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Group-Regular Rings

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Abstract. We propose different generalizations of unit-regularity of elements in general rings (non necessarily unital rings). We then study general rings for which all elements have these properties. We notably compare them with unit-regular ideals and general rings with stable range one. We also prove that these rings are morphic rings.

1. Introduction

In this paper, R denotes a general ring (that may or may not contain an identity), and $E(R)$ its set of idempotents. To emphasize the distinction between rings with or without an identity, we call the former unital rings, the latter non-unital rings and by a ring we always mean a general ring (without necessarily an identity). While every unital ring may be regarded as the endomorphism ring of a module (thus making all the module-theoretical statements available), non-unital rings cannot be endomorphism rings. Thus new approaches (generally elementwise statements with more "elementary" proofs) have to be introduced.

We say that a is (von Neumann) regular in R if $a \in aRa$. A particular solution to $axa = a$ is called an inner inverse of a. A solution to $xax = x$ is called a weak (or outer) inverse. Finally, an element that satisfies $axa = a$ and $xax = x$ is called an inverse (or reflexive inverse, or relative inverse) of a. A commuting inverse, if it exists, is unique and denoted by $a^\#$. It is the unique solution to:

$$ax = xa, axa = a, xax = x.$$

It is usually called the group inverse of a, for a is group invertible if and only if it belongs to a subgroup G_a of the multiplicative semigroup $(R, .)$ (for instance the commutative subgroup $G_a = \{a, a^\#, aa^\#\}$, whith $aa^\#$ is the identity of the group). We let $R^\#$ denote the set of group invertible elements. These are exactly the strongly regular elements of R, where $a \in R$ is strongly regular if $a \in a^2R \cap Ra^2$.

We will use without further comment that for a group element $x \in R$, $xR = x^\#R = eR$ with $e = xx^\# = x^\#x \in E(R)$, and dually.

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If R is unital, we say a is unit-regular if $a \in aU(R)a$, where $U(R)$ denotes the multiplicative group of invertible elements (units). A ring is regular if all its elements are regular, and a unital ring is unit-regular if all its elements are unit-regular.

Many papers have recently focussed on the extension of notions defined for unital rings to general rings, see for instance [3], [28], where the authors define exchange elements and clean elements in general rings. They use the adjoint semigroup of the general ring $(R, ., +)$, defined as the set R with adjoint operation $x * y = x + y + xy$. The semigroup $(R, *)$ is actually a monoid with identity 0, and we define $Q(R) = \{q \in R | \exists q' \in R, q + q' + qq' = q + q' + q'q = 0\}$, group of units of $(R, *)$. Elements of $Q(R)$ are called quasiregular elements of R, and play a prominent role in the theory of Jacobson radicals. For an element $q \in Q(R)$, by q' we will always mean its (unique) inverse in $(R, *)$. If R is unital, then the monoids $(R, .)$ and $(R, *)$ are isomorphic through the map $x \mapsto x - 1$, and so are $U(R)$ and $Q(R)$. Generalizations of unit-regular elements have not been defined in this manner. We do so in Section 2. In this section we also propose a second method, inspired by semigroup theory ([13], [16], [23]) that uses group-invertible elements instead of solely the group of units, that may even not exist. Section 3 relates these notions of regularity with unit-regularity in ideal extensions. In Section 4 we study general rings where all elements have such properties, and show that they define a unique class of general rings (called group-regular rings afterwards). We then characterize these rings by means of unitizations. This allows us to compare group-regular rings with two other classes of rings: rings with stable range 1 [32] and unit-regular ideals [9]. An example shows that, in lack of an identity, group-regular rings need not have stable range 1. Finally in Section 5 we give a characterization of group-regular rings in terms of isomorphic idempotents. We then deduce that group-regular rings are morphic (see [27] for the notion), thus recovering one half of the equivalence of Ehrlich[11]: a unital ring is unit-regular if and only if it is regular and (left) morphic.

We will make use of the natural partial order (minus partial order) on a regular semigroup, defined independently by Hartwig[18] and Nambooripad[26] and extended by Mitsch[25] to non-regular semigroups. It is defined for a regular $b \in R$ by $a \leq b \Leftrightarrow \exists e, f \in E(R), a = eb = bf$. For idempotents, this reduces to $e \leq f \Leftrightarrow ef = fe = e$. We will also use corner rings eRe ($e \in E(R)$).

Finally, we recall the following definitions and results about stable range 1, regular rings, unit-regular rings and corner rings.

A unital ring R has stable range 1 if for all $a, b \in R$, $aR + bR = R$ implies that $(a + bc)R = R$ for some $c \in R$ (equivalently, $a + bc \in U(R)$ by [32] Theorem 2.6). A (general) ring R has stable range 1 if for all $a \in R, b \in \hat{R}$, $(1 + a)\hat{R} + b\hat{R} = \hat{R}$ implies that $(1 + a + bc)\hat{R} = \hat{R}$ for some $c \in \hat{R}$ and some (all by [32] Theorem 3.6) unitization $\hat{R}$ of R.

Lemma 1.1 ([12] Lemma 2). *A regular ring is the directed union of its corner rings.*

Lemma 1.2 ([14] Theorem). *A regular ring admits a regular unitization.*

The next lemma is well-known in the literature. A proof based on module cancellation is due to Ehrlich[11] and Handelman[17]. The first ring theoretical (and element wise) proof is probably due to Kaplansky, as explained in [20]. The link between unit-regularity in a ring and unit-regularity in a corner ring is then precisely studied in [22].

Lemma 1.3 ([20] Proposition 8). *Corner rings of a unit-regular ring are unit-regular.*

Proposition 1.4 (Fuchs and Kaplansky [15] Proposition 4.12). *A unital ring R is unit-regular iff it is regular with stable range 1.*

Lemma 1.5 ([24] Lemma 1.4, [8] Lemma 1). *Let R be a regular ring. Then the following are equivalent:*

1. *For each idempotent $e \in E(R)$ the corner ring eRe is unit-regular.*
2. *R admits a unit-regular unitization.*
3. *R has stable range 1.*

2. Group-regular, group-dominated and Q-unit-regular elements

There are many equivalent characterizations of a unit-regular element a in R unital ring. By definition, $aua = a$ for some unit $u \in U(R)$. But also $a = eu = vf$ for idempotents e, f and units u, v. One can also characterize unit-regularity by means of an inequality: a is unit-regular if and only if $a \leq u$ for some unit u ([31] Theorem 4.3, see also Proposition 2.2 below), that is we can use the same unit in the previous statement. Finally, since $u \mapsto u-1$ is a group isomorphism from $U(R)$ to $Q(R)$, $a(1+q)a = a$ for a quasi-regular element $q \in Q(R)$.

Definition 2.1. *Let R be a general ring and $a \in R$. We say that:*

1. *a is group-regular if $a = axa$ for some $x \in R^{\#}$;*
2. *a is intra group-regular if there exists $x \in R^{\#}$ such that $axa = a$ and $a^2 = axx^{\#}a$;*
3. *a is group-dominated if $a \leq x^{\#}$ for some $x \in R^{\#}$;*
4. *a is Q-unit-regular if $a^2 + aqa = a$ for some $q \in Q(R)$.*

R is group-regular (resp. intra group-regular, group-dominated,Q-unit-regular) if every element of R is group-regular (resp. intra group-regular, group-dominated,Q-unit-regular).

We have the following characterizations of group-domination. The third one is a generalization of [7] and [6] Theorem 5. The fifth one goes back to [19] for unit-regular elements.

Proposition 2.2. *Let $a, x \in R$ with $x \in R^{\#}$. The following statements are equivalent:*

(1) $a \leq x^{\#}$;
(2) $Ra \subseteq Rx$, $aR \subseteq xR$ and $axa = a$;
(3) $a \in xR \cap Rx$ and $aR \cap (a - x^{\#})R = \{0\}$ (or $Ra \cap R(a - x^{\#}) = \{0\}$);
(4) a is unit-regular in the corner ring $xx^{\#}Rxx^{\#}$ with inverse x;
(5) $a = ex^{\#}$ for some $e \in E(R)$ such that $e \leq xx^{\#}$ ($a = x^{\#}f$ for some $f \in E(R)$ such that $f \leq xx^{\#}$).

Proof. Let $a \in R$, $x \in R^{\#}$.

(1) ⇒ (2) Assume (1) and let $e, f \in E(R)$ such that $a = ex^{\#} = x^{\#}f$. Then $Ra = Re(x^{\#})^2x \subseteq Rx$, $aR = x(x^{\#})^2fR \subseteq xR$, and $axa = ex^{\#}xx^{\#}f = ex^{\#}f = eex^{\#} = a$.

(2) ⇒ (3) Assume (2) and let $b = ac = (a - x^{\#})d$ for some $c, d \in R$. First $ax \in E(R)$, and as $Ra \subseteq Rx = Rxx^{\#}$ then $a = axa = yxx^{\#}$ for some $y \in R$, and then $a = axx^{\#}$ since $xx^{\#} \in E(R)$. Then $b = axx^{\#}c = axaxx^{\#}c = axb = ax(a - x^{\#})d = (axa - axx^{\#})d = 0$. The other statement is dual.

(3) ⇒ (4) Assume (3) and pose $e = xx^{\#} \in E(R)$. As $Rx = Re$ and $xR = eR$ then $a \in eR \cap Re = eRe$ (in particular $axx^{\#} = a = xx^{\#}a$). Also $xx^{\#} = x^{\#}x = e$ by definition and x is a unit of eRe. We now compute $axa - a = ax(a-x^{\#}) = (a-x^{\#}+x^{\#})x(a-x^{\#}) = (a-x^{\#})x(a-x^{\#})+a-x^{\#} = (a-x^{\#})[x(a-x^{\#})+xx^{\#}] \in aR \cap (a-x^{\#})R = \{0\}$.

(4) ⇒ (5) Assume (4). Then $axx^{\#} = a = x^{\#}xa$ and $axax = ax$. Thus $e = ax \in E(R)$, and $(ax)(xx^{\#}) = ax = (xx^{\#})(ax)$ ($ax \leq xx^{\#}$). The other statement is dual.

(5) ⇒ (1) Assume (5) and let $a = ex^{\#}$ with $e \in E(R), e \leq xx^{\#}$. Pose $f = xex^{\#}$. Then $f^2 = xex^{\#}xex^{\#} = xex^{\#} = f \in E(R)$ and $x^{\#}f = x^{\#}xex^{\#} = ex^{\#} = a$.

□

Proposition 2.2 shows that group-domination is a localized version of unit-regularity, that proved important in the theory of Leavitt path algebras [1], [30]. It claims in particular that every group-dominated element a of a ring R lies in some corner ring eRe in which it is unit-regular. It does not not claim however that if $a \in fRf, f \in E(R)$, it is unit-regular in this corner ring. Indeed [22] gives an example of a element $a \in eRe \subseteq R$ ring of two by two matrices over a special ring S such that a is unit-regular in R but not in eRe.

We have the following implications:

Corollary 2.3. *Let $a \in R$ and consider the following statements:*

1. *a is group-dominated;*
2. *a is intra group-regular;*
3. *a is group-regular;*
4. *a is Q-unit-regular.*

Then (1) ⇒ (2) ⇒ (3) ⇒ (4).

Proof. The first implication follows from Proposition 2.2, and the second is tautological. We now prove the third one.
Let a be group-regular with $x \in R^\#$ such that $axa = a$ and pose $q = x - xx^\# - xa + xaxx^\#$. Pose also $q' = x^\# - xx^\# + xx^\# a - xx^\# axx^\#$. Then $qx^\# = xx^\# - x^\#, qxx^\# = x - xx^\#$ and computations give $qq' = xx^\# - x^\# - x + xx^\# + xa - xx^\# a - xaxx^\# + xx^\# axx^\# = -q - q'$. On the other hand, $x^\# q = -q'$ so that $xx^\# q = q$ and $(xx^\# a)q = (xx^\# a)(xx^\# q)$ and $q'q = -q' - q$. This proves that q is quasiregular. We finally compute $a^2 + aqa = a^2 + a(x - xx^\# - xa + xaxx^\#)a = a^2 + axa - axx^\# a - axa^2 + axaxx^\# a$ and since $axa = a$, then $a^2 + aqa = a$. Thus a is Q-unit-regular. □

In the next example, we construct a ring R with an element $a \in R$ group-regular but not group-dominated, so that (some of) the reverse implications do not hold in general.

Example 2.4. *This example is inspired by Example 4.8 and Remark 4.9 in [29]. Let F be a field. Consider the semigroup S quotient of the free semigroup with two generators a, e by the relations* $aea = a, e^2 = e$ *and let* $R = F[S]$ *be the semigroup ring of S over F. Equivalently, R is the non-unital free algebra on two indeterminates a, e (excluding the empty word) over F quotiented by the above relations. These relations form a reduction system, whose ambiguities are all resolvable in the sense of [4]. Equivalently, by Theorem 1.2 of [4], every element of R is reduction unique (it has a unique canonical form, where all occurences of the left-hand side of the relations are replaced by their right-hand side). First of all, a (as an element of R,* $a = 1a$*) is group-regular with inner inverse* $e = 1e$*. We now prove that a is not group-dominated because it does not belong to any corner ring. Contrary to [29], we cannot work only on monomials inner inverses of a here, because we would need to consider all (not just one) inner inverses. And some inner inverses are not monomials: for instance* $e + a - ea^2$ *satisfies* $a(e + a - ea^2)a = a + a^3 - aea^3 = a$*.*
So assume that there exists an idempotent $f \in E(R)$ *such that* $a \in fRf$*, that is* $fa = af = a$*. As ea is reduced then* $\lambda ea \neq a$ *for any* $\lambda \in F$ *and f is not a multiple of e. By applying the reductions above, we thus see that f can be written as*

$$f = \lambda e + \sum_{k=1}^{n} \left(\lambda_k^0 a^k + \lambda_k^l e a^k + \lambda_k^r a^k e + \lambda_k^{lr} e a^k e \right)$$

for some $n \geq 1$*, with all coefficients in F, and one of* $\lambda_n^0, \lambda_n^l, \lambda_n^r, \lambda_n^{lr}$ *non zero. After applying the reductions, we get that*

$$fa = \lambda_1^r a + (\lambda + \lambda_1^{lr}) ea + \sum_{k=2}^{n} \left[(\lambda_{k-1}^0 + \lambda_k^r) a^k + (\lambda_{k-1}^l + \lambda_k^{lr}) e a^k \right]$$

and dually

$$af = \lambda_1^l a + (\lambda + \lambda_1^{lr}) ae + \sum_{k=2}^{n} \left[(\lambda_{k-1}^0 + \lambda_k^l) a^k + (\lambda_{k-1}^r + \lambda_k^{lr}) a^k e \right].$$

We deduce that the coefficients of f satisfy:

$$\begin{cases} \lambda + \lambda_1^{lr} & = 0 \\ \lambda_1^l = \lambda_1^r & = 1 \\ \lambda_k^l = \lambda_k^r & = -\lambda_{k-1}^0 \ (\forall k \geq 2) \\ \lambda_k^{lr} = -\lambda_{k-1}^r & = -\lambda_{k-1}^l \ (\forall k \geq 2) \end{cases}$$

We finally consider the coefficients of rank $n \geq 1$ of f. If $\lambda_n^0 \neq 0$ then $f^2 = f$ contains a^{2n} hence $2n \geq n < 2n$ whence a contradiction. Thus $\lambda_n^0 = 0$. Also if $\lambda_n^l \neq 0$ then as $\lambda_n^l = \lambda_n^r$ then $f^2 = f$ contains $ea^{2n}e$ whence a contradiction. Thus $\lambda_n^l = \lambda_n^r = 0$. It follows that $\lambda_n^{lr} \neq 0$, and $f^2 = f$ contains $ea^{2n-1}e$. Thus $2n - 1 \leq n$ and $n = 1$. But we then get a contradiction since $\lambda_1^l = 1 = 0$.
Finally such an f does not exist, and a cannot be group-dominated.

In Example 2.4, the ring is not unital. This is actually a necessary condition because it happens that in unital rings, all the previous concepts are equivalent elementwise. Example 2.4 thus shows that the situation is drastically different for non-unital rings than for unital ones.

Corollary 2.5. *Let $a \in R$ unital ring. Then the following statements are equivalent:*

(1) a is unit-regular;
(2) a is group-dominated (locally unit-regular);
(3) a is intra group-regular;
(4) a is group-regular;
(5) a is Q-unit-regular.

Proof. That (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) is straightforward by Corollary 2.3. We prove that (5) ⇒ (1). Let $a \in R$ and assume that $a = a^2 + aqa$ for some $q \in Q(R)$. Then $a = axa$ where $x = 1 + q \in U(R)$. □

In particular, we recover directly Arens-Kaplansky's result that completely regular rings are unit-regular. The particular case of group-dominated elements can actually be deduced from Proposition 2.2 and results from two other papers: the main Theorem in [22] and Lemma 3 in [30].

Corner rings of general rings provide unital rings. We will use without further comments that if a has any of the previous properties in a corner ring $eRe, e \in E(R)$ then it has the property in R, because of the inclusions $(eRe)^\# \subseteq R^\# \cap eRe$ and $Q(eRe) \subseteq Q(R) \cap eRe$.

Also, we deduce from Corollary 2.5 a converse to Corollary 2.3 in the special case of semicentral idempotents. Recall that an idempotent $e \in E(R)$ is right semicentral [5] if one of the following equivalent conditions hold:

$$(1)\, eRe = eR,\ (2)\, (\forall r \in R)\ ere = er,\ (3)\, Re \text{ is an ideal.}$$

Proposition 2.6. *Let $e \in E(R)$ be a right semicentral idempotent of R and $a \in eRe = eR$. If a is Q-unit-regular (in R), then it is unit-regular in eRe, and in particular group-dominated (in R).*

Proof. Let $e \in E(R)$ such that $eRe = eR$, and let $a \in eRe$. Assume that a is Q-unit-regular and let $q \in Q(R)$ such that $a^2 + aqa = a$. Let q' be the inverse of q in the group $Q(R)$. As $ae = ea = a$ then $a(e + eq)a = a$. As $eq, eq' \in eR = eRe$, then $eqe = eq$ and $eq'e = eq'$. Pose $x = e + eq$ and $x' = e + eq'$. As $q + q' - qq' = q' + q - q'q = 0$ then $xx' = x'x = e$ and $xx'x = x, x'xx' = x$. It follows that x, x' are inverses in $(eRe)^{-1}$ and a is unit-regular in the unital corner ring eRe since $axa = a$. □

3. Ideal extensions

Our next results deal with ideal extensions.

Lemma 3.1. *Let $a \in R$. Consider the following statements:*

(1) a is Q-unit-regular.
(2) Whenever $R \triangleleft T$ where T is a unital ring, a is unit-regular in T.
(3) $(0, a)$ is unit-regular in $\mathbb{Z} \oplus R$ (Dorroh unitization of R).
(4) a or $-a$ is Q-unit-regular.

Then (1) ⇒ (2) ⇒ (3) ⇒ (4).

Proof. Let $a \in R$.

(1) ⇒ (2) Assume a is Q-unit-regular, with $a^2 + aqa = a$ for some $q \in Q(R)$. Then $q \in Q(T)$ and $u = 1 + q$ is a unit of Tthat satisfies $axa = a$.

(2) ⇒ (3) Follows from $R \triangleleft T = \mathbb{Z} \oplus R$.

(3) ⇒ (4) Assume $(0, a)$ is unit-regular in $\mathbb{Z} \oplus R$. Then exists $u = (n, r) \in U(\mathbb{Z} \oplus R)$ such that $(0, a)(n, r)(0, a) = (0, a)$. First, as u is a unit then $n = 1$ or $n = -1$ and $na^2 + ara = a$. Assume that $n = 1$. Then $q = u - 1 = (0, r) \in Q(\mathbb{Z} \oplus R)$. Thus exists $q' = (m, r')$, $q + q' + qq' = q + q' + q'q = 0$ and it follows that $m = 0$ and $r + r' + rr' = r + r' + r'r = 0$ in R, so that $r \in Q(R)$. Finally $a = a^2 + ara$ is Q-unit-regular. Assume now that $n = -1$. Then $-a$ is unit-regular with inverse $-u = (1, -r)$ and the previous arguments prove that $-a$ is Q-unit-regular. Finally in both cases a or $-a$ is Q-unit-regular.

□

As in a unital ring T, $a \in T$ is unit-regular (with $aua = a, u \in T^{-1}$) if and only if $-a$ is unit-regular ($-a = (-a)(-u)(-a), -u \in T^{-1}$) we deduce:

Corollary 3.2. *Let $a \in R$. Then the following statements are equivalent:*

(1) a or $-a$ is Q-unit-regular.

(2) Whenever $R \triangleleft T$ where T is a unital ring, a and $-a$ are unit-regular in T.

(3) $(0, a)$ and $(0, -a)$ are unit-regular in $\mathbb{Z} \oplus R$.

It is very likely that, in lack of a identity, a Q-unit-regular does not imply $-a$ Q-unit-regular.
We now consider group-domination.

Lemma 3.3. *Let $a \in R$. Then the following statements are equivalent:*

(1) a is group-dominated.

(2) There exists $e \in E(R)$ such that $a \in eRe$ and whenever $R \triangleleft T$ where T is a unital ring, $1 + a - e$ is unit-regular in T.

(3) There exists $e \in E(R)$ such that $a \in eRe$ and $(1, a - e)$ is unit-regular in the Dorroh extension $\hat{R} = \mathbb{Z} \oplus R$ of R.

(4) There exists $e \in E(R)$ such that $a \in eRe$, and there exists a unital ring S such that R is a left and right module over S and $(1, a - e)$ is unit-regular in $S \oplus R$.

Proof. Let $a \in R$.

(1) ⇒ (2) Assume $a \leq x^\#$ for some $x \in R^\#$ and pose $xx^\# = e$. Then $1 + x - xx^\# = (1 - e) + x$ is a unit of T and $(1 - e + a)(1 - e + x)(1 - e + a) = (1 - e)^3 + axa = 1 - e + a$.

(2) ⇒ (3) Follows from $R \triangleleft T = \mathbb{Z} \oplus R$.

(3) ⇒ (4) Take $S = \mathbb{Z}$.

(4) ⇒ (1) Let $e \in E(R)$ such that $a \in eRe$ and S be a ring such that R is a left and right module over S and $(1, a - e)$ is unit-regular in $T = S \oplus R$. Then $(0, a) + [1 - (0, e)]$ is unit-regular with $(0, a) \in (0, e)T(0, e)$ and by [22], it is unit-regular in the corner ring $(0, e)T(0, e) = (0, eRe)$. Finally a is locally unit-regular, that is group-dominated.

□

We finally consider the case R has a unit-regular unitization:

Lemma 3.4. *Let R be a left (right, two-sided) ideal of T where T is a unit-regular ring, and $a \in R$. Then a is group-dominated.*

Proof. First, as T is regular then R is regular. Indeed, let $a \in R, x \in T$ such that $axa = a$. Then $a = a(xax)a$ is regular in R. Let now $a \in R$. As R is regular, it is the directed union of its corner rings and exists $e \in E(R)$ such that $a \in eRe$. As corner rings of unit-regular rings are unit-regular, then a is unit-regular in $eTe = eRe$ since R is a left (right, two-sided) ideal and $e \in R$. Denote by $x \in U(eRe)$ its inverse and pose $x^\# = x_{eRe}^{-1}$ (inverse in the corner ring). Then $a \leq x^\#$ and a is group-dominated. □

4. Group-regular rings

If R is a ring without identity, Example 2.4 shows that the previous concepts are different elementwise. It happens however that they provide a unique global characterization, that is group-regular rings, group-dominated rings and Q-unit-regular rings are the same. This is based on the following observation. If an element is group-regular or Q-unit-regular in a corner ring, then it is actually unit-regular in this corner ring hence group-dominated in the whole ring. This happens for instance if any finite set of elements of the ring lies in a corner ring. Such rings are usually called rings with "local units" ([2, Definition 1]), but the terminology may however have other meanings. By Lemma 1.1 this is the case for von Neumann regular general rings. Thus we deduce:

Theorem 4.1. *Let R be a (general) ring. Then the following statements are equivalent:*

(1) R is group-dominated ($\forall a \in R$, a is group-dominated);
(2) R is group-regular ($\forall a \in R$, a is group-regular);
(3) R is Q-unit-regular ($\forall a \in R$, a is Q-unit-regular).

Proof. Let R be a ring.

(1) ⇒ (2) Straightforward;

(2) ⇒ (3) Follows directly from the last implication of Corollary 2.3.

(3) ⇒ (1) Assume (3) and let $a \in R$. Then there exists $q \in Q(R)$ such that $a^2 + aqa = a$. Left multiplication by a gives $a^3 + a^2qa = a^2$ and we deduce that $a = a^2 + aqa = a^3 + a^2qa + aqa = a(a + aq + q)a$ is regular. Thus the whole ring R is regular and there exists $e \in E(R)$ such that $a \in eRe$. Also there exists $f \in E(R)$ such that $e, a, q \in fRf$. As a is Q-unit-regular in the unital ring fRf then it is unit-regular in fRf, an therefore group-dominated in R by Proposition 2.2.

□

Obviously, this is also equivalent with being an intra-group-regular ring by Corollary 2.3.

Example 4.2. *Let R be the ideal of all linear operators of finite rank on a vector space (over a field F) of countably infinite dimension E, and T be the ring of all linear operators on E. It is known that T is regular but not unit-regular [10]. Thus R is regular, but we cannot use Lemma 3.4 directly to show that R is group-regular. Let $a \in R$. As R is regular, then $a \in eRe = eTe$ with $e \in E(R)$ of finite rank n. But eRe is isomorphic to $\mathcal{M}_n(\mathbb{K})$ which is unit-regular, and a is group-regular.*

In Example 4.2, all corner rings of R are unit-regular so that R has stable range 1 by Lemma 1.5. And if R has stable range 1, then it admits a unit-regular unitization (still by Lemma 1.5) and by Lemma 3.4 R is group-regular. Thus we may wonder wether the two concepts are equivalent. This is not the case, as shown in the next example.

Example 4.3. *Let T_0 be a regular non unit-regular unital ring. Define iteratively $T_{n+1} = \mathcal{M}_2(T_n)$ for all $n \in \mathbb{N}$, and embed each $T_n, n \in \mathbb{N}$ as the $1-1$ corner of T_{n+1}. Then define $R = \varinjlim T_n$, direct limit of T_n. We claim that R is group-regular, but has not stable range 1. Indeed, we first deduce by induction that each $T_n, n \in \mathbb{N}$ is regular since matrix rings over regular rings are regular (Theorem 24 in [21]). Also R has not stable range 1 since some corner rings are not unit-regular: for instance, T_0 is a non unit-regular corner ring by assumption. Let now $a \in R$. Then $a \in T_n$ for some $n \in \mathbb{N}$, and as T_n is regular then there exists $b \in T_n$ such that $aba = a$. Pose $B = \begin{pmatrix} b & b-1 \\ 1 & 1 \end{pmatrix} \in T_{n+1}$. Then B is group-invertible in R with group inverse $B^\# = \begin{pmatrix} 1 & 1-b \\ -1 & b \end{pmatrix} \in T_{n+1}$ and $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} B \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. It follows that a is group-regular, and R is group-regular.*

Group-regular rings share many properties with unit-regular rings. For instance, they are dependent, a result due to Ehrlich ([10] Theorem 6) for unit-regular rings.

Corollary 4.4. *Let R be a group-regular ring, and $a, b \in R$. Then there exist $s, t \in R$ with s or t non zero such that $sa + tb = 0$.*

Proof. Let $a, b \in R$. As R is group-regular it is group dominated by Theorem 4.1 and there exists $x, y \in R^\#$ such that $a \leq x^\#$ and $b \leq y^\#$, and in particular x and y are inner inverses of a and b respectively. Also, as R is Von Neumann regular, then $a, b, x, x^\#, y$ and $y^\#$ lie in a common corner ring eRe for some $e \in E(R)$. If $e = xx^\# = yy^\#$, then by Proposition 2.2, a and b are unit-regular in the unital ring eRe, and by [10] Theorem 6 a and b are dependent. So assume that $e \neq xx^\#$ (the other case is symmetric). Then $s = e - xx^\# \neq 0$. But as $a \leq x^\#$ then $sa = (e - xx^\#)a = ea - xx^\# a = a - a = 0$. Letting $t = 0$ gives the desired property. □

Using Corollary 2.5, Corollary 2.3, Lemma 3.1 on ideal extensions and Theorem 4.1 we also deduce a characterization of group-regular rings in terms of their unitizations.

Corollary 4.5. *Let R be a ring. Then the following statements are equivalent:*

1. *R is group-regular;*
2. *For any unitization T of R, all elements of R are unit-regular in T;*
3. *All elements of R are unit-regular in the Dorroh extension $\hat{R} = \mathbb{Z} \oplus R$ of R.*

Proof. Let R be a ring.

(1) ⇒ (2) Assume that R is group-regular, and let T be a unitization of R. Let also $a \in R$. As a is group-regular in R it is group-regular in T and by Corollary 2.5, a is unit-regular in T.

(2) ⇒ (3) The Dorroh extension $\hat{R} = \mathbb{Z} \oplus R$ is a unitization of R.

(3) ⇒ (1) Assume (3) and let $a \in R$. By (3) there exists $u \in \hat{R}$ such that $aua = a$, and $a = a(uau)a$ is regular in R since $uau \in R$ ideal of $\hat{R}$. Thus the whole ring R is regular. We now prove that a is Q-regular in R. By (3) $a = (0, a)$ is unit-regular in $\hat{R}$ and by Lemma 3.1, a or $-a$ is Q-unit-regular in R. Assume that $-a$ is Q-unit-regular and let $q \in R$ such that $a^2 + aqa = -a$. As R is regular then exists $f \in E(R)$ such that a, q and q' lie in the (unital) corner ring fRf. By Corollary 2.5, $-a$ is then group-regular. Let $x \in R^\#$ be an inner of $-a$. Then $-x \in R^\#$ is an inner inverse of a, and a is group-regular hence Q-unit regular by Corollary 2.3. Finally in both cases a is Q-unit-regular, and the whole ring is Q-unit-regular. By Theorem 4.1, R is a group-regular ring.

□

In [9], Chen et al. define an ideal R of a unital ring T to be a unit-regular ideal if all elements of R are unit-regular in T. Corollary 4.5 thus claims that R is group-regular if and only if it is a unit-regular ideal of any of its unitizations, if and only if it is a unit-regular ideal of its Dorroh extension $\hat{R} = \mathbb{Z} \oplus R$. In particular, Lemma 1 and Theorem 5 in [9] together with Corollary 4.5 give the following equivalences:

Corollary 4.6. *Let R be a regular ring, and $\hat{R} = \mathbb{Z} \oplus R$ be the Dorroh extension of R. Then the following statements are equivalent:*

1. *R is group-regular;*
2. *For any unitization T of R and any $a \in R, b \in T$, if $aT + bT = T$ then exists $c \in T$ such that $a + bc \in U(T)$;*
3. *For any $a \in R, b \in \hat{R}$, if $a\hat{R} + b\hat{R} = \hat{R}$ then exists $c \in \hat{R}$ such that $a + bc \in U(\hat{R})$;*
4. *For any unitization T of R and any $a, b \in R$, if $aR \cong bR$ then a and b are equivalent in T ($(\exists u, v \in U(T))\ a = ubv$);*
5. *For any $a, b \in R$, if $aR \cong bR$ then a and b are equivalent in $\hat{R}$ ($(\exists u, v \in U(\hat{R}))\ a = ubv$).*

5. Group-regular rings, isomorphic idempotents and the morphic property

From the results of Ehrlich [11] and Handelman [17] regarding module cancellation in unit-regular rings, it was deduced that unit-regular rings can be characterized in terms of isomorphic idempotents: a unital ring is unit-regular if and only if it is regular and complementary idempotents of isomorphic idempotents are isomorphic. Recall that two idempotents $e, f \in E(R)$ are isomorphic (or Kaplansky equivalent, or

Murray-von Neumann equivalent), denoted by $e \sim f$, if $Re \cong Rf$ as left modules, or equivalently if $e = ab$ and $f = ba$ for some $a, b \in R$. We can always choose a and b to be reflexive inverses, $aba = a, bab = b$. Then Calugareanu [6] gave a purely ring theoretical, elementwise proof of the result. We can restate his elementwise result as follows.

Theorem 5.1 (Ehrlich, Handelman, Calugareanu).
Let $e = ab$, $f = ba$ be isomorphic idempotents with $aba = a, bab = b$ in a unital ring R. Then:

1. *if a is unit-regular then $1 - e \sim 1 - f$;*
2. *if $1 - e \sim 1 - f$ then a is unit-regular.*

We deduce from all the previous results the following characterization of group-regular rings in terms of isomorphic idempotents.

Theorem 5.2. *Let R be a regular ring. Then the following statements are equivalent:*

1. *R is group-regular;*
2. *for all $e, f \in E(R)$, if $e \sim f$ then $1 - e \sim 1 - f$ in any unitization T of R;*
3. *for all $e, f \in E(R)$, if $e \sim f$ then $1 - e \sim 1 - f$ in the Dorroh extension $\hat{R} = \mathbb{Z} \oplus R$.*

Proof.

(1) ⇒ (2) Let T be a unitization of R group-regular ring, and let $e = ab$, $f = ba$ be isomorphic idempotents with $aba = a, bab = b$. As R is group-regular a is actually group-dominated by Theorem 4.1, and exists $x \in R^\#$, $a \leq x^\#$. Then $1 + x - xx^\#$ is a unit in T, and an inverse of a, so that a is unit-regular in T. It then follows by Theorem 5.1 that $1 - e \sim 1 - f$ in T.

(2) ⇒ (3) Straightforward.

(3) ⇒ (1) Let $a \in R$. As a is regular, it admits a reflexive inverse x, and $e = ax \sim xa = f$ are isomorphic idempotents. By assumption, $1 - e \sim 1 - f$ in the Dorroh extension $\hat{R} = \mathbb{Z} \oplus R$ and by Theorem 5.1, a is unit regular in $\hat{R}$. By Lemma 3.1, a or $-a$ is Q-unit-regular in R. Assume that $-a$ is Q-unit-regular with inverse q. As R is regular then exists $g \in E(R)$, $a, q, q' \in gRg$ and by Corollary 2.5, $-a$ is group-dominated in the unital ring gRg. But then $a = -(-a)$ is also group-dominated, hence Q-unit-regular in R by Corollary 2.3. Finally, in the two-cases a is Q-unit-regular in R. Thus R is a Q-unit-regular ring hence a group-regular ring by Theorem 4.1.

□

Our proof relies heavily on Theorem 5.1. However, we note here that Theorem 5.2 can be proved by another road, that combines Corollary 4.6 and a lemma on generalized inverses due to Hartwig and Luh ([19] Lemma 3).

We finally consider the morphic property in group-regular rings. The isomorphism theorem states that $R/l(a) \cong Ra$ (as left modules) for any element a of a ring R (whether or not is has an identity), where as usual $l(a)$ denotes the left annihilator of a: $l(a) = \{r \in R, ra = 0\}$. A fundamental result due to Ehrlich[11], see also [27], states that R is a unit-regular ring if and only if it is regular and the dual isomorphism theorem $R/Ra \cong l(a)$ holds for all $a \in R$. In [27], Nicholson and Sánchez Campos call an element with the property $R/Ra \cong l(a)$ left morphic (The ring itself is called a left morphic ring if every element is left morphic), and give a short proof of the result elementwise:

Proposition 5.3 (Ehrlich, Nicholson, Sánchez Campos). *Let R be a unital ring. Then a is unit-regular if and only if it is regular and morphic.*

We conclude the article by proving that group-regular rings are morphic. We first note that idempotents are always left morphic:

Lemma 5.4. *Let $e \in E(R)$, R general ring. Then e is morphic.*

Proof. Define $\phi : R/Re \to l(e)$ by $\phi : x + Re \mapsto (x - xe)$. Then ϕ is well-defined and invertible ($x - y \in Re \Leftrightarrow x - y = (x - y)e \Leftrightarrow x - xe = y - ye$). Thus e is left morphic. It is right morphic by duality. □

Theorem 5.5. *Let R be a group-regular ring. Then:*

1. *for all $e, f \in E(R)$, if $e \sim f$ then $l(e) \cong l(f)$;*
2. *for all $a \in R$, a is morphic.*

Proof. 1. Let $e, f \in E(R)$ be isomorphic idempotents, and let T be a unitization of R. Then $1 - e \sim 1 - f$ by Theorem 5.2 and exists $c, d \in T$ such that $1 - e = cd, 1 - f = dc, cdc = c, dcd = d$. We consider the map $\phi : l(e) \to l(f)$ (left annihilitors in R) defined by $x \mapsto xc$. It is well defined since R is a ideal of T, and for all $x \in R$, $xe = 0 \Rightarrow x = xcd \Rightarrow xcf = xc - xc(1 - f) = xc - xcdc = 0$. It is bijective with reciprocal $y \mapsto yd$ and a morphism of left modules since it acts by right multiplication. Thus $l(e) \cong l(f)$.

2. Let now $a \in R$ and assume that a is group-dominated by $x^\# \in R^\#$. Then $e = ax$ and $f = xa$ are isomorphic idempotents such that $Ra = Rf$ and $l(e) = l(a)$. Indeed, as $a = ex^\# = x^\# f$ then $Rf = Rxx^\# f \subseteq Ra = Rx^\# f \subseteq Rf$, and dually $(\forall z \in R)\, za = 0 \Leftrightarrow zex^\# = 0 \Leftrightarrow ze = 0$ (since $ex^\# x = e$). Finally

$$R/Ra = R/Rf \cong l(f) \cong l(e) = l(a)$$

and a is left morphic. It is right morphic by dual arguments.

□

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