TM is slant. Holomorphic and totally real submanifolds are special cases of slant submanifolds with Wirtinger angle 0 and $\pi/2$ respectively. A slant submanifold is called *proper slant* if it is neither holomorphic nor totally real.

If M is a slant submanifold of an almost Hermitian manifold \bar{M} with Wirtinger angle θ then we have

$$P^2 = -(\cos^2 \theta)I. \tag{10}$$

This gives

$$g(PU, PV) = \cos^2 \theta g(U, V) \tag{11}$$

and

$$g(FU, FV) = \sin^2 \theta g(U, V), \tag{12}$$

for U, V tangent to M [9].

A Hermitian metric g on a complex manifold (\bar{M}, J) is called a *Kaehler metric* if the fundamental 2-form Ω defined as $\Omega(U, V) = g(JU, V)$ is closed. A complex manifold with a Kaehler metric is called a *Kaehler manifold*. If $\bar{\nabla}$ is the Levi-Civita connection of g then $(\bar{M}, J, g, \bar{\nabla})$ is Kaehler if and only if $\bar{\nabla}J = 0$.

On a submanifold M of a Kaehler manifold \bar{M} , by using equations (2)-(7), it is easy to deduce that

$$(\bar{\nabla}_U P)V = A_{FV}U + th(U, V), \tag{13}$$

$$(\bar{\nabla}_U F)V = fh(U, V) - h(U, PV). \tag{14}$$

A proper slant submanifold is said to be *Kaehlerian slant* if the $(1,1)$ tensor field P is parallel that is $\bar{\nabla}P = 0$. A Kaehlerian slant submanifold of an almost Hermitian manifold is a Kaehler manifold with respect to the induced metric and the almost complex structure $J^0 = (\sec \theta)P$, where θ is the Wirtinger angle. B.-Y. Chen [9] established the following characterization for submanifolds of an almost Hermitian manifold with $\bar{\nabla}P = 0$.

Theorem 2.1. *Let M be a submanifold of an almost Hermitian manifold \bar{M} . Then $\bar{\nabla}P = 0$ if and only if M is locally a Riemannian product $M_1 \times M_2 \times \dots \times M_k$, where each M_i is either a Kaehler submanifold, a totally real submanifold or a Kaehlerian slant submanifold.*

In particular, we have

Corollary 2.2. *Let M be an irreducible submanifold of an almost Hermitian manifold \bar{M} . If M is neither holomorphic nor totally real in \bar{M} , then M is Kaehlerian slant if and only if P is parallel, i.e. $\bar{\nabla}P = 0$.*

N. Papaghiuc [21] introduced semi-slant submanifolds of an almost Hermitian manifold as a generalized version of a CR-submanifold. He formulated the definition as:

A submanifold M of an almost Hermitian manifold is said to be a *semi-slant submanifold* if it is endowed with a pair of orthogonal complementary distributions D and D^θ on M such that D is holomorphic and D^θ is slant.

Thus a CR-submanifold is a special case of a semi-slant submanifold with $\theta = \pi/2$.

J. L. Cabrerizo et al.[6] defined *bi-slant submanifolds* of an almost contact metric manifold as the submanifold admitting two orthogonal distributions D^{θ_1} and D^{θ_2} such that $TM = D^{\theta_1} \oplus D^{\theta_2} \oplus \langle \xi \rangle$, where D^{θ_i} is a slant distribution with Wirtinger angle $\theta_i (i = 1, 2)$ and $\langle \xi \rangle$ is the one dimensional distribution spanned by the structure vector field ξ . One of the interesting classes of bi-slant submanifolds is the class of anti-slant (or pseudo-slant) submanifolds. In fact, if one of the θ_i 's is equal to $\pi/2$ then a bi-slant submanifold reduces to an anti-slant submanifold.

A. Carriazo [7] extended the notion of anti-slant submanifolds to the Kaehlerian settings as:

A submanifold M of an almost Hermitian manifold \bar{M} is said to be a *hemi-slant submanifold* if there exist two orthogonal complementary distributions D^θ and D^\perp such that D^θ is slant with Wirtinger angle θ and D^\perp is totally real.

For $\theta = 0$, a hemi-slant submanifold reduces to a CR-submanifold. A hemi-slant submanifold is proper if $\theta \neq 0, \pi/2$.

Since our aim is to study hemi-slant submanifolds which are warped product submanifolds in an almost Hermitian manifold, we recall in the following paragraphs the notion of warped product manifolds and some intrinsic geometric properties of these manifolds.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively and f be a positive differentiable function on M_1 . Then the *warped product* $M_1 \times_f M_2$ is the manifold $M_1 \times M_2$ endowed with Riemannian metric g defined as

$$g = \pi_1^*(g_1) + (f \circ \pi_1)^2 \pi_2^*(g_2) \tag{15}$$

where $\pi_i (i = 1, 2)$ are the projection maps of M onto $M_i (i = 1, 2)$. The function f , in this case is known as the *warping function*. If the warping function f is constant, then the warped product is simply a Riemannian product, known as a *trivial warped product*. Given vector fields U_1 on M_1 and U_2 on M_2 , we may obtain their horizontal lifts $\widetilde{U}_1, \widetilde{U}_2$ such that $d\pi_1 \widetilde{U}_1 = U_1$ and $d\pi_2 \widetilde{U}_2 = U_2$, nevertheless we identify $\widetilde{U}_1, \widetilde{U}_2$ by the same symbols U_1, U_2 respectively.

On a warped product manifold $M = M_1 \times_f M_2$, some of the relevant formulae revealing some geometric aspects are as follows:

$$\nabla_{U_1} U_2 = \nabla_{U_2} U_1 = (U_1 \ln f) U_2 \tag{16}$$

and

$$nor(\nabla_{U_2} V_2) = -g(U_2, V_2) \nabla \ln f \tag{17}$$

for any $U_1 \in TM_1$ and $U_2, V_2 \in TM_2$, where $nor(\nabla_{U_2} V_2)$ denotes the component of $\nabla_{U_2} V_2$ in TM_1 and ∇f is the gradient of f defined as

$$g(\nabla f, U) = Uf \tag{18}$$

for any $U \in TM$.

As an immediate consequence of the above formulae, we have

Proposition 2.3. [4] *On a warped product manifold $M = M_1 \times_f M_2$,*

- (i) M_1 is totally geodesic in M ,
- (ii) M_2 is totally umbilical in M .

A warped product manifold isometrically immersed into a Riemannian manifold is known as *warped product submanifold*.

3. Hemi-slant submanifolds of a Kaehler manifold

B. Sahin [23] initiated to seek some differential geometric properties of hemi-slant submanifolds of a Kaehler manifold and constructed various examples of these submanifolds. In his investigations, he worked out integrability conditions of the two distributions on the submanifold and studied mixed totally geodesic hemi-slant warped product submanifolds of a Kaehler manifold. Our aim in the present note is to study hemi-slant submanifolds of Kaehler manifolds and seek conditions in terms of the structure tensors P and F under which the underlying submanifold reduces to a warped product submanifold. To achieve the objective, we first fix up the basic notations and obtain some relevant differential geometric properties of a hemi-slant submanifold of a Kaehler manifold.

Throughout, we denote by M , a proper hemi-slant submanifold of a Kaehler manifold $(\bar{M}, J, g, \bar{\nabla})$. That is M is assumed to admit a slant distribution D^θ such that its orthogonal complementary distribution D^\perp is totally real. That is,

$$TM = D^\theta \oplus D^\perp.$$

Let B and C be the canonical projections of TM onto D^θ and D^\perp respectively. That is, for any $U \in TM$, we have

$$U = BU + CU, \tag{19}$$

where $BU \in D^\theta$ and $CU \in D^\perp$. It is straightforward to see that

$$PU = PB U, PC U = 0; FU = FB U + JC U.$$

Thus, for any $U \in TM$, $PU \in D^\theta$ and the normal bundle $T^\perp M$ admits the following orthogonal direct decomposition

$$T^\perp M = FD^\theta \oplus JD^\perp \oplus \nu,$$

where ν is the orthogonal complement of $FD^\theta \oplus JD^\perp$ in $T^\perp M$ and it is easy to notice that ν is an invariant normal sub bundle of $T^\perp M$ under J . Further, as D^θ and D^\perp are assumed to be orthogonal complementary distributions, FD^θ and JD^\perp are orthogonal in $T^\perp M$.

A hemi-slant submanifold of a Kaehler manifold is *mixed totally geodesic* if the second fundamental form h satisfies

$$h(X, Z) = 0$$

for each $X \in D^\theta$ and $Z \in D^\perp$.

The totally real distribution D^\perp and the slant distribution D^θ are integrable in view of the following results.

Proposition 3.1. [23] *Let M be a hemi-slant submanifold of a Kaehler manifold \bar{M} . Then the totally real distribution D^\perp is involutive on M .*

Proposition 3.2. [23] *The slant distribution D^θ on a hemi-slant submanifold of a Kaehler manifold is involutive if and only if*

$$t[h(X, PY) - h(PX, Y) + \nabla_X^\perp FY - \nabla_Y^\perp FX]$$

lies in D^θ for all $X, Y \in D^\theta$.

Let Ω_M be the restriction of the fundamental 2-form Ω on a submanifold M of a Kaehler manifold \bar{M} . Then $\text{Ker}(\Omega_M)$ is the set of all vector fields Z on M such that $\Omega_M(Z, U) = 0$, for each $U \in TM$. If $\text{Ker}(\Omega_M)$ has constant rank over M then it defines a totally real distribution on M with respect to the almost complex structure J on \bar{M} .

The integrability of the totally real distribution on an l.c.K. manifold (and therefore on a Kaehler manifold) also follows from the following theorem.

Theorem 3.3. [20] *If Ω is a differential p -form on a differentiable manifold M such that $d\Omega = \Omega \wedge \alpha$ where α is a pfaffian form, then the distribution generated by set of sections of $\text{Ker}(\Omega)$ is completely integrable.*

Now, for $X, Y \in D^\theta$ and $Z \in D^\perp$, by using (13) and (6), we obtain

$$g(\nabla_X PY, Z) = g(h(X, Z), FY) - g(h(X, Y), FZ),$$

and hence, it is deduced that

Proposition 3.4. *On a hemi-slant submanifold of a Kaehler manifold, the slant distribution D^θ is parallel if and only if*

$$g(h(X, Z), FY) = g(h(X, Y), FZ)$$

for each $X, Y \in D^\theta$ and $Z \in D^\perp$.

Also, from (2), (4) and (5), we have

$$g(h(Z, W), FX) = g(h(Z, X), JW) - g(\nabla_Z W, PX) \tag{20}$$

Now, if M_\perp be a leaf of D^\perp in M and h' be the second fundamental form of the immersion of M_\perp into M , then by (4),

$$g(\nabla_Z W, PX) = g(h'(Z, W), PX). \tag{21}$$

The above equations lead to the following result.

Proposition 3.5. *Let M be hemi-slant submanifold of a Kaehler manifold \bar{M} . Then the leaves of the totally real distribution D^\perp are totally umbilical in M if and only if*

$$g(h(Z, W), FX) = g(h(Z, X), JW) - g(Z, W)g(\mu, PX),$$

for each $X \in D^\theta$ and $Z, W \in D^\perp$, where μ is the mean curvature vector of M_\perp in M .

4. Hemi-slant warped product submanifolds

The realization of the fact that warped product manifolds provide a natural frame work for time dependent mechanical system and excellent setting to model space-time near black holes or bodies with high gravitational fields gave impetus to the studies of warped product manifolds with extrinsic geometric point of view [2, 13, 15]. B.-Y. Chen [10, 11] initiated the study by exploring CR-submanifolds of Kaehler manifolds as warped product submanifolds. Since the existence of totally real and slant submanifolds of a Kaehler manifold are ensured by virtue of Proposition 3.1 and 3.2, our aim in this section is to explore conditions under which a hemi-slant submanifold of a Kaehler manifold reduces to a warped product submanifold.

The following important result by S. Heipko [14] will be used later to obtain the desired characterizations for hemi-slant submanifolds.

Theorem 4.1. [14] *Let F be a vector sub bundle in the tangent bundle of a Riemannian manifold M and let F^\perp be its normal bundle. Assume that the two distributions are both involutive and the integral manifold of F (resp. F^\perp) are extrinsic spheres (resp. totally geodesic). Then M is locally isometric to a warped product $M_1 \times_f M_2$. Moreover, if M is simply connected and complete, there exists a global isometry of M with a warped product.*

Let \bar{M} be a Kaehler manifold, M_θ and M_\perp be respectively slant and totally real submanifolds of \bar{M} . Then with these factors there are two possible warped product submanifolds namely (i) $M = M_\perp \times_f M_\theta$ and (ii) $M = M_\theta \times_f M_\perp$. The non-trivial warped product submanifolds of type (i) do not exist in a Kaehler manifold in view of the following theorem.

Theorem 4.2. [23] *Let \bar{M} be a Kaehler manifold. Then there exist no non-trivial warped product submanifolds $M = M_{\perp} \times_f M_{\theta}$ of \bar{M} such that M_{\perp} is totally real submanifold and M_{θ} is a proper slant submanifold of \bar{M} .*

However, non-trivial warped product submanifolds of type (ii) do exist in a Kaehler manifold, called in the sequel as *hemi-slant warped product submanifolds*. Some examples of these submanifolds are constructed in [23].

On a hemi-slant warped product submanifold, using formula (16), we can write

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z, \tag{22}$$

for each $X \in TM_{\theta}$ and $Z \in TM_{\perp}$.

Lemma 4.3. *Let M be a hemi-slant warped product submanifold of a Kaehler manifold \bar{M} . Then*

$$g(h(U, V), JZ) = g(h(U, Z), FV) - (PV \ln f)g(CU, Z)$$

for each $U, V \in TM$ and $Z \in TM_{\perp}$.

Proof. Using Gauss-Weingarten formulas, (19) and the fact that \bar{M} is Kaehler, we may write

$$\begin{aligned} g(h(U, V), JZ) &= -g(\bar{\nabla}_U J V, Z) \\ &= -g(\nabla_U P V, Z) + g(A_{FV} U, Z) \\ &= -g(\nabla_{BU} P V, Z) - g(\nabla_{CU} P V, Z) + g(h(U, Z), FV) \end{aligned}$$

The first term in the right hand side is zero by virtue of the fact that $PV \in D^{\theta}$ and M_{θ} is totally geodesic in M . Further, on taking account of formula (22), the above equation takes the form

$$g(h(U, V), JZ) = g(h(U, Z), FV) - (PV \ln f)g(CU, Z). \tag{23}$$

□

The above lemma provides a generalization to Proposition 3.5 and to [23, Lemma 5.1].

Lemma 4.4. *Let M be a hemi-slant warped product submanifold of a Kaehler manifold \bar{M} . Then*

$$(\bar{\nabla}_X P)V = 0,$$

for each $X \in TM_{\theta}$ and $V \in TM$.

Proof. If ∇' is the induced Riemannian connection on M_{θ} and h' is the second fundamental form of the immersion of M_{θ} into M , then by (4) and (6), we may write

$$(\bar{\nabla}_X P)Y = \nabla'_X P Y + h'(X, P Y) - P \nabla'_X Y - P h'(X, Y)$$

for $X, Y \in TM_{\theta}$. Denoting $\nabla'_X P Y - P \nabla'_X Y$ by $(\nabla_X P)Y$ and taking account of the fact that M_{θ} is totally geodesic in M , we deduce from the above equation that

$$(\bar{\nabla}_X P)Y = (\nabla_X P)Y.$$

As M_{θ} is a slant submanifold of \bar{M} , by Corollary 2.2, $\nabla_X P = 0$. Hence, the last equation reduces to

$$(\bar{\nabla}_X P)Y = 0. \tag{24}$$

Now, in view of the fact that $PZ = 0$ for each $Z \in TM_{\perp}$, by (6) and (22), we have

$$(\bar{\nabla}_X P)Z = 0. \tag{25}$$

From (24) and (25), we obtain that

$$(\bar{\nabla}_X P)V = 0, \tag{26}$$

for all $V \in TM$. □

In the next two theorems, we analyse hemi-slant submanifolds in a Kaehler manifold and the outcomes are interesting characterizations of these submanifolds which reduces them to warped product submanifolds.

Theorem 4.5. *A hemi-slant submanifold M of a Kaehler manifold is a warped product submanifold if and only if there exists a real valued function μ on M with $Z\mu = 0$ for every $Z \in D^\perp$ satisfying*

$$(\bar{\nabla}_U P)V = g(CU, CV)P\nabla\mu + (PBV\mu)CU \tag{27}$$

for each $U, V \in TM$.

Proof. Let M_θ and M_\perp be respectively slant (with Wirtinger angle θ) and totally real submanifolds of a Kaehler manifold \bar{M} such that the warped product $M = M_\theta \times_f M_\perp$ admits an isometric immersion in \bar{M} . Then for $Z, W \in TM_\perp$ and $X \in TM_\theta$, using (6), (22) and the fact that M_\perp is totally real submanifold, we have

$$(\bar{\nabla}_Z P)X = (PX\ln f)Z. \tag{28}$$

Again using (17) and (18), we get

$$g((\bar{\nabla}_Z P)W, X) = -(PX\ln f)g(Z, W).$$

The last equation gives

$$B((\bar{\nabla}_Z P)W) = g(Z, W)P\nabla\ln f. \tag{29}$$

It is also easy to see that

$$C((\bar{\nabla}_Z P)W) = 0. \tag{30}$$

Hence,

$$(\bar{\nabla}_Z P)W = g(Z, W)P\nabla\ln f. \tag{31}$$

Combining (28) and (31), we have

$$(\bar{\nabla}_Z P)V = (PV\ln f)Z + g(Z, V)P\nabla\ln f. \tag{32}$$

Now, from (26) and (32), we deduce (27), where $\mu = \ln f$.

Conversely, suppose that M is a hemi-slant submanifold of a Kaehler manifold \bar{M} satisfying (27). Then for $Z, W \in D^\perp$ and $X \in D^\theta$, (27) gives

$$g((\bar{\nabla}_Z P)W, X) = g(Z, W)g(P\nabla\mu, X)$$

or

$$g(\nabla_Z W, PX) = -g(Z, W)g(\nabla\mu, PX).$$

If M_\perp is a leaf of D^\perp and h'' is the second fundamental form of the immersion of M_\perp into M , then from the above relation, it follows that

$$h''(Z, W) = -g(Z, W)\nabla\mu.$$

This means M_\perp is totally umbilical in M . Further, as $Z\mu = 0$ for every $Z \in D^\perp$, M_\perp is extrinsic sphere in M . Now, for any $X, Y \in D^\theta$, by (27), we find

$$(\bar{\nabla}_X P)Y = 0.$$

In view of (6), it yields

$$\nabla_X PY = P\nabla_X Y \in D^\theta.$$

This proves that D^θ is parallel. In other words, D^θ is involutive and its leaves are totally geodesic in M . If M_θ denotes a leaf of D^θ then by virtue of Theorem 4.1, M is isometric to a warped product submanifold $M_\theta \times_f M_\perp$. This proves the theorem completely. \square

Let M be a hemi-slant submanifold of a Kaehler manifold satisfying (27) then for $X \in D^\theta$ and $Z \in D^\perp$, $(\bar{\nabla}_X P)Z = 0$ and $(\bar{\nabla}_Z P)X = (PX\mu)Z$. Taking account of these observations in (13), we get,

$$A_{FZ}X = -th(X, Z) \tag{33}$$

and

$$A_{FX}Z = (PX\mu)Z - th(X, Z). \tag{34}$$

That gives

$$A_{FX}Z - A_{FZ}X = (PX\mu)Z. \tag{35}$$

Hence, in terms of the shape operator of the immersion of M into \bar{M} , we arrive at the following characterization:

Corollary 4.6. *A hemi-slant submanifold M of a Kaehler manifold is a hemi-slant warped product submanifold if and only if there exists a smooth function μ on M such that*

$$A_{FZ}PX - A_{FPX}Z = \cos^2\theta(X\mu)Z,$$

for each $X \in D^\theta$ and $Z \in D^\perp$ satisfying $Z\mu = 0$.

If the submanifold M is a mixed totally geodesic proper hemi-slant submanifold, then (33) and (34) respectively reduce to

$$A_{JZ}X = 0 \quad \text{and} \quad A_{FPX}Z = -\cos^2\theta(X\mu)Z$$

for each $X \in D^\theta$ and $Z \in D^\perp$.

The above relations were established by B. Sahin [23] as the necessary and sufficient condition for mixed totally geodesic hemi-slant submanifold of a Kaehler manifold to be a hemi-slant warped product submanifold.

If M is a CR-submanifold of a Kaehler manifold, then formula (35) yields the following characterization.

Corollary 4.7. *A CR- submanifold M of a Kaehler manifold is a CR-warped product if and only if there exists a smooth function μ on M satisfying $Z\mu = 0$ such that*

$$A_{JZ}X = -(JX\mu)Z,$$

for each $X \in D$ and $Z \in D^\perp$.

The above characterization was proved by B.-Y.Chen in [10].

Theorem 4.8. *Let M be a hemi-slant submanifold of a Kaehler manifold \bar{M} . Then M is a hemi-slant warped product if and only if for each $U, V \in TM$ and $W \in D^\perp$*

$$g((\bar{\nabla}_U F)V, JW) = -(BV\mu)\cos^2\theta g(U, W) - g(A_{FPV}U, W) \tag{36}$$

where μ is a C^∞ -function on M with $Z\mu = 0$ for each $Z \in D^\perp$.

Proof. Let $M = M_\theta \times_f M_\perp$ be a hemi-slant warped product submanifold of a Kaehler manifold \bar{M} . Then for any $U, V \in TM$ and $W \in TM_\perp$, by (14), we have

$$g((\bar{\nabla}_U F)V, JW) = -g(h(U, PV), JW).$$

On applying Lemma 4.3 on the right hand side, the above equation takes the form

$$\begin{aligned} g((\bar{\nabla}_U F)V, JW) &= -g(h(U, W), FPV) + (P^2BV\ln f)g(CU, W) \\ &= -g(A_{FPV}U, W) - \cos^2\theta(BV\ln f)g(CU, W). \end{aligned}$$

This proves (36).

Conversely, suppose that M is a hemi-slant submanifold of a Kaehler manifold \bar{M} such that (36) holds for a C^∞ -function μ on M with $Z\mu = 0$ for each $Z \in D^\perp$. Then for $X, Y \in D^\theta$

$$g((\bar{\nabla}_X F)Y, JW) = -g(A_{FPY}X, W).$$

Making use of formula (14), while taking account of the fact that $g(fh(X, Y), JW) = 0$, the above equation reduces to

$$g(h(X, PY), JW) = g(h(X, W), FPY).$$

Therefore by Proposition 3.4, D^θ is parallel. In other words, each leaf M_θ of D^θ is totally geodesic in M . Now, by (36) for $X \in D^\theta$ and $Z \in D^\perp$, we have

$$g((\bar{\nabla}_Z F)X, JW) = -(X\mu)\cos^2\theta g(Z, W) - g(h(Z, W), FPX). \tag{37}$$

On making use of (14) and the fact that $g(fh(X, Z), JW) = 0$, the left hand side of the above equation reduces to $-g(h(PX, Z), JW)$, whereas on using (4), (10) and the Kaehler condition, the second term in the right hand side of (37) is written as:

$$\begin{aligned} g(h(Z, W), FPX) &= -g(\bar{\nabla}_Z JW, PX) + \cos^2\theta g(\nabla_Z W, X) \\ &= g(h(PX, Z), JW) + \cos^2\theta g(\nabla_Z W, X). \end{aligned}$$

Thus, (37) reduces to

$$g(\nabla_Z W, X) = -(X\mu)g(Z, W).$$

Let M_\perp be a leaf of D^\perp and h'' , the second fundamental form of the immersion of M_\perp into M then on taking account of (4) and (18), the above equation yields

$$h''(Z, W) = -g(Z, W)\nabla\mu$$

which shows that M_\perp is totally umbilical in M with $-\nabla\mu$ as the mean curvature vector. Further, as $Z\mu = 0$ for all $Z \in D^\perp$, $-\nabla\mu$ is parallel. This means M_\perp is an extrinsic sphere in M . Hence by Theorem 4.1, M is locally a warped product submanifold $M_\theta \times_f M_\perp$ of \bar{M} . \square

If M is a CR- submanifold (a special case of hemi-slant submanifolds) then $\theta = 0$ and $FPV = 0$ for all $V \in TM$, therefore (36) reduces to

$$g((\bar{\nabla}_U F)V, JW) = -(BV\mu)g(U, W).$$

Hence, we may state:

Corollary 4.9. *A CR- submanifold M of a Kaehler manifold is a CR-warped product if and only if there exists a smooth function μ on M satisfying $Z\mu = 0$ such that*

$$g((\bar{\nabla}_U F)V, JW) = -(BV\mu)g(U, W),$$

for each $U, V \in TM$ and $W \in D^\perp$.

The above characterization was proved in [19].

References

- [1] F. R. Al-Solamy, V. A. Khan and S. Uddin, Geometry of warped product semi-slant submanifolds of nearly Kaehler manifolds, *Results in Mathematics* 71 (2017) 783–799.
- [2] J. K. Beem, P. Ehrlich and T. G. Powell, *Warped product manifolds in relativity*, Selected studies, North-Holland, Amsterdam-New York, 1982.
- [3] A. Bejancu, CR-submanifold of a Kaehler manifold I, *Proceedings of the American Mathematical Society* 89 (1978), 135–142.
- [4] R. L. Bishop and B. O'Neill, Manifolds of negative curvature, *Transactions of the American Mathematical Society* 145 (1969), 1–49.
- [5] V. Bonanzinga and K. Matsumoto, Warped product CR-submanifolds in locally conformal Kaehler manifolds, *Periodica Mathematica Hungarica* 48 (2004), 207–221.
- [6] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez, Semi-slant submanifolds of a Sasakian Manifold, *Geometriae Dedicata* 78 (1999), 183–199.
- [7] A. Carriazo, Bi-slant immersions, *Proc. ICRAMS 2000*, Kharagpur India, (2000), 88–97.
- [8] B.-Y. Chen, CR-submanifolds of a Kaehler Manifold I, *Journal of Differential Geometry* 16 (1981), 305–322.
- [9] B.-Y. Chen, Slant immersions, *Bulletin of the Australian Mathematical Society* 41 (1990), 135–147.
- [10] B.-Y. Chen, Geometry of warped product CR-submanifolds in Kaehler manifolds I, *Monatshefte fur Mathematik* 133 (2001), 177–195.
- [11] B.-Y. Chen, Geometry of warped product CR-submanifolds in Kaehler manifolds II, *Monatshefte fur Mathematik* 134 (2001), 103–119.
- [12] B.-Y. Chen, *Differential geometry of warped product manifolds and submanifolds*, World Scientific, Hackensack, NJ, 2017.
- [13] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*, Cambridge Univ. Press, Cambridge, 1973.
- [14] S. Heipko, Eine Innere Kennzeichnung der verzerrten Produkte, *Annals of Mathematics* 241 (1979), 209–215.
- [15] S. T. Hong, Warped products and black holes, *Nuovo Cim. B120* (2005), 1227–1234.
- [16] V. A. Khan and M. A. Khan, Semi-slant submanifolds of a nearly Kaehler manifold, *Turkish Journal of Mathematics* 31 (2007), 341–353.
- [17] V. A. Khan and K. A. Khan, Generic warped product submanifolds of nearly Kaehler manifolds, *Contributions to Algebra and Geometry* 50(2) (2009), 337–352.
- [18] V. A. Khan, K. A. Khan and S. Uddin, Warped product CR-submanifolds in nearly Kaehler manifolds, *SUT Journal of Mathematics* 43(2) (2007), 201–213.
- [19] V. A. Khan, K. A. Khan and S. Uddin, CR-warped product submanifolds in a Kaehler manifold, *Southeast Asian Bulletin of Mathematics* 33 (2009), 875–878.
- [20] P. Libermann and C. M. Marle, *Symplectic Geometry and Analytical Mechanics*, D. Reidel Pub. Co. 1987.
- [21] N. Papaghiuc, Semi-slant submanifolds of Kaehler manifold, *Analele Stiintifice ale Universitatii Al. I. Cuza din Iasi* 40 (1994), 55–61.
- [22] B. Sahin, Non-existence of warped product semi-slant submanifolds of Kaehler manifold, *Geometriae Dedicata* 117 (2006), 195–202.
- [23] B. Sahin, Warped product submanifolds of Kaehler manifolds with a slant factor, *Annales Polonici Mathematici* 95(3) (2009), 207–226.
- [24] S. Uddin, B.-Y. Chen and F. R. Al-Solamy, Warped product bi-slant immersions in Kaehler manifolds, *Mediterranean Journal of Mathematics* (2017), 14:95. doi:10.1007/s00009-017-0896-8.