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# $AB$-wavelet Frames in $L^2(\mathbb{R}^n)$

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**Abstract.** In order to provide a unified treatment for the continuum and digital realm of multivariate data, Guo, Labate, Weiss and Wilson [*Electron. Res. Announc. Amer. Math. Soc.* **10** (2004), 78-87] introduced the notion of $AB$-wavelets in the context of multiscale analysis. We continue and extend their work by studying the frame properties of $AB$-wavelet systems $\left\{D_A D_B T_k \psi^\ell \quad (k \in \mathbb{Z}^n;\ 1 \leqq \ell \leqq L)\right\}$ in $L^2(\mathbb{R}^n)$. More precisely, we establish four theorems giving sufficient conditions under which the $AB$-wavelet system constitutes a frame for $L^2(\mathbb{R}^n)$. The proposed conditions are stated in terms of the Fourier transforms of the generating functions.

## 1. Introduction

A complete representation of non-stationary signals requires frequency analysis that is local in time, resulting in the time-frequency analysis of signals. Although time-frequency analysis of signals had its origin almost sixty years ago, there has been major development of the time-frequency distributions approach in the last three decades. The basic idea of the method is to develop a joint function of time and frequency, known as a time-frequency distribution, that can describe the energy density of a signal simultaneously in both time and frequency. Unlike the case with orthonormal bases, series expansions of functions with frames give such information at fixed discrete points in the time-frequency plane under relatively flexible criterion. The theory of frames was initiated by Duffin and Shaeffer [7] with regard to certain interesting problems in non-harmonic Fourier series, and more precisely with the query to determine when a family of exponentials $\left\{e^{i\alpha_n t} : n \in \mathbb{Z}\right\}$ is complete for $L^2[a, b]$. Apparently, the significance of the idea was not realized by the mathematical community; at least it took almost 30 years prior to the next printed treatment. In 1986, Daubechies *et al.* [4] revisited the theory of frames and pointed out that orthonormal basis like series expansions of functions in $L^2(\mathbb{R})$ can be obtained via frames. This pioneering treatment had a profound impact and the theory of frames began to be investigated in a broader context, particularly in the more specialized areas of wavelet frames and Gabor frames. Nowadays, the theory of frames has attained respectable status

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in the realm of mathematics with numerous implications in signal and image processing, harmonic analysis, Banach space theory, sampling theory and filter banks, and is a pivotal tool in the modern time-frequency analysis [1, 5, 6].

It is a fact that wavelets serve as a promising and powerful analyzing tool for time-frequency analysis and have been applied in a number of fields mainly due to the reason that wavelets offer good properties like symmetry, regularity, continuity, and compact support. However, the standard orthogonal wavelets suffer from three major limitations: poor directionality, shift sensitivity and lack of phase information. These disadvantages severely restrict its scope for certain classes of singular integral operators, signal and image processing applications such as edge detection, image segmentation, motion estimation, and so on. To overcome these limitations, Guo *et al.* [8, 9] introduced the notion of $AB$-wavelet systems in $L^2(\mathbb{R}^n)$ as a directional representation system in order to provide a unified treatment of the continuum and digital realm. These systems have the following form:

$$\begin{aligned}\mathcal{W}_{AB}(\psi, j, k) &= \left\{D_A D_B T_k \psi^\ell : A \in \mathcal{A}, B \in \mathcal{B}, k \in \mathbb{Z}^n, 1 \leqq \ell \leqq L\right\} \\ &= \left\{\psi^\ell_{j,k}(x) = q^{j/2}\psi\left(A^j B^\ell x - k\right) : j \in \mathbb{Z}, k \in \mathbb{Z}^n, 1 \leqq \ell \leqq L\right\}, \qquad (1.1)\end{aligned}$$

where $L = \min\{m : B^m = I, m \geqq 1, m \in \mathbb{Z}\}$, $T_k$ are the translations, defined by $T_k f(x) = f(x-k)$, $D_A$ are the dilations, defined by

$$D_A f(x) = q^{1/2} f(Ax) \quad (q = |\det A|),$$

and the sets $\mathcal{A}$ and $\mathcal{B}$, which may not be commuting matrix sets, are denumerable subsets of $GL_n(\mathbb{R})$. Generally, certain constraints are put on the sets $\mathcal{A}$ and $\mathcal{B}$. For instance, $\mathcal{A}$ is typically chosen to be a collection of invertible matrices with eigenvalues $|\lambda| > 1$ and $\mathcal{B}$ to be a group of uni-modular matrices. Nevertheless, such constraints are not always necessary (see [14]).

The $AB$-wavelet system $\mathcal{W}_{AB}(\psi, j, k)$ is referred to as a $AB$-wavelet frame, provided there exist scalars $C$ and $D$ $(0 < C \leqq D < \infty)$ such that the following inequality:

$$C\|f\|_2^2 \leqq \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n}\left|\left\langle f, \psi^\ell_{j,k}\right\rangle\right|^2 \leqq D\|f\|_2^2 \qquad (1.2)$$

holds true for every $f \in L^2(\mathbb{R}^n)$. We call the optimal constants $C$ and $D$ the lower and upper frame bounds,respectively. A tight $AB$-wavelet frame refers to the case when $C = D$. Furthermore, a *Parseval frame* refers to the case when $C = D = 1$.

The redundancy and flexibility offered by wavelet frames have impelled their implications in numerous areas of mathematics, physics and engineering. Owing to the increasing number of applications, significant attention has been paid to find the necessary and sufficient conditions for the wavelet systems to constitute frames in $L^2(\mathbb{R}^n)$ in recent years. For instance, Daubechies [3] proved the first result on the necessary and sufficient conditions for the following conventional wavelet system $\left\{\psi_{j,k} := a^{j/2}\psi\left(a^j x - kb\right) : j,k \in \mathbb{Z}\right\}$ to constitute a frame for $L^2(\mathbb{R}^n)$, Chui and Shi [2] refined the result of Daubechies in [3], Christenson [1] established a stronger version of Daubechies sufficient condition for wavelet frames. Recently, these conditions have been further refined and investigated by several authors (see, for example, [10–13, 15]). Therefore, the main objective of this article is to establish conditions on the wavelet function $\psi$ and the dilation and translation parameters so that the corresponding $AB$-wavelet system $\mathcal{W}_{AB}(\psi, j, k)$ given by (1.1) constitutes a frame for $L^2(\mathbb{R}^n)$. Particularly, we derive four sufficient conditions for the $AB$-wavelet system $\mathcal{W}_{AB}(\psi, j, k)$ to be a frame for $L^2(\mathbb{R}^n)$ using the machinery of the Fourier transforms.

The rest of this article is organized as follows. In Section 2, we first present some notations and prerequisites related to the $AB$-wavelets. We then establish, in Section 2 itself, four sufficient conditions for the $AB$-wavelet frames in $L^2(\mathbb{R}^n)$ via Fourier transforms.

**2. $AB$-Wavelet Frames in $L^2(\mathbb{R}^n)$**

We shall use the following conventions throughout this paper. We adopt the notation that the time domain is represented by $\mathbb{R}^n$, and its elements will be column vectors denoted by letters of the Roman alphabet, $x = (x_1, x_2, \ldots, x_n)^t \in \mathbb{R}^n$. The elements of the frequency domain will be row vectors, $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$. We denote by $T^n = [-1/2, 1/2]^n$ the $n$-dimensional torus and hence, clearly, the subsets of $\mathbb{R}^n$ are invariant under $\mathbb{Z}^n$ translations and the subsets of $T^n$ are often identified. We use the Fourier transform in the form given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \xi x} dx. \tag{2.1}$$

As such, the Fourier transform of the $AB$-wavelet system $\mathcal{W}_{AB}(\psi, j, k)$ is given by

$$\hat{\psi}^{\ell}_{j,k}(\xi) = q^{-j/2} \hat{\psi}\Big(A^{*-j} B^{*-\ell} \xi\Big) e^{-2\pi i B^{-\ell} A^{-j} k \xi}, \tag{2.2}$$

where $A^*$ and $B^*$ denotes the transpose of $A$ and $B$, respectively. Before proceeding further, it is useful to state a basic lemma the proof of which can be found in Christensen [1].

**Lemma 1.** *Suppose that $\{f_k\}_{k=1}^{\infty}$ is a family of elements in a Hilbert space $\mathcal{H}$ such that there exist constants $0 < C \leqq D < \infty$ satisfying the following inequality*:

$$C\big\|f\big\|_2^2 \leqq \sum_{k=1}^{\infty} \Big|\Big\langle f, f_k \Big\rangle\Big|^2 \leqq D\big\|f\big\|_2^2, \tag{2.3}$$

*for all $f$ belonging to a dense subset $\mathfrak{D}$ of $\mathcal{H}$. Then the same inequalities* (2.3) *are true for all $f \in \mathcal{H}$, that is, $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$.*

In view of Lemma 1, we will consider the following set of functions:

$$\mathcal{D} = \Big\{ f \in L^2(\mathbb{R}^n) : \hat{f} \in L^{\infty}(\mathbb{R}^n) \text{ and } \hat{f} \text{ has compact support in } \mathbb{R}^n \setminus \{0\} \Big\}.$$

It is clear that $\mathcal{D}$ is a dense subspace of $L^2(\mathbb{R}^n)$. Therefore, it is sufficient to verify that the $AB$-wavelet system $\mathcal{W}_{AB}(\psi, j, k)$ given by (1.1) is a frame for $L^2(\mathbb{R}^n)$ if (1.2) holds true for all $f \in \mathcal{D}$.

We now prove a lemma which will be used in the proofs of our main results.

**Lemma 2.** *Suppose that the AB-wavelet system $\mathcal{W}_{AB}(\psi, j, k)$ is defined by* (1.1). *If $f \in \mathcal{D}$ and* ess $\sup \Big\{ \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \Big|\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\Big|^2 : 1 \leqq \xi \leqq q \Big\} < \infty$, *then*

$$\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \Big|\Big\langle f, \psi^{\ell}_{j,k} \Big\rangle\Big|^2 = \int_{\mathbb{R}^n} \big|\hat{f}(\xi)\big|^2 \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \Big|\hat{\psi}\Big((A^*)^j (B^*)^{\ell} \xi\Big)\Big|^2 d\xi + R_{\psi}(f), \tag{2.4}$$

*where*

$$R_{\psi}(f) = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{s \in \mathbb{Z}^n \setminus \{0\}} \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big) \hat{f}\Big(\xi + (A^*)^j (B^*)^{\ell} s\Big)\, \overline{\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi + s\Big)} d\xi. \tag{2.5}$$

*Furthermore, the iterated series in* (2.5) *is absolutely convergent.*

*Proof.* For fixed $j \in \mathbb{Z}$ and $f \in \mathcal{D}$, we first examine the expression

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^n} \left|\left\langle f, \psi^\ell_{j,k}\right\rangle\right|^2 &= \sum_{k\in\mathbb{Z}^n} \left|\left\langle \hat{f}, \hat{\psi}^\ell_{j,k}\right\rangle\right|^2 \\
&= \sum_{k\in\mathbb{Z}^n} \left|\int_{\mathbb{R}^n} \hat{f}(\xi)\, q^{-j/2}\overline{\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)} e^{2\pi i B^{-\ell}A^{-j}k\xi}\, d\xi\right|^2 \\
&= \sum_{k\in\mathbb{Z}^n} q^j \left|\int_{\mathbb{R}^n} \hat{f}\Big((A^*)^j(B^*)^\ell\xi\Big)\overline{\hat{\psi}(\xi)}\, e^{2\pi ik\xi}\, d\xi\right|^2 \\
&= q^j \sum_{k\in\mathbb{Z}^n} \int_{\mathbb{R}^n} \hat{f}\Big((A^*)^j(B^*)^\ell\xi\Big)\overline{\hat{\psi}(\xi)}\, e^{2\pi ik\xi}\, d\xi \int_{\mathbb{R}^n} \overline{\hat{f}\Big((A^*)^j(B^*)^\ell\xi\Big)}\hat{\psi}(\xi)\, e^{-2\pi ik\xi}\, d\xi \\
&= q^j \sum_{k\in\mathbb{Z}^n}\sum_{s\in\mathbb{Z}^n} \int_{[0,1]^n} \hat{f}\Big((A^*)^j(B^*)^\ell(\xi+s)\Big)\overline{\hat{\psi}(\xi+s)}\, e^{2\pi ik\xi}\, d\xi \int_{\mathbb{R}^n} \overline{\hat{f}\Big((A^*)^j(B^*)^\ell\xi\Big)}\hat{\psi}(\xi)\, e^{-2\pi ik\xi}\, d\xi \\
&= q^j \int_{\mathbb{R}^n} \overline{\hat{f}\Big((A^*)^j(B^*)^\ell\xi\Big)}\hat{\psi}(\xi) \left\{\sum_{s\in\mathbb{Z}^n} \hat{f}\Big((A^*)^j(B^*)^\ell(\xi+s)\Big)\overline{\hat{\psi}(\xi+s)}\right\} d\xi \\
&= \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big) \left\{\sum_{s\in\mathbb{Z}^n} \hat{f}\Big(\xi+(A^*)^j(B^*)^\ell s\Big)\overline{\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi+s\Big)}\right\} d\xi.
\end{aligned}$$

We then have

$$\begin{aligned}
\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} \left|\left\langle f, \psi^\ell_{j,k}\right\rangle\right|^2 &= \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big) \left\{\sum_{s\in\mathbb{Z}^n} \hat{f}\Big(\xi+(A^*)^j(B^*)^\ell s\Big)\overline{\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi+s\Big)}\right\} d\xi \\
&= Q_\psi(f) + R_\psi(f), \qquad (2.6)
\end{aligned}$$

where

$$Q_\psi(f) = \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\right|^2 d\xi, \qquad (2.7)$$

$$R_\psi(f) = \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{s\in\mathbb{Z}^n\setminus\{0\}} \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\hat{f}\Big(\xi+(A^*)^j(B^*)^\ell s\Big)\overline{\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi+s\Big)} d\xi. \qquad (2.8)$$

Implementing our assumption, it follows that $Q_\psi(f)$ is convergent.
We now turn to prove that the iterated series in (2.5) is absolutely convergent. We note that

$$\left|\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi+s\Big)\right| \leqq \frac{1}{2}\left[\left|\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\right|^2 + \left|\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi+s\Big)\right|^2\right].$$

It is easy to verify that the convergence of $|R_\psi(f)|$ follows from the convergence of

$$\begin{aligned}
&\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{s\in\mathbb{Z}^n\setminus\{0\}} \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|\left|\hat{f}\Big(\xi+(A^*)^j(B^*)^\ell s\Big)\right|\left|\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\right|^2 d\xi \\
&\leqq \int_{\mathbb{R}^n} \left\{\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{s\in\mathbb{Z}^n\setminus\{0\}} q^j \left|\hat{f}\Big((A^*)^j(B^*)^\ell\xi\Big)\right|\left|\hat{f}\Big((A^*)^j(B^*)^\ell\xi+(A^*)^j(B^*)^\ell s\Big)\right|\right\} \left|\hat{\psi}(\xi)\right|^2 d\xi.
\end{aligned}$$

Since $s \neq 0$ and $f \in \mathcal{D}$, there exists a constant $J > 0$ such that, for all $j \geqq J$, we have

$$\hat{f}\Big((A^*)^j(B^*)^\ell\xi\Big)\hat{f}\Big((A^*)^j(B^*)^\ell\xi+(A^*)^j(B^*)^\ell s\Big) = 0.$$

On the other hand, for each fixed $j \leqq J$, the number of $s \in \mathbb{Z}^n$ for which the above product is nonzero is less than or equal to $Mq^{-j}$ for some constant $M$. We thus find that

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{s\in\mathbb{Z}^n\setminus\{0\}} q^j \left|\hat{f}\left((A^*)^j(B^*)^\ell\xi\right)\right|\left|\hat{f}\left((A^*)^j(B^*)^\ell\xi+(A^*)^j(B^*)^\ell s\right)\right| \leqq M\sum_{\ell=1}^{L}\sum_{j\leqq J}\left\|\hat{f}\right\|_\infty^2 \mathbf{1}_S\left((A^*)^j(B^*)^\ell\xi\right), \tag{2.9}$$

where $\mathbf{1}_S(\xi)$ is the characteristic function on a compact set $S$ in $\mathbb{R}^n \setminus \{0\}$. Since $A$ is an expansive matrix and $B$ is a rotation matrix, so we observe that $M_1 < \left|(A^*)^j(B^*)^\ell\xi\right| < M_2$, where $M_1$ and $M_2$ are constants. Therefore, we can find a constant $K > 0$ such that the inequality (2.9) is less than $MK\left\|\hat{f}\right\|_\infty^2$. Thus, the iterated series in (2.5) is absolutely convergent. The proof of Lemma 2 is complete. □

In order to derive the first sufficient condition for $AB$-wavelet frame in $L^2(\mathbb{R}^n)$, we set

$$\Delta_\psi(m) = \operatorname{ess}\sup_{1\leqq\xi\leqq q}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\left|\beta_\psi\left(m,(A^*)^{-j}(B^*)^{-\ell}\xi\right)\right|, \tag{2.10}$$

where

$$\beta_\psi(m,\xi) = \sum_{k\neq 0}\hat{\psi}\left((A^*)^k(B^*)^\ell\xi\right)\overline{\hat{\psi}\left((A^*)^k(B^*)^\ell(\xi+m)\right)}. \tag{2.11}$$

We also use the following set:

$$\Lambda = \left\{(AB)n+\ell : n\in\mathbb{Z}^n, 1\leqq \ell\leqq q\right\}.$$

**Theorem 1.** *Let $\psi \in L^2(\mathbb{R}^n)$ be such that*

$$C_1(\psi) = ess\inf_{1\leqq\xi\leqq q}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left((A^*)^{-j}(B^*)^{-\ell}\xi\right)\right|^2 - \sum_{m\in\Lambda}\left[\Delta_\psi(m)\cdot\Delta_\psi(-m)\right]^{1/2} > 0,$$

$$D_1(\psi) = ess\sup_{1\leqq\xi\leqq q}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left((A^*)^{-j}(B^*)^{-\ell}\xi\right)\right|^2 + \sum_{m\in\Lambda}\left[\Delta_\psi(m)\cdot\Delta_\psi(-m)\right]^{1/2} < \infty.$$

*Then the AB-wavelet system $\mathcal{W}_{AB}(\psi, j, k)$ given by* (1.1) *constitutes a frame for $L^2(\mathbb{R}^n)$ with bounds $C_1(\psi)$ and $D_1(\psi)$.*

*Proof.* Since the last series in (2.5) is absolutely convergent for every $f \in \mathcal{D}$, we can estimate $R_\psi(f)$ by rearranging the series, changing the order of summation and integration by Levi's Lemma as follows:

$$\begin{aligned} R_\psi(f) &= \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}\overline{\hat{f}(\xi)}\,\hat{\psi}\left((A^*)^{-j}(B^*)^{-\ell}\xi\right)\left[\sum_{n\neq 0}\hat{f}\left(\xi+(A^*)^j(B^*)^\ell n\right)\overline{\hat{\psi}\left((A^*)^{-j}(B^*)^{-\ell}\xi+n\right)}\right]d\xi \\ &= \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}\overline{\hat{f}(\xi)}\left[\sum_{k\neq 0}\sum_{m\in\Lambda}\hat{\psi}\left((A^*)^{-j}(B^*)^{-\ell}\xi\right)\hat{f}\left(\xi+(A^*)^{j+k}(B^*)^\ell m\right)\overline{\hat{\psi}\left((A^*)^{-j}(B^*)^{-\ell}\xi+(A^*)^k(B^*)^\ell m\right)}\right]d\xi \\ &= \int_{\mathbb{R}^n}\overline{\hat{f}(\xi)}\left[\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\neq 0}\sum_{m\in\Lambda}\hat{\psi}\left((A^*)^{-j+k}(B^*)^{-\ell}\xi\right)\hat{f}\left(\xi+(A^*)^j(B^*)^\ell m\right)\overline{\hat{\psi}\left((A^*)^{-j+k}(B^*)^{-\ell}\xi+(A^*)^k(B^*)^\ell m\right)}\right]d\xi \end{aligned}$$

$$= \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \left[ \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\Lambda} \hat{f}\Big(\xi + (A^*)^j (B^*)^\ell m\Big) \sum_{k\neq 0} \hat{\psi}\Big((A^*)^{-j+k}(B^*)^{-\ell}\xi\Big) \overline{\hat{\psi}\Big((A^*)^k (B^*)^{-\ell}\Big((A^*)^{-j}\xi + m\Big)} \right] d\xi$$

$$= \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \left[ \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\Lambda} \hat{f}\Big(\xi + (A^*)^j (B^*)^\ell m\Big) \beta_\psi\Big(m, (A^*)^{-j}(B^*)^{-\ell}\xi\Big) \right] d\xi$$

$$= \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\Lambda} \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \hat{f}\Big(\xi + (A^*)^j (B^*)^\ell m\Big) \beta_\psi\Big(m, (A^*)^{-j}(B^*)^{-\ell}\xi\Big) d\xi.$$

We further deduce that

$$\begin{aligned}
&\Big|R_\psi(f)\Big| \\
&\leq \int_{\mathbb{R}^n} \Big|\hat{f}(\xi)\Big| \left[ \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\Lambda} \Big|\hat{f}\Big(\xi + (A^*)^j (B^*)^\ell m\Big)\Big| \Big|\beta_\psi\Big(m, (A^*)^{-j}(B^*)^{-\ell}\xi\Big)\Big| \right] d\xi \\
&= \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\Lambda} \int_{\mathbb{R}^n} \left[ \Big|\hat{f}(\xi)\Big| \Big|\beta_\psi\Big(m, (A^*)^{-j}(B^*)^{-\ell}\xi\Big)\Big|^{1/2} \right] \left[ \Big|\hat{f}\Big(\xi + (A^*)^j (B^*)^\ell m\Big)\Big| \Big|\beta_\psi\Big(m, (A^*)^{-j}(B^*)^{-\ell}\xi\Big)\Big|^{1/2} \right] d\xi \\
&\leqq \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\Lambda} \left[ \int_{\mathbb{R}^n} \Big|\hat{f}(\xi)\Big|^2 \Big|\beta_\psi\Big(m, (A^*)^{-j}(B^*)^{-\ell}\xi\Big)\Big| d\xi \right]^{1/2} \left[ \int_{\mathbb{R}^n} \Big|\hat{f}\Big(\xi + (A^*)^j (B^*)^\ell m\Big)\Big|^2 \Big|\beta_\psi\Big(m, (A^*)^{-j}(B^*)^{-\ell}\xi\Big)\Big| d\xi \right]^{1/2} \\
&\leqq \sum_{m\in\Lambda} \left[ \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \Big|\hat{f}(\xi)\Big|^2 \Big|\beta_\psi\Big(m, (A^*)^{-j}(B^*)^{-\ell}\xi\Big)\Big| d\xi \right]^{1/2} \left[ \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \Big|\hat{f}\Big(\xi + (A^*)^j (B^*)^\ell m\Big)\Big|^2 \Big|\beta_\psi\Big(m, (A^*)^{-j}(B^*)^{-\ell}\xi\Big)\Big| d\xi \right]^{1/2} \\
&= \sum_{m\in\Lambda} \left[ \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \Big|\hat{f}(\xi)\Big|^2 \Big|\beta_\psi\Big(m, (A^*)^{-j}(B^*)^{-\ell}\xi\Big)\Big| d\xi \right]^{1/2} \left[ \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \Big|\hat{f}(\eta)\Big|^2 \Big|\beta_\psi\Big(-m, (A^*)^{-j}(B^*)^{-\ell}\eta\Big)\Big| d\eta \right]^{1/2} \\
&\leqq \sum_{m\in\Lambda} \left[ \int_{\mathbb{R}^n} \Big|\hat{f}(\xi)\Big|^2 \Delta_\psi(m)\, d\xi \right]^{1/2} \left[ \int_{\mathbb{R}^n} \Big|\hat{f}(\xi)\Big|^2 \Delta_\psi(-m)\, d\xi \right]^{1/2} \\
&= \int_{\mathbb{R}^n} \Big|\hat{f}(\xi)\Big|^2 d\xi \sum_{m\in\Lambda} \Big[\Delta_\psi(m) \cdot \Delta_\psi(-m)\Big]^{1/2}.
\end{aligned}$$

Consequently, it follows from the expression (2.4) in Lemma 2 that

$$\int_{\mathbb{R}^n} \Big|\hat{f}(\xi)\Big|^2 \left\{ \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \Big|\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\Big|^2 - \sum_{m\in\Lambda} \Big[\Delta_\psi(m) \cdot \Delta_\psi(-m)\Big]^{1/2} \right\} d\xi \leqq \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \Big|\Big\langle f, \psi_{j,k}^\ell \Big\rangle\Big|^2 \tag{2.12}$$

and

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \Big|\Big\langle f, \psi_{j,k}^\ell \Big\rangle\Big|^2 \leqq \int_{\mathbb{R}^n} \Big|\hat{f}(\xi)\Big|^2 \left\{ \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \Big|\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\Big|^2 + \sum_{m\in\Lambda} \Big[\Delta_\psi(m) \cdot \Delta_\psi(-m)\Big]^{1/2} \right\} d\xi. \tag{2.13}$$

Taking the infimum in (2.12) and the supremum in (2.13), we see that

$$C_1(\psi)\big\|f\big\|_2^2 \leqq \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \Big|\Big\langle f, \psi_{j,k}^\ell \Big\rangle\Big|^2 \leqq D_1(\psi)\big\|f\big\|_2^2$$

holds true for every $f \in \mathcal{D}$. The proof of Theorem 1 is complete. □

**Theorem 2.** *Let $\psi \in L^2(\mathbb{R}^n)$ be such that*

$$C_2(\psi) = \operatorname{ess}\inf_{1\leqq\xi\leqq q} \left\{ \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} \left|\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi\big)\right|^2 - \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\neq 0} \left|\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi\big)\overline{\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi+k\big)}\right| \right\} > 0,$$

$$D_2(\psi) = \operatorname{ess}\sup_{1\leqq\xi\leqq q} \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\neq 0} \left|\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi\big)\overline{\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi+k\big)}\right| < \infty.$$

*Then the AB-wavelet system $\mathcal{W}_{AB}(\psi, j, k)$ given by* (1.1) *is a frame for $L^2(\mathbb{R}^n)$ with bounds $C_2(\psi)$ and $D_2(\psi)$.*

*Proof.* We apply Lemma 2 to estimate $R_\psi(f)$ in (2.5) for $f \in \mathcal{D}$ by using another technique. We first deduce that

$$\begin{aligned}
\big|R_\psi(f)\big| &= \left|\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\neq 0}\int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi\big)\hat{f}\left(\xi+(A^*)^{j}(B^*)^{\ell}k\right)\overline{\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi+k\big)}d\xi\right| \\
&\leqq \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\neq 0}\left\{\int_{\mathbb{R}^n}\big|\hat{f}(\xi)\big|^2\left|\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi\big)\overline{\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi+k\big)}\right|d\xi\right\}^{1/2} \\
&\qquad \left\{\int_{\mathbb{R}^n}\left|\hat{f}\left(\xi+(A^*)^{j}(B^*)^{\ell}k\right)\right|^2\left|\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi\big)\overline{\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi+k\big)}\right|d\xi\right\}^{1/2} \\
&\leqq \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\left\{\sum_{k\neq 0}\int_{\mathbb{R}^n}\big|\hat{f}(\xi)\big|^2\left|\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi\big)\overline{\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi+k\big)}\right|d\xi\right\}^{1/2} \\
&\qquad \left\{\sum_{k\neq 0}\int_{\mathbb{R}^n}\left|\hat{f}\left(\xi+(A^*)^{j}(B^*)^{\ell}k\right)\right|^2\left|\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi\big)\overline{\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi+k\big)}\right|d\xi\right\}^{1/2} \\
&= \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\left\{\sum_{k\neq 0}\int_{\mathbb{R}^n}\big|\hat{f}(\xi)\big|^2\left|\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi\big)\overline{\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi+k\big)}\right|d\xi\right\}^{1/2} \\
&\qquad \left\{\sum_{k\neq 0}\int_{\mathbb{R}^n}\big|\hat{f}(\eta)\big|^2\left|\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\eta-k\big)\overline{\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\eta\big)}\right|d\eta\right\}^{1/2} \\
&\leqq \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\neq 0}\int_{\mathbb{R}^n}\big|\hat{f}(\xi)\big|^2\left|\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi\big)\overline{\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi+k\big)}\right|d\xi.
\end{aligned}$$

Using Levi's Lemma once again, we obtain

$$\big|R_\psi(f)\big| \leq \int_{\mathbb{R}^n}\big|\hat{f}(\xi)\big|^2\left\{\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\neq 0}\left|\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi\big)\overline{\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi+k\big)}\right|\right\}d\xi.$$

Also, by applying (2.4), we have

$$\int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \left\{ \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \left|\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\right|^2 - \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\neq 0} \left|\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\overline{\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi + k\Big)}\right| \right\} d\xi$$

$$\leqq \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \left|\left\langle f, \psi_{j,k}^{\ell}\right\rangle\right|^2 \tag{2.14}$$

and

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \left|\left\langle f, \psi_{j,k}^{\ell}\right\rangle\right|^2 \leqq \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \left\{ \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\neq 0} \left|\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\overline{\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi + k\Big)}\right| \right\} d\xi. \tag{2.15}$$

Now, by taking the infimum in (2.14) and the supremum in (2.15), we observe again that

$$C_2(\psi)\|f\|_2^2 \leqq \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \left|\left\langle f, \psi_{j,k}^{\ell}\right\rangle\right|^2 \leqq D_2(\psi)\|f\|_2^2.$$

This completes the proof of Theorem 2. □

In order to state our next sufficient condition, we first introduce some further notations. Given the dilation matrix $A$ and the rotation matrix $B$, similar to the $a$-adic number [1], the $AB$-adic vector is defined by

$$\Gamma = \Big\{\alpha \in \mathbb{R}^n : \exists\, (j, m) \in \mathbb{Z}\times\mathbb{Z}^n, \alpha = (A^*)^{-j}(B^*)^{-\ell}m, 1 \leqq \ell \leqq L\Big\}. \tag{2.16}$$

Also, for all $\alpha \in \Gamma$, we define

$$I(\alpha) = \Big\{(j, m) \in \mathbb{Z}\times\mathbb{Z}^n : \alpha = (A^*)^{-j}(B^*)^{-\ell}m, 1 \leqq \ell \leqq L\Big\}, \tag{2.17}$$

$$\Upsilon_\alpha^+(\xi) = \sum_{\ell=1}^{L} \sum_{(j,m)\in I(\alpha)} \hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\overline{\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi + m\Big)} \tag{2.18}$$

and

$$\Upsilon_\alpha^-(\xi) = \sum_{\ell=1}^{L} \sum_{(j,m)\in I(\alpha)} \hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\overline{\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi - m\Big)}. \tag{2.19}$$

**Theorem 3.** *Let $\psi \in L^2(\mathbb{R}^n)$ be such that*

$$C_3(\psi) = \operatorname{ess}\inf_{1\leqq\xi\leqq q} \Big\{\Upsilon_0^+(\xi) - \sum_{\alpha\in\Gamma\setminus\{0\}} \left|\Upsilon_\alpha^+(\xi)\right|\Big\} > 0,$$

$$D_3(\psi) = \operatorname{ess}\sup_{1\leqq\xi\leqq q} \sum_{\alpha\in\Gamma\setminus\{0\}} \left|\Upsilon_\alpha^+(\xi)\right| < \infty.$$

*Then the AB-wavelet system $\mathcal{W}_{AB}(\psi, j, k)$ given by* (1.1) *is a frame for $L^2(\mathbb{R}^n)$ with bounds $C_3(\psi)$ and $D_3(\psi)$.*

*Proof.* We first note that

$$\Upsilon_0^+(\xi) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \left|\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\right|^2.$$

We apply Lemma 2 to re-estimate $R_\psi(f)$ for $f \in \mathcal{D}$ as follows:

$$\begin{aligned}
R_\psi(f) &= \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big) \left\{\sum_{m\neq 0} \hat{f}\left(\xi + (A^*)^j(B^*)^\ell m\right) \overline{\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi + m\Big)}\right\} d\xi \\
&= \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\neq 0} \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\hat{f}\left(\xi + (A^*)^j(B^*)^\ell m\right) \hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\overline{\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi + m\Big)} d\xi \\
&= \sum_{\ell=1}^{L} \sum_{\alpha\in\Gamma\setminus\{0\}} \sum_{(j,k)\in I(\alpha)} \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\hat{f}\left(\xi + (A^*)^j(B^*)^\ell k\right) \hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\overline{\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi + k\Big)} d\xi \\
&= \sum_{\alpha\in\Gamma\setminus\{0\}} \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\hat{f}(\xi+\alpha) \left\{\sum_{\ell=1}^{L} \sum_{(j,k)\in I(\alpha)} \hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\overline{\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi + k\Big)}\right\} d\xi \\
&= \sum_{\alpha\in\Gamma\setminus\{0\}} \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\hat{f}(\xi+\alpha)\, \Upsilon_\alpha^+(\xi)\, d\xi. \qquad (2.20)
\end{aligned}$$

Using the Cauchy-Schwartz inequality, Eq. (2.20) becomes

$$\begin{aligned}
\left|R_\psi(f)\right| &\leq \sum_{\alpha\in\Gamma\setminus\{0\}} \left\{\int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \left|\Upsilon_\alpha^+(\xi)\right| d\xi\right\}^{1/2} \left\{\int_{\mathbb{R}^n} \left|\hat{f}(\xi+\alpha)\right|^2 \left|\Upsilon_\alpha^+(\xi)\right| d\xi\right\}^{1/2} \\
&\leqq \left\{\sum_{\alpha\in\Gamma\setminus\{0\}} \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \left|\Upsilon_\alpha^+(\xi)\right| d\xi\right\}^{1/2} \left\{\sum_{\alpha\in\Gamma\setminus\{0\}} \int_{\mathbb{R}^n} \left|\hat{f}(\xi+\alpha)\right|^2 \left|\Upsilon_\alpha^+(\xi)\right| d\xi\right\}^{1/2}. \qquad (2.21)
\end{aligned}$$

For $\eta = \xi + \alpha$, we deduce from $\alpha = A^{*j}B^{*\ell}k$ for $(j,k) \in I(\alpha)$ that

$$\begin{aligned}
\Upsilon_\alpha^+(\xi) &= \sum_{\ell=1}^{L} \sum_{(j,k)\in I(\alpha)} \hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\overline{\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi + k\Big)} \\
&= \sum_{\ell=1}^{L} \sum_{(j,k)\in I(\alpha)} \hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}(\eta-\xi)\Big)\overline{\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}(\eta-\xi) + k\Big)} \\
&= \sum_{\ell=1}^{L} \sum_{(j,k)\in I(\alpha)} \hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\eta - k\Big)\overline{\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\eta\Big)} \\
&= \overline{\Upsilon_\alpha^-(\eta)}.
\end{aligned}$$

Therefore, we have

$$\sum_{\alpha\in\Gamma\setminus\{0\}} \left|\Upsilon_\alpha^+(\xi)\right| = \sum_{\alpha\in\Gamma\setminus\{0\}} \left|\overline{\Upsilon_\alpha^-(\xi)}\right|. \qquad (2.22)$$

Replacing $\xi + \alpha$ by $\eta$ in the last integration of (2.21), we find from (2.21) and (2.22) that

$$\left|R_\psi(f)\right| \leq \left\{\sum_{\alpha\in\Gamma\setminus\{0\}} \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \left|\Upsilon_\alpha^+(\xi)\right| d\xi\right\}^{1/2} \left\{\sum_{\alpha\in\Gamma\setminus\{0\}} \int_{\mathbb{R}^n} \left|\hat{f}(\eta)\right|^2 \left|\overline{\Upsilon_\alpha^-(\eta)}\right| d\eta\right\}^{1/2} \leqq \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \left\{\sum_{\alpha\in\Gamma\setminus\{0\}} \left|\Upsilon_\alpha^+(\xi)\right|\right\} d\xi. \qquad (2.23)$$

Lemma 2 and the inequality (2.23) imply that

$$\int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \left\{ \Upsilon_0^+(\xi) - \sum_{\alpha\in\Gamma\setminus\{0\}} \left|\Upsilon_\alpha^+(\xi)\right| \right\} d\xi \leqq \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \left|\left\langle f, \psi_{j,k}^\ell \right\rangle\right|^2 \tag{2.24}$$

and

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \left|\left\langle f, \psi_{j,k}^\ell \right\rangle\right|^2 \leqq \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \left\{ \Upsilon_0^+(\xi) + \sum_{\alpha\in\Gamma\setminus\{0\}} \left|\Upsilon_\alpha^+(\xi)\right| \right\} d\xi$$

or, equivalently,

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \left|\left\langle f, \psi_{j,k}^\ell \right\rangle\right|^2 \leqq \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \left\{ \sum_{\alpha\in\Gamma} \left|\Upsilon_\alpha^+(\xi)\right| \right\} d\xi. \tag{2.25}$$

Upon taking the infimum in (2.24) and the supremum in (2.25), we see again that

$$C_3(\psi)\left\|f\right\|_2^2 \leqq \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \left|\left\langle f, \psi_{j,k}^\ell \right\rangle\right|^2 \leqq D_3(\psi)\left\|f\right\|_2^2.$$

The proof of Theorem 3 is complete. □

With the notations in (2.18) and (2.19), we define the new sets as follows:

$$\Omega_\alpha^+ = \operatorname{ess\,sup}\left\{ \left|\Upsilon_\alpha^+(\xi)\right| : 1 \leqq \xi \leqq q \right\}, \qquad \Omega_\alpha^- = \operatorname{ess\,sup}\left\{ \left|\Upsilon_\alpha^-(\xi)\right| : 1 \leqq \xi \leqq q \right\}. \tag{2.26}$$

**Theorem 4.** *Let $\psi \in L^2(\mathbb{R}^n)$ be such that*

$$C_4(\psi) = \operatorname{ess}\inf_{1\leqq\xi\leqq q} \left\{ \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \left|\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\right|^2 \right\} - \sum_{\alpha\in\Gamma\setminus\{0\}} \left[\Pi_\alpha^+ \, \Pi_\alpha^-\right]^{1/2} > 0 \tag{2.27}$$

and

$$D_4(\psi) = \operatorname{ess}\inf_{1\leqq\xi\leqq q} \left\{ \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \left|\hat{\psi}\Big((A^*)^{-j}(B^*)^{-\ell}\xi\Big)\right|^2 \right\} + \sum_{\alpha\in\Gamma\setminus\{0\}} \left[\Pi_\alpha^+ \, \Pi_\alpha^-\right]^{1/2} < \infty. \tag{2.28}$$

*Then the AB-wavelet system $\mathcal{W}_{AB}(\psi, j, k)$ given by* (1.1) *is a frame for $L^2(\mathbb{R}^n)$ with bounds $C_4(\psi)$ and $D_4(\psi)$.*

*Proof.* By using the equation (2.20), we have

$$\begin{aligned}
\left|R_\psi(f)\right| &\leqq \sum_{\alpha\in\Gamma\setminus\{0\}} \left\{ \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \left|\Upsilon_\alpha^+(\xi)\right| d\xi \right\}^{1/2} \left\{ \int_{\mathbb{R}^n} \left|\hat{f}\Big(\xi+\alpha\Big)\right|^2 \left|\Upsilon_\alpha^+(\xi)\right| d\xi \right\}^{1/2} \\
&= \sum_{\alpha\in\Gamma\setminus\{0\}} \left\{ \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \left|\Upsilon_\alpha^+(\xi)\right| d\xi \right\}^{1/2} \left\{ \int_{\mathbb{R}^n} \left|\hat{f}(\omega)\right|^2 \left|\overline{\Upsilon_\alpha^-(\omega)}\right| d\xi \right\}^{1/2} \\
&\leqq \left[\Pi_\alpha^+ \, \Pi_\alpha^-\right]^{1/2} \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 d\xi.
\end{aligned}$$

Proceeding similarly as in Theorem 1, we have

$$\int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \left( \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \left|\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi\big)\right|^2 - \sum_{\alpha\in\Gamma\setminus\{0\}} \left[\Pi_\alpha^+ \, \Pi_\alpha^-\right]^{1/2} \right) d\xi \leqq \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \left|\left\langle f, \psi_{j,k}^\ell \right\rangle\right|^2 \tag{2.29}$$

and

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \left|\left\langle f, \psi_{j,k}^\ell \right\rangle\right|^2 \leqq \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \left( \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \left|\hat{\psi}\big((A^*)^{-j}(B^*)^{-\ell}\xi\big)\right|^2 + \sum_{\alpha\in\Gamma\setminus\{0\}} \left[\Pi_\alpha^+ \, \Pi_\alpha^-\right]^{1/2} \right) d\xi. \tag{2.30}$$

The last two inequalities (2.29) and (2.30) imply that

$$C_4(\psi)\|f\|_2^2 \leqq \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \left|\left\langle f, \psi_{j,k}^\ell \right\rangle\right|^2 \leqq D_4(\psi)\|f\|_2^2.$$

This completes the proof of Theorem 4. □

## 3. Concluding Remarks and Observations

The present investigation was motivated by the recent work on a unified treatment for the continuum and digital realm of multivariate data by Guo *et al.* [8], who introduced the notion of *AB*-wavelets in the context of multiscale analysis. Here, in this sequel, we have continued and extended their work by studying the frame properties of the *AB*-wavelet systems $\left\{D_A D_B T_k \psi^\ell \quad (k \in \mathbb{Z}^n;\ 1 \leqq \ell \leqq L)\right\}$ in $L^2(\mathbb{R}^n)$. More precisely, we have established four theorems (see Theorems 1 to 4 of the preceding section), each of which is intended to give sufficient conditions under which the *AB*-wavelet system constitutes a frame for $L^2(\mathbb{R}^n)$. We have chosen to state the proposed conditions in terms of the Fourier transforms of the generating functions. Our results are believed to be potentially useful in several areas of the mathematical, physical and engineering sciences.


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