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New Modified Baskakov Operators Based On The Inverse Pólya-Eggenberger Distribution

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Abstract. In the present article we introduce some modifications of the Baskakov operators in sense of the Lupaş operators based on the inverse Pólya-Eggenberger distribution. For these new modifications we derive some direct results concerning the uniform convergence and the asymptotic formula, as well as some quantitative type theorems.

1. Introduction

Urn models have been among the most popular as well as useful schemes and have received a lot of attention in the literature. One significant point in this area is the Pólya urn model and its generalizations. In 1923, Eggenberger and Pólya [9] devised the original Pólya-Eggenberger urn model (usually simplified as Pólya urn) to study processes such as the spread of contagious diseases. In one of its simplest form, the Pólya-Eggenberger urn model contains w white balls and b black balls. A ball is drawn at random and then replaced together with s balls of the same color. This procedure is repeated n times and noting the distribution of the random variable X representing the number of times a white ball is drawn. The distribution of X is given by

$$Pr(X = k) = \binom{n}{k} \frac{w(w+s)\cdot\ldots\cdot(w+\overline{k-1}s)b(b+s)\cdot\ldots\cdot(b+\overline{n-k-1}s)}{(w+b)(w+b+s)\cdot\ldots\cdot(w+b+\overline{n-1}s)}, \tag{1}$$

for $k = 0, 1, \ldots, n$ and $\overline{k-1}s = (k-1)s$. The distribution (1) is known as Pólya-Eggenberger distribution with parameters (n, w, b, s) and contains binomial and hypergeometric distribution as particular cases. The inverse Pólya-Eggenberger distribution is defined by

$$Pr(N = n + k) = \binom{n+k-1}{k} \frac{w(w+s)\cdot\ldots\cdot(w+\overline{n-1}s)b(b+s)\cdot\ldots\cdot(b+\overline{k-1}s)}{(w+b)(w+b+s)\cdot\ldots\cdot(w+b+\overline{n+k-1}s)}, \tag{2}$$

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for $k = 0, 1, \ldots$ and is the distribution of the number N of drawings needed to obtain n white balls. More details about Pólya-Eggenberger distributions (1) and (2) can be found in [17]. Based on the Pólya-Eggenberger distribution (1), Stancu [24] introduced a new class of positive linear operators associated to a real-valued function $f : [0, 1] \to \mathbb{R}$, given by

$$P_n^{[\alpha]}(f; x) = \sum_{k=0}^{n} p_{n,k}(x, \alpha) f\left(\frac{k}{n}\right) = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}} f\left(\frac{k}{n}\right), \tag{3}$$

where α is a non-negative parameter which may depend only on the natural number n and $t^{[n,h]} = t(t-h)(t-2h)\cdot \ldots \cdot (t-\overline{n-1}h)$, $t^{[0,h]} = 1$ represents the factorial power of t with increment h. In the case when $\alpha = 0$ operators (3) reduce, obviously, to the original Bernstein operators [4] and for $\alpha = \frac{1}{n}$ we get a special case

$$P_n^{\left[\frac{1}{n}\right]}(f; x) = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{\left[k,-\frac{1}{n}\right]}(1-x)^{\left[n-k,-\frac{1}{n}\right]}}{1^{\left[n,-\frac{1}{n}\right]}} f\left(\frac{k}{n}\right), \tag{4}$$

introduced by Lupaş and Lupaş [19]. Concerning the operators defined by the relations (3) and (4), the reader is invited to see the papers [20], [21], where some results of the recalled operators are revised. In 2014, Gupta and Rassias [15] introduced the Durrmeyer type integral modification of the operators (4) and studied the asymptotic approximation, local and global results respectively. Further modifications in sense of the Lupaş operators (4) having as start point the paper of Gupta and Rassias are given in [1], [2] and [16]. Using the inverse Pólya-Eggenberger distribution (2), Stancu [25] introduced a generalization of the Baskakov operators for a real-valued function bounded on $[0, +\infty)$, given by

$$V_n^{[\alpha]}(f; x) = \sum_{k=0}^{\infty} v_{n,k}(x, \alpha) f\left(\frac{k}{n}\right) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{1^{[n,-\alpha]} x^{[k,-\alpha]}}{(1+x)^{[n+k,-\alpha]}} f\left(\frac{k}{n}\right). \tag{5}$$

The operators (5) include as a special case ($\alpha = 0$) the Baskakov operators [3]

$$V_n(f, x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right). \tag{6}$$

Taking the aforesaid papers [1], [2], [15], [16] into account, we remark that there exists a high interest for research of some modifications of certain operators in sense of the Lupaş operators (4). In the following we introduce some modifications of the Baskakov operators based on the inverse Pólya-Eggenberger distribution (2). For a real-valued function bounded on $[0, +\infty)$ we give the first modification

$$V_n^{\left[\frac{1}{n}\right]}(f; x) = \sum_{k=0}^{\infty} v_{n,k}\left(x, \tfrac{1}{n}\right) f\left(\frac{k}{n}\right) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{1^{[n,-\frac{1}{n}]} x^{[k,-\frac{1}{n}]}}{(1+x)^{[n+k,-\frac{1}{n}]}} f\left(\frac{k}{n}\right), \tag{7}$$

called Baskakov-Lupaş operators based on the inverse Pólya-Eggenberger distribution. Using the definition of the factorial power we can establish an explicit representation of the operators defined at (7), such that

$$V_n^{\left[\frac{1}{n}\right]}(f; x) = \frac{(2n)!}{2(n)!} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{(nx)^{(k)}}{(n+nx)^{(n+k)}} f\left(\frac{k}{n}\right),$$

where $t^{(n)} = t(t+1)(t+2)\cdot \ldots \cdot (t+n-1)$ represents the rising factorial. For any bounded and integrable function f defined on $[0, +\infty)$ we introduce the next modifications, given by

$$K_n^{\left[\frac{1}{n}\right]}(f; x) = n \sum_{k=0}^{\infty} v_{n,k}\left(x, \tfrac{1}{n}\right) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt = n \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{1^{[n,-\frac{1}{n}]} x^{[k,-\frac{1}{n}]}}{(1+x)^{[n+k,-\frac{1}{n}]}} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt \tag{8}$$

and

$$D_n^{\left[\frac{1}{n}\right]}(f;x) = (n-1)\sum_{k=0}^{\infty} v_{n,k}\left(x,\tfrac{1}{n}\right)\int_0^{\infty} v_{n,k}\left(t,\tfrac{1}{n}\right) f(t)\,dt, \tag{9}$$

called Kantorovich-Baskakov-Lupaş operators and Durrmeyer-Baskakov-Lupaş operators based on the inverse Pólya-Eggenberger distribution respectively. These type of summation-integral operators have been also considered in various papers, for instance [5], [13], [15], [16].

The aim of this paper is to present these new modifications of the Baskakov operators based on the inverse Pólya-Eggenberger distribution, studying in each case the uniform convergence and the asymptotic formula. In order to get the degree of approximation, some quantitative type theorems will be established. The next part of our article is divided into four sections. A general result concerning the Korovkin theorem for unbounded intervals, as well as the definitions of moduli of continuity, weighted modulus of continuity and K-functional are given in auxiliary results section. Each of the following remaining three sections is dedicated to the one modification of the Baskakov operators. In the last section we give only the definition of the Durrmeyer-Baskakov-Lupaş operators and we let an open gate for further research.

2. Auxiliary results

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The monomials $e_k(x) = x^k$, for $k \in \mathbb{N}_0$ called also test functions play an important role in the uniform approximation by linear positive operators. In order to establish the uniform convergence for certain operators we apply the Korovkin theorem, which says that if a sequence of linear positive operators approximates uniformly the test functions e_0, e_1 and e_2, then it approximates all continuous functions defined on a bounded interval. Since an immediate analog of the Korovkin theorem does not hold in the unbounded interval, some restrictions are needed. Let $B[0,+\infty)$ be the space of all functions f defined on $[0,+\infty)$ satisfying the inequality $|f(x)| \leq M_f(1+x^2)$, where M_f is a positive constant depending only on the function f. Denote by

$$C_B[0,+\infty) = B[0,+\infty) \cap C[0,+\infty) \text{ and } C^*[0,+\infty) = \left\{ f \in C_B[0,+\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^2} = K_f < \infty \right\}$$

the spaces endowed with the norm

$$\|f\| = \sup_{x\geq 0} \frac{|f(x)|}{1+x^2}.$$

As it follows from the Gadzhiev papers [11], [12], the Korovkin type theorem for positive linear operators does not hold in the space $C_B[0,+\infty)$, but holds in the space $C^*[0,+\infty)$ and has the following form:

Theorem 2.1. *For a sequence of positive linear operators L_n which satisfy the conditions*

$$\lim_{n\to\infty} \left\| L_n(e_i;x) - x^i \right\| = 0, \quad i = 0,1,2,$$

we get

$$\lim_{n\to\infty} \|L_n f - f\| = 0 \text{ for any function } f \in C^*[0,+\infty).$$

The main tools to measure the approximation degree of linear positive operators towards the identity operators are moduli of continuity.

Definition 2.2. Let $f \in C_B[0,+\infty)$ be given and $\delta \geq 0$. The modulus of continuity of the function f is defined by

$$\omega(f,\delta) = \sup\{\left|f(x) - f(y)\right| : x, y \in [0,+\infty), |x-y| \leq \delta\}, \tag{10}$$

where $C_B[0,+\infty)$ is the space of all real-valued functions continuous and bounded on $[0,+\infty)$.

Definition 2.3. For any $f \in C[0,+\infty)$ and $\delta \geq 0$

$$\omega_1(f,\delta) = \sup\{|f(x+h)-f(x)| : x, x+h \in [0,+\infty),\ 0 \leq h \leq \delta\} \tag{11}$$

and

$$\omega_2(f,\delta) = \sup\{|f(x+h)-2f(x)+f(x-h)| : x, x\pm h \in [0,+\infty),\ 0 \leq h \leq \delta\} \tag{12}$$

are the moduli of smoothness of first and second order.

Definition 2.4. Let the function $f \in C_B[0,+\infty)$ endowed with the norm $\|f\| = \sup\limits_{x\in[0,+\infty)} |f(x)|$ and let us consider Peetre's K-functional

$$K_2(f,\delta) = \inf_{g\in W_\infty^2}\left\{\left\|f-g\right\| + \delta\left\|g''\right\|\right\}, \tag{13}$$

where $\delta > 0$ and $W_\infty^2 = \{g \in C_B[0,+\infty) : g', g'' \in C_B[0,+\infty)\}$. According with ([7], p. 177, Theorem 2.4) there exists an absolute constant $C > 0$, such that

$$K_2(f,\delta) \leq C\omega_2\left(f, \sqrt{\delta}\right). \tag{14}$$

For $0 \leq \lambda \leq 1$, $\varphi(x) = \sqrt{x(x+1)}$ and $f \in C_B[0,+\infty)$, the weighted modulus of continuity and K-functional are defined in [14] or [18] as

$$\omega_{\varphi^\lambda}^2(f,\delta) = \sup_{0<h\leq t}\ \sup_{x\pm h\varphi^\lambda(x)\in[0,+\infty)}\left|\Delta_{h\varphi^\lambda}^2 f(x)\right|, \tag{15}$$

where

$$\Delta_{h\varphi^\lambda}^2 f(x) = f\left(x+h\varphi^\lambda(x)\right) - 2f(x) + f\left(x-h\varphi^\lambda(x)\right).$$

The appropriate K-functional is given by

$$K_{\varphi^\lambda}^2\left(f,t^2\right) = \inf_{g\in D_\lambda^2}\left\{\left\|f-g\right\| + t^2\left\|\varphi^{2\lambda}g''\right\|\right\}, \tag{16}$$

where

$$D_\lambda^2 = \left\{f \in C_B[0,\infty) : f' \in AC_{\mathrm{loc}}[0,+\infty),\ \left\|\varphi^{2\lambda}f''\right\| < +\infty\right\}.$$

There exists a connection between the weighted modulus of continuity and the appropriate K-functional given by the following relation

$$\omega_{\varphi^\lambda}^2(f,t) \sim K_{\varphi^\lambda}^2\left(f,t^2\right). \tag{17}$$

3. Direct results for the Baskakov-Lupaş operators based on the inverse Pólya-Eggenberger distribution

In introduction, for a real-valued function bounded on $[0,+\infty)$ we presented the first modification of the Baskakov operators (5) defined by

$$V_n^{\left[\frac{1}{n}\right]}(f;x) = \sum_{k=0}^{\infty} v_{n,k}\left(x,\tfrac{1}{n}\right) f\left(\frac{k}{n}\right) = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{1^{[n,-\frac{1}{n}]}x^{[k,-\frac{1}{n}]}}{(1+x)^{[n+k,-\frac{1}{n}]}} f\left(\frac{k}{n}\right).$$

In order to compute the images of the test functions by Baskakov-Lupaş operators we write them in the following useful form

$$V_n^{\left[\frac{1}{n}\right]}(f;x) = \frac{n^{[n,-1]}}{(nx+n)^{[n,-1]}} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{n^{[k,-1]} \cdot (nx)^{[k,-1]}}{(nx+2n)^{[k,-1]}} f\left(\frac{k}{n}\right) \tag{18}$$

and recall the definition of hypergeometric series

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \cdot \frac{a^{[k,-1]} \cdot b^{[k,-1]}}{c^{[k,-1]}},$$

where the parameters a, b, c satisfy the conditions $a, b > 0, a+b < c$. If $z = 1$, then the following representation for the hypergeometric series in terms of Gamma function holds

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c) \cdot \Gamma(c-a-b)}{\Gamma(c-a) \cdot \Gamma(c-b)},$$

where $\Gamma(r) = \int_0^{+\infty} u^{r-1} e^{-u} du$ and $\Gamma(r+n) = r(r+1) \cdot \ldots \cdot (r+n-1)\Gamma(r)$, for $n \in \mathbb{N}$.

Lemma 3.1. *For the Baskakov-Lupaş operators* (7) *hold*

$$V_n^{\left[\frac{1}{n}\right]}(e_0;x) = 1, \quad V_n^{\left[\frac{1}{n}\right]}(e_1;x) = x + \frac{x}{n-1}, \quad V_n^{\left[\frac{1}{n}\right]}(e_2;x) = x^2 + \frac{(2n-1)x(2x+1)}{(n-1)(n-2)}.$$

Proof. Using the representation (18) it follows

$$\begin{aligned} V_n^{\left[\frac{1}{n}\right]}(e_0;x) &= \frac{n^{[n,-1]}}{(nx+n)^{[n,-1]}} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{n^{[k,-1]} \cdot (nx)^{[k,-1]}}{(nx+2n)^{[k,-1]}} = \frac{n^{[n,-1]}}{(nx+n)^{[n,-1]}} \cdot \frac{\Gamma(nx+2n) \cdot \Gamma(n)}{\Gamma(nx+n) \cdot \Gamma(2n)} \\ &= \frac{n^{[n,-1]}}{(nx+n)^{[n,-1]}} \cdot \frac{(nx+n)^{[n,-1]} \cdot \Gamma(nx+n) \cdot \Gamma(n)}{\Gamma(nx+n) \cdot n^{[n,-1]} \cdot \Gamma(n)} = 1. \end{aligned}$$

$$\begin{aligned} V_n^{\left[\frac{1}{n}\right]}(e_1;x) &= \frac{n^{[n,-1]}}{(nx+n)^{[n,-1]}} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{k}{n} \cdot \frac{n^{[k,-1]} \cdot (nx)^{[k,-1]}}{(nx+2n)^{[k,-1]}} \\ &= \frac{n^{[n,-1]}}{(nx+n)^{[n,-1]}} \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \cdot \frac{(n+1)^{[k-1,-1]} \cdot (nx)(nx+1)^{[k-1,-1]}}{(nx+2n)(nx+2n+1)^{[k-1,-1]}} \\ &= \frac{n^{[n,-1]}}{(nx+n)^{[n,-1]}} \cdot \frac{nx}{nx+2n} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{(n+1)^{[k,-1]} \cdot (nx+1)^{[k,-1]}}{(nx+2n+1)^{[k,-1]}} \\ &= \frac{n^{[n,-1]}}{(nx+n)^{[n,-1]}} \cdot \frac{nx}{nx+2n} \cdot \frac{\Gamma(nx+2n+1) \cdot \Gamma(n-1)}{\Gamma(nx+n) \cdot \Gamma(2n)} \\ &= \frac{n^{[n,-1]}}{(nx+n)^{[n,-1]}} \cdot \frac{nx}{nx+2n} \cdot \frac{(nx+2n)(nx+n)^{[n,-1]} \cdot \Gamma(nx+n) \cdot \Gamma(n-1)}{\Gamma(nx+n) \cdot (n-1)n^{[n,-1]} \cdot \Gamma(n-1)} \\ &= \frac{nx}{n-1} = x + \frac{x}{n-1}. \end{aligned}$$

$$
\begin{aligned}
V_n^{\left[\frac{1}{n}\right]}(e_2;x) &= \frac{n^{[n,-1]}}{(nx+n)^{[n,-1]}} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{k^2}{n^2} \cdot \frac{n^{[k,-1]} \cdot (nx)^{[k,-1]}}{(nx+2n)^{[k,-1]}} \\
&= \frac{n^{[n,-1]}}{(nx+n)^{[n,-1]}} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{k(k-1)+k}{n^2} \cdot \frac{n^{[k,-1]} \cdot (nx)^{[k,-1]}}{(nx+2n)^{[k,-1]}} \\
&= \frac{n^{[n,-1]}}{(nx+n)^{[n,-1]}} \cdot \frac{n+1}{n} \sum_{k=2}^{\infty} \frac{1}{(k-2)!} \cdot \frac{(n+2)^{[k-2,-1]} \cdot (nx)(nx+1)(nx+2)^{[k-2,-1]}}{(nx+2n)(nx+2n+1)(nx+2n+2)^{[k-2,-1]}} + \frac{1}{n} V_n^{\left[\frac{1}{n}\right]}(e_1;x) \\
&= \frac{n^{[n,-1]}}{(nx+n)^{[n,-1]}} \cdot \frac{(n+1)(nx)(nx+1)}{n(nx+2n)(nx+2n+1)} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{(n+2)^{[k,-1]} \cdot (nx+2)^{[k,-1]}}{(nx+2n+2)^{[k,-1]}} + \frac{1}{n} V_n^{\left[\frac{1}{n}\right]}(e_1;x) \\
&= \frac{n^{[n,-1]}}{(nx+n)^{[n,-1]}} \cdot \frac{(n+1)(nx)(nx+1)}{n(nx+2n)(nx+2n+1)} \cdot \frac{\Gamma(nx+2n+2) \cdot \Gamma(n-2)}{\Gamma(nx+n) \cdot \Gamma(2n)} + \frac{1}{n} V_n^{\left[\frac{1}{n}\right]}(e_1;x) \\
&= \frac{(n+1)x(nx+1)}{(n-1)(n-2)} + \frac{x}{n-1} = x^2 + \frac{(2n-1)x(2x+1)}{(n-1)(n-2)}.
\end{aligned}
$$

□

Remark 3.2. *The images of the test functions by the Baskakov-Lupaş operators* (7) *could be also derived knowing previously the images of the test functions by the Stancu-Baskakov operators* (5).

Corollary 3.3. *The central moments up to the second order of the Baskakov-Lupaş operators* (7) *are*

$$V_n^{\left[\frac{1}{n}\right]}(e_1-x;x) = \frac{x}{n-1}, \quad V_n^{\left[\frac{1}{n}\right]}\left((e_1-x)^2;x\right) = \frac{2nx(x+1)+x(2x-1)}{(n-1)(n-2)}.$$

Proof. Taking Lemma 3.1 into account, it follows the desired equalities. □

Proposition 3.4. *Let f be a bounded function defined on* $[0,+\infty)$ *with* $\|f\| = \sup_{x\in[0,+\infty)} |f(x)|$, *then*

$$\left|V_n^{\left[\frac{1}{n}\right]}(f;x)\right| \le \|f\|.$$

Proof. Starting with the definition of the Baskakov-Lupaş operators (7) and using the fact that they preserve constants, it follows

$$\left|V_n^{\left[\frac{1}{n}\right]}(f;x)\right| = \left|\sum_{k=0}^{\infty} v_{n,k}\left(x,\tfrac{1}{n}\right) f\left(\frac{k}{n}\right)\right| \le \sum_{k=0}^{\infty} v_{n,k}\left(x,\tfrac{1}{n}\right) \left|f\left(\frac{k}{n}\right)\right| \le \|f\| \cdot V_n^{\left[\frac{1}{n}\right]}(e_0;x) = \|f\|.$$

□

We are able to prove a result concerning the uniform approximation of the functions defined on $C^*[0,+\infty)$.

Theorem 3.5. *For each function $f \in C^*[0,+\infty)$*

$$\lim_{n\to\infty} \left\|V_n^{\left[\frac{1}{n}\right]} f - f\right\| = 0.$$

Proof. Using Theorem 2.1 we remark that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \left\|V_n^{\left[\frac{1}{n}\right]}(e_i;x) - x^i\right\| = 0, \quad i = 0,1,2.$$

In view of Lemma 3.1 the above three conditions are fulfilled, hence applying Theorem 2.1 we get the desired result. □

We establish the asymptotic behavior of the Baskakov-Lupaş operators (7) giving a Voronovskaja type theorem.

Theorem 3.6. *Let f be a bounded function on $[0,+\infty)$. If there exists first and second derivative of the function f bounded at a fixed point $x \in [0,+\infty)$, then*

$$\lim_{n\to\infty} n\left(V_n^{\left[\frac{1}{n}\right]}(f;x) - f(x)\right) = xf'(x) + x(1+x)f''(x)$$

Proof. Using Taylor's expansion formula of the function f, it follows

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2!}(t-x)^2 f''(x) + \varepsilon(t,x)(t-x)^2,$$

where $\varepsilon(t,x) := \varepsilon(t-x)$ is a bounded function and $lim_{t\to x}\varepsilon(t,x) = 0$. Taking the linearity of the Baskakov-Lupaş operators (7) into account and then apply the operators $V_n^{\left[\frac{1}{n}\right]}$ on both sides of the above equation, we get

$$V_n^{\left[\frac{1}{n}\right]}(f;x) - f(x) = V_n^{\left[\frac{1}{n}\right]}(e_1 - x;x)\, f'(x) + \frac{1}{2}V_n^{\left[\frac{1}{n}\right]}\left((e_1-x)^2;x\right) f''(x) + V_n^{\left[\frac{1}{n}\right]}\left(\varepsilon(t,x)\cdot(e_1-x)^2;x\right).$$

Therefore using Corollary 3.3, it follows

$$\lim_{n\to\infty} n\left(V_n^{\left[\frac{1}{n}\right]}(f;x) - f(x)\right) = xf'(x) + x(1+x)f''(x) + \lim_{n\to\infty} n\left(V_n^{\left[\frac{1}{n}\right]}\left(\varepsilon(t,x)\cdot(e_1-x)^2;x\right)\right). \tag{19}$$

We estimate the last term on the right hand side of the above equality applying the Cauchy-Schwarz inequality, such that

$$V_n^{\left[\frac{1}{n}\right]}\left(\varepsilon(t,x)\cdot(e_1-x)^2;x\right) \le \sqrt{V_n^{\left[\frac{1}{n}\right]c}\left(\varepsilon^2(t,x);x\right)}\sqrt{V_n^{\left[\frac{1}{n}\right]}\left((e_1-x)^4;x\right)}. \tag{20}$$

Because $\varepsilon^2(x,x) = 0$ and $\varepsilon^2(\cdot,x) \in C^*[0,+\infty)$, using the uniform convergence from Theorem 3.5, we get

$$\lim_{n\to\infty} V_n^{\left[\frac{1}{n}\right]}\left(\varepsilon^2(t,x);x\right) = \varepsilon^2(x,x) = 0. \tag{21}$$

Therefore, from (20) and (21) yields

$$\lim_{n\to\infty} n\left(V_n^{\left[\frac{1}{n}\right]}\left(\varepsilon(t,x)\cdot(e_1-x)^2;x\right)\right) = 0$$

and using (19) we obtain the asymptotic behavior of the Baskakov-Lupaş operators (7). □

We derive some quantitative upper estimates in terms of modulus of continuity and Peetre's K-functional.

Theorem 3.7. *Let be $f \in C_B[0,+\infty)$, then for any $x \in [0,+\infty)$ and $\delta > 0$, it follows*

$$\left|V_n^{\left[\frac{1}{n}\right]}(f;x) - f(x)\right| \le 2\cdot\omega(f,\delta), \;\; with \;\; \delta = \left(V_n^{\left[\frac{1}{n}\right]}\left((e_1-x)^2;x\right)\right)^{\frac{1}{2}}.$$

Proof. Taking into account the fact that Baskakov-Lupaş operators (7) preserve constants, according with Lemma 3.1 and using the well-known property of the modulus of continuity

$$|f(x) - f(y)| \le \omega(f,|x-y|) \le \left(1 + \frac{1}{\delta}|x-y|\right)\cdot\omega(f,\delta),$$

it follows

$$\left|V_n^{\left[\frac{1}{n}\right]}(f;x)-f(x)\right| \le \sum_{k=0}^{\infty} v_{1n,k}\left(x,\frac{1}{n}\right)\left|f\left(\frac{k}{n}\right)-f(x)\right| \le \left(1+\frac{1}{\delta}\sum_{k=0}^{\infty} v_{n,k}\left(x,\frac{1}{n}\right)\left|\frac{k}{n}-x\right|dt\right)\cdot\omega(f,\delta).$$

Applying the Cauchy-Schwarz inequality for linear positive operators, we get

$$\begin{aligned}\left|V_n^{\left[\frac{1}{n}\right]}(f;x)-f(x)\right| &\le \left(1+\frac{1}{\delta}\left(\sum_{k=0}^{\infty} v_{n,k}\left(x,\frac{1}{n}\right)\right)^{1/2}\left(\sum_{k=0}^{\infty} v_{n,k}\left(x,\frac{1}{n}\right)\left(\frac{k}{n}-x\right)^2\right)^{1/2}\right)\cdot\omega(f,\delta)\\ &= \left(1+\frac{1}{\delta}\left(V_n^{\left[\frac{1}{n}\right]}(e_0;x)\right)^{1/2}\left(V_n^{\left[\frac{1}{n}\right]}((e_1-x)^2;x)\right)^{1/2}\right)\cdot\omega(f,\delta)=2\cdot\omega(f,\delta),\end{aligned}$$

with $\delta := \left(V_n^{\left[\frac{1}{n}\right]}((e_1-x)^2;x)\right)^{1/2}$. □

Theorem 3.8. *Let be $f\in C[0,+\infty)$, then for any $x\in[0,+\infty)$ it follows*

$$\left|V_n^{\left[\frac{1}{n}\right]}(f;x)-f(x)\right| \le M\cdot\omega_2\left(f,\frac{1}{2}\delta_n(x)\right)+\omega(f,\delta_\omega),$$

where M is an absolute constant and $\delta_n(x)=\left(V_n^{\left[\frac{1}{n}\right]}\left((e_1-x)^2;x\right)+\left(V_n^{\left[\frac{1}{n}\right]}(e_1-x;x)\right)^2\right)^{\frac{1}{2}}$, $\delta_\omega=\left|V_n^{\left[\frac{1}{n}\right]}(e_1-x;x)\right|$.

Proof. For $x\in[0,+\infty)$ we define the operators

$$\tilde{V}_n^{\left[\frac{1}{n}\right]}(f;x)=V_n^{\left[\frac{1}{n}\right]}(f;x)-f\left(\frac{nx}{n-1}\right)+f(x). \tag{22}$$

We remark that $\tilde{V}_n^{\left[\frac{1}{n}\right]}(e_0;x)=1$ and $\tilde{V}_n^{\left[\frac{1}{n}\right]}(e_1;x)=x$, i.e. the defined operators (22) preserve constants as well as linear functions. Therefore

$$\tilde{V}_n^{\left[\frac{1}{n}\right]}(e_1-x;x)=0. \tag{23}$$

Let be $g\in W_\infty^2$ and $x,t\in[0,+\infty)$. By Taylor's expansion formula, we have

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t (t-u)g''(u)\,du.$$

Applying the operators (22) on both sides of the above equation, we get

$$\begin{aligned}\tilde{V}_n^{\left[\frac{1}{n}\right]}(g;x)-g(x) &= g'(x)\cdot\tilde{V}_n^{\left[\frac{1}{n}\right]}(e_1-x;x)+\tilde{V}_n^{\left[\frac{1}{n}\right]}\left(\int_x^t (t-u)g''(u)\,du;x\right)\\ &= V_n^{\left[\frac{1}{n}\right]}\left(\int_x^t (t-u)g''(u)\,du;x\right)-\int_x^{\frac{nx}{n-1}}\left(\frac{nx}{n-1}-u\right)g''(u)du.\end{aligned}$$

On the other hand

$$\left|\int_x^t (t-u)g''(u)\,du\right| \le (t-x)^2\cdot\|g''\|,$$

then

$$\left|\tilde{V}_n^{\left[\frac{1}{n}\right]}(g;x)-g(x)\right| \le \left(V_n^{\left[\frac{1}{n}\right]}((e_1-x)^2;x)+\left(V_n^{\left[\frac{1}{n}\right]}(e_1-x;x)\right)^2\right)\cdot\|g''\|.$$

Using the definition (23) of the operators $\tilde{V}_n^{[\frac{1}{n}]}$ and Proposition 3.4, it follows

$$\begin{aligned}\left|V_n^{[\frac{1}{n}]}(f;x)-f(x)\right| &\leq \left|\tilde{V}_n^{[\frac{1}{n}]}(f-g;x)\right|+\left|\tilde{V}_n^{[\frac{1}{n}]}(g;x)-g(x)\right|+\left|g(x)-f(x)\right|+\left|f\left(\tfrac{nx}{n-1}\right)-f(x)\right|\\ &\leq 4\left\|f-g\right\|+\delta_n^2(x)\left\|g''\right\|+\omega(f,\delta_\omega),\end{aligned}$$

with $\delta_n^2(x)=V_n^{[\frac{1}{n}]}\left((e_1-x)^2;x\right)+\left(V_n^{[\frac{1}{n}]}(e_1-x;x)\right)^2$ and $\delta_\omega=\left|V_n^{[\frac{1}{n}]}(e_1-x;x)\right|$.
Now, taking infimum on the right-hand side over all $g\in W_\infty^2$ and using the relation (14), we get

$$\left|V_n^{[\frac{1}{n}]}(f;x)-f(x)\right|\leq 4K_2\left(f,\tfrac{1}{4}\delta_n^2(x)\right)+\omega(f,\delta_\omega)\leq M\cdot\omega_2\left(f,\tfrac{1}{2}\delta_n(x)\right)+\omega(f,\delta_\omega),$$

where M is an absolute constant. □

Our last result of this section proposed for study is to obtain the degree of approximation with the help of the Ditzian-Totik modulus of continuity.

Theorem 3.9. *Let be $f\in C_B[0,+\infty)$, then for any $x\in[0,+\infty)$ we have*

$$\left|V_n^{[\frac{1}{n}]}(f;x)-f(x)\right|\leq 4\cdot\omega_{\varphi^\lambda}^2\left(f,\frac{1}{2}\sqrt{B}(n-1)^{-\frac{1}{2}}\varphi^{(1-\lambda)}(x)\right)+\omega\left(f,\left|\frac{x}{n-1}\right|\right),$$

where B is an absolute constant.

Proof. We consider again the auxiliary operators defined at (22), given by the following relation

$$\tilde{V}_n^{[\frac{1}{n}]}(f;x)=V_n^{[\frac{1}{n}]}(f;x)-f\left(\frac{nx}{n-1}\right)+f(x).$$

Taking the definition of the Baskakov-Lupaş operators (7) and Lemma 3.1 into account, for $g\in D_\lambda^2$ we get as in Theorem 3.8 that

$$\left|\tilde{V}_n^{[\frac{1}{n}]}(g;x)-g(x)\right|\leq V_n^{[\frac{1}{n}]}\left(\int_x^t|t-u|\left|g''(u)\right|du;x\right)+\int_x^{\frac{nx}{n-1}}\left|\frac{nx}{n-1}-u\right|\left|g''(u)\right|du. \tag{24}$$

For $t<u<x$, in ([8], p.141) is proved the following inequality

$$\frac{|t-u|}{\varphi^{2\lambda}(u)}\leq\frac{|t-x|}{\varphi^{2\lambda}(x)}. \tag{25}$$

If we use this inequality in the relation (24), then we get

$$\begin{aligned}\left|\tilde{V}_n^{[\frac{1}{n}]}(g;x)-g(x)\right| &\leq V_n^{[\frac{1}{n}]}\left(\int_x^t\varphi^{-2\lambda}(u)|t-u|du;x\right)\cdot\left\|\varphi^{2\lambda}g''\right\|+\left\|\varphi^{2\lambda}g''\right\|\cdot\int_x^{\frac{nx}{n-1}}\varphi^{-2\lambda}(u)\left|\frac{nx}{n-1}-u\right|du \\ &\leq \varphi^{-2\lambda}(x)\left\|\varphi^{2\lambda}g''\right\|\left(V_n^{[\frac{1}{n}]}\left((e_1-x)^2;x\right)+\left(V_n^{[\frac{1}{n}]}(e_1-x;x)\right)^2\right),\end{aligned} \tag{26}$$

where

$$\int_x^t|t-x|\left|g''(u)\right|du\leq(t-x)^2\cdot\left\|g''\right\|.$$

Based on Corollary 3.3 we can give the following estimations

$$\left(V_n^{[\frac{1}{n}]}(e_1-x;x)\right)^2=\left(\frac{x}{n-1}\right)^2\leq\frac{x(x+1)}{(n-1)^2}=\frac{\varphi^2(x)}{(n-1)^2},$$

$$V_n^{\left[\frac{1}{n}\right]}\left((e_1-x)^2;x\right)=\frac{2nx(x+1)+x(2x-1)}{(n-1)(n-2)}=\frac{2x(x+1)(n+1)-3x}{(n-1)(n-2)}\le\frac{2x(x+1)(n+1)}{(n-1)(n-2)}\le\frac{A}{n-1}\varphi^2(x),$$

and

$$V_n^{\left[\frac{1}{n}\right]}\left((e_1-x)^2;x\right)+\left(V_n^{\left[\frac{1}{n}\right]}(e_1-x;x)\right)^2=\frac{\varphi^2(x)}{n-1}\left(A+\frac{1}{n-1}\right)\le\frac{B}{n-1}\varphi^2(x), \tag{27}$$

where A and B are absolute constants. The relations (26) and (27) provide

$$\left|\tilde{V}_n^{\left[\frac{1}{n}\right]}(g;x)-g(x)\right|\le\frac{B}{n-1}\varphi^{2(1-\lambda)}(x)\left\|\varphi^{2\lambda}g''\right\|. \tag{28}$$

Furthermore, for all bounded functions f defined on $[0,+\infty)$, applying the result from Proposition 3.4 on the auxiliary operators $\tilde{V}_n^{\left[\frac{1}{n}\right]}$, it follows

$$\left|\tilde{V}_n^{\left[\frac{1}{n}\right]}(f;x)\right|\le\left|V_n^{\left[\frac{1}{n}\right]}(f;x)\right|+\left|f\left(\frac{nx}{n-1}\right)\right|+\left|f(x)\right|\le 3\|f\|. \tag{29}$$

Let f be a bounded function defined on $[0,+\infty)$ and $g\in D_\lambda^2$, then taking the relations (28) and (29) into account, we get

$$\begin{aligned}\left|V_n^{\left[\frac{1}{n}\right]}(f;x)-f(x)\right|&=\left|\tilde{V}_n^{\left[\frac{1}{n}\right]}(f;x)-f(x)+f\left(\frac{nx}{n-1}\right)-f(x)\right|\\&\le\left|\tilde{V}_n^{\left[\frac{1}{n}\right]}(f-g;x)\right|+\left|f(x)-g(x)\right|+\left|\tilde{V}_n^{\left[\frac{1}{n}\right]}(g;x)-g(x)\right|+\left|f\left(\frac{nx}{n-1}\right)-f(x)\right|\\&\le 4\left\|f-g\right\|+\frac{B}{n-1}\varphi^{2(1-\lambda)}(x)\left\|\varphi^{2\lambda}g''\right\|+\omega\left(f,\left|\frac{x}{n-1}\right|\right).\end{aligned}$$

Taking infimum on the right-hand side over all $g\in D_\lambda^2$ and using the relation (17), it follows

$$\begin{aligned}\left|V_n^{\left[\frac{1}{n}\right]}(f;x)-f(x)\right|&\le 4\left(\left\|f-g\right\|+\frac{B}{4(n-1)}\varphi^{2(1-\lambda)}(x)\left\|\varphi^{2\lambda}g''\right\|\right)+\omega\left(f,\left|\frac{x}{n-1}\right|\right)\\&=4\cdot K_{\varphi^\lambda}^2\left(f,\frac{1}{4}B(n-1)^{-1}\varphi^{2(1-\lambda)}(x)\right)+\omega\left(f,\left|\frac{x}{n-1}\right|\right)\\&=4\cdot\omega_{\varphi^\lambda}^2\left(f,\frac{1}{2}\sqrt{B}(n-1)^{-\frac{1}{2}}\varphi^{(1-\lambda)}(x)\right)+\omega\left(f,\left|\frac{x}{n-1}\right|\right),\end{aligned}$$

where B is an absolute constant. □

4. Direct results for Kantorovich-Baskakov-Lupaş operators based on the inverse Pólya-Eggenberger distribution

First modification of the Baskakov operators in sense of Kantorovich operators was given by Ditzian and Totik [8], in order to approximate Lebesgue integrable functions. Several researchers introduced and studied new forms of Kantorovich operators being in close connection with Baskakov, Stancu and Lupaş operators, for instance [23], [22], [10], [14], [16], [6]. Being motivated by the aforesaid works, in introduction for any bounded and integrable function f defined on $[0,+\infty)$ we gave the second modification (8) called Kantorovich-Baskakov-Lupaş operators based on the inverse Pólya-Eggenberger distribution, defined by

$$K_n^{\left[\frac{1}{n}\right]}(f;x)=n\sum_{k=0}^{\infty}v_{n,k}\left(x,\tfrac{1}{n}\right)\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(t)\,dt=n\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{1^{[n,-\frac{1}{n}]}x^{[k,-\frac{1}{n}]}}{(1+x)^{[n+k,-\frac{1}{n}]}}\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(t)\,dt.$$

Using some well-known rules for computation with factorial powers, the Kantorovich-Baskakov-Lupaş operators (8) could be written in the following useful form

$$K_n^{\left[\frac{1}{n}\right]}(f;x)=\frac{n\cdot n^{[n,-1]}}{(nx+n)^{[n,-1]}}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{n^{[k,-1]}\cdot(nx)^{[k,-1]}}{(nx+2n)^{[k,-1]}}\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(t)dt. \tag{30}$$

Lemma 4.1. *For the Kantorovich-Baskakov-Lupaş operators* (8) *hold*

$$K_n^{\left[\frac{1}{n}\right]}(e_0;x)=1,\quad K_n^{\left[\frac{1}{n}\right]}(e_1;x)=x+\frac{(2x+1)n-1}{2n(n-1)},\quad K_n^{\left[\frac{1}{n}\right]}(e_2;x)=x^2+\frac{2(2n-1)x^2+3(n-1)x}{(n-1)(n-2)}+\frac{1}{3n^2}.$$

Proof. Using the representation (30) and Lemma 3.1, for the Kantorovich-Baskakov-Lupaş operators it follows

$$\begin{aligned}K_n^{\left[\frac{1}{n}\right]}(e_0;x)&=\frac{n\cdot n^{[n,-1]}}{(nx+n)^{[n,-1]}}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{n^{[k,-1]}\cdot(nx)^{[k,-1]}}{(nx+2n)^{[k,-1]}}\int_{\frac{k}{n}}^{\frac{k+1}{n}}dt\\&=\frac{n^{[n,-1]}}{(nx+n)^{[n,-1]}}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{n^{[k,-1]}\cdot(nx)^{[k,-1]}}{(nx+2n)^{[k,-1]}}=V_n^{\left[\frac{1}{n}\right]}(e_0;x)=1.\end{aligned}$$

$$\begin{aligned}K_n^{\left[\frac{1}{n}\right]}(e_1;x)&=\frac{n\cdot n^{[n,-1]}}{(nx+n)^{[n,-1]}}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{n^{[k,-1]}\cdot(nx)^{[k,-1]}}{(nx+2n)^{[k,-1]}}\int_{\frac{k}{n}}^{\frac{k+1}{n}}t\,dt\\&=\frac{n\cdot n^{[n,-1]}}{(nx+n)^{[n,-1]}}\sum_{k=0}^{\infty}\frac{1}{k!}\cdot\frac{2k+1}{2n^2}\cdot\frac{n^{[k,-1]}\cdot(nx)^{[k,-1]}}{(nx+2n)^{[k,-1}}\\&=V_n^{\left[\frac{1}{n}\right]}(e_1;x)+\frac{1}{2n}V_n^{\left[\frac{1}{n}\right]}(e_0;x)=x+\frac{(2x+1)n-1}{2n(n-1)}.\end{aligned}$$

$$\begin{aligned}K_n^{\left[\frac{1}{n}\right]}(e_2;x)&=\frac{n\cdot n^{[n,-1]}}{(nx+n)^{[n,-1]}}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{n^{[k,-1]}\cdot(nx)^{[k,-1]}}{(nx+2n)^{[k,-1]}}\int_{\frac{k}{n}}^{\frac{k+1}{n}}t^2\,dt\\&=\frac{n\cdot n^{[n,-1]}}{(nx+n)^{[n,-1]}}\sum_{k=0}^{\infty}\frac{1}{k!}\cdot\frac{3k^2+3k+1}{3n^3}\cdot\frac{n^{[k,-1]}\cdot(nx)^{[k,-1]}}{(nx+2n)^{[k,-1}}\\&=V_n^{\left[\frac{1}{n}\right]}(e_2;x)+\frac{1}{n}V_n^{\left[\frac{1}{n}\right]}(e_1;x)+\frac{1}{3n^2}V_n^{\left[\frac{1}{n}\right]}(e_0;x)=x^2+\frac{2(2n-1)x^2+3(n-1)x}{(n-1)(n-2)}+\frac{1}{3n^2}.\end{aligned}$$

□

Corollary 4.2. *The central moments up to the second order of Kantorovich-Baskakov-Lupaş operators* (8) *are*

$$K_n^{\left[\frac{1}{n}\right]}(e_1-x;x)=\frac{(2x+1)n-1}{2n(n-1)},\quad K_n^{\left[\frac{1}{n}\right]}\left((e_1-x)^2;x\right)=\frac{2x(1+x)n^2+2x(nx-1)}{n(n-1)(n-2)}+\frac{1}{3n^2}.$$

Proof. Taking Lemma 4.1 into account, it follows the above equalities. □

Proposition 4.3. *Let f be a bounded function defined on* $[0,+\infty)$, *with* $\|f\|=\sup_{x\in[0,+\infty)}|f(x)|$, *then*

$$\left|K_n^{\left[\frac{1}{n}\right]}(f;x)\right|\le\|f\|.$$

Proof. The definition of Kantorovich-Baskakov-Lupaş operators (8) and the fact that they preserve constants provide

$$\left|K_n^{\left[\frac{1}{n}\right]}(f;x)\right|=\left|n\cdot\sum_{k=0}^{\infty}v_{n,k}\left(x,\frac{1}{n}\right)\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(t)dt\right|\le n\cdot\sum_{k=0}^{\infty}v_{n,k}\left(x,\frac{1}{n}\right)\int_{\frac{k}{n}}^{\frac{k+1}{n}}|f(t)|dt\le\|f\|\cdot K_n^{\left[\frac{1}{n}\right]}(e_0;x)=\|f\|.$$

□

The uniform approximation of the functions defined on $C^*[0,+\infty)$ is given in the following:

Theorem 4.4. *For each function $f \in C^*[0,+\infty)$ yields*

$$\lim_{n\to\infty} \left\| K_n^{\left[\frac{1}{n}\right]} f - f \right\| = 0.$$

Proof. Taking Lemma 4.1 into account, the following three conditions hold

$$\lim_{n\to\infty} \left\| K_n^{\left[\frac{1}{n}\right]}(e_i;x) - x^i \right\| = 0, \quad i = 0,1,2.$$

Next, applying Theorem 2.1 we get the desired result. □

Theorem 4.5. *Let f be a bounded and integrable function on $[0,+\infty)$. If there exists first and second derivative of the function f in a fixed point $x \in [0,+\infty)$, then*

$$\lim_{n\to\infty} n\left(K_n^{\left[\frac{1}{n}\right]}(f;x) - f(x)\right) = \left(x + \frac{1}{2}\right) f'(x) + x(1+x) f''(x).$$

Proof. Using Taylor's expansion formula of function f, it follows

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2!}(t-x)^2 f''(x) + \varepsilon(t,x)(t-x)^2,$$

where $\varepsilon(t,x) := \varepsilon(t-x)$ is a bounded function and $lim_{t\to x}\varepsilon(t,x) = 0$. Taking the linearity of Kantorovich-Baskakov-Lupaş operators (8) into account and then by applying the operators $K_n^{\left[\frac{1}{n}\right]}$ on both sides of the above equation, we get

$$K_n^{\left[\frac{1}{n}\right]}(f;x) - f(x) = K_n^{\left[\frac{1}{n}\right]}(e_1 - x;x) f'(x) + \frac{1}{2} K_n^{\left[\frac{1}{n}\right]}\left((e_1-x)^2;x\right) f''(x) + K_n^{\left[\frac{1}{n}\right]}\left(\varepsilon(t,x)\cdot(e_1-x)^2;x\right).$$

Therefore using Corollary 4.2, we get

$$\lim_{n\to\infty} n\left(K_n^{\left[\frac{1}{n}\right]}(f;x) - f(x)\right) = \left(x + \frac{1}{2}\right) f'(x) + x(1+x) f''(x) + \lim_{n\to\infty} n\left(K_n^{\left[\frac{1}{n}\right]}\left(\varepsilon(t,x)\cdot(e_1-x)^2;x\right)\right). \tag{31}$$

We estimate the last term on the right hand side of the above equality by applying the Cauchy-Schwarz inequality, such that

$$K_n^{\left[\frac{1}{n}\right]}\left(\varepsilon(t,x)\cdot(e_1-x)^2;x\right) \le \sqrt{K_n^{\left[\frac{1}{n}\right]}(\varepsilon^2(t,x);x)}\sqrt{K_n^{\left[\frac{1}{n}\right]}\left((e_1-x)^4;x\right)}. \tag{32}$$

Because $\varepsilon^2(x,x) = 0$ and $\varepsilon^2(\cdot,x) \in C^*[0,+\infty)$, using the convergence from Theorem 4.4, we get

$$\lim_{n\to\infty} K_n^{\left[\frac{1}{n}\right]}\left(\varepsilon^2(t,x);x\right) = \varepsilon^2(x,x) = 0. \tag{33}$$

Therefore, from (32) and (33) yields

$$\lim_{n\to\infty} n\left(K_n^{\left[\frac{1}{n}\right]}\left(\varepsilon(t,x)\cdot(e_1-x)^2;x\right)\right) = 0$$

and using (31) we obtain the asymptotic behavior of the Kantorovich-Baskakov-Lupaş operators (8). □

Theorem 4.6. *Let be $f \in C[0,+\infty)$, then for any $x \in [0,+\infty)$ yields*

$$\left|K_n^{\left[\frac{1}{n}\right]}(f;x) - f(x)\right| \le N\cdot\omega_2\left(f, \frac{1}{2}\gamma_n(x)\right) + \omega(f,\gamma_\omega),$$

where N is an absolute constant and $\gamma_n(x) = \left(K_n^{\left[\frac{1}{n}\right]}\left((e_1-x)^2;x\right) + \left(K_n^{\left[\frac{1}{n}\right]}(e_1-x;x)\right)^2\right)^{\frac{1}{2}}$, $\gamma_\omega = \left|K_n^{\left[\frac{1}{n}\right]}(e_1-x;x)\right|$.

Proof. For $x \in [0, +\infty)$, we define the operators

$$\tilde{K}_n^{\left[\frac{1}{n}\right]}(f;x) = K_n^{\left[\frac{1}{n}\right]}(f;x) - f\left(\frac{2n^2x+n-1}{2n(n-1)}\right) + f(x). \tag{34}$$

We remark that $\tilde{K}_n^{\left[\frac{1}{n}\right]}(e_0;x) = 1$ and $\tilde{K}_n^{\left[\frac{1}{n}\right]}(e_1;x) = x$, i.e. the defined operators (34) preserve constants as well as linear functions. Therefore

$$\tilde{K}_n^{\left[\frac{1}{n}\right]}(e_1 - x; x) = 0. \tag{35}$$

Let be $g \in W_\infty^2$ and $x, t \in [0, +\infty)$. By Taylor's expansion formula, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)\,du.$$

Applying the Kantorovich-Baskakov-Lupaş operators (8) on both sides of the above equation, we get

$$\begin{aligned}\tilde{K}_n^{\left[\frac{1}{n}\right]}(g;x) - g(x) &= g'(x)\cdot\tilde{K}_n^{\left[\frac{1}{n}\right]}(e_1-x;x) + \tilde{K}_n^{\left[\frac{1}{n}\right]}\left(\int_x^t (t-u)g''(u)\,du;x\right)\\ &= K_n^{\left[\frac{1}{n}\right]}\left(\int_x^t (t-u)g''(u)\,du;x\right) - \int_x^{\frac{2n^2x+n-1}{2n(n-1)}}\left(\frac{2n^2x+n-1}{2n(n-1)} - u\right)g''(u)du.\end{aligned}$$

On the other hand

$$\left|\int_x^t (t-u)g''(u)\,du\right| \le (t-x)^2\cdot\|g''\|,$$

then

$$\left|\tilde{K}_n^{\left[\frac{1}{n}\right]}(g;x) - g(x)\right| \le \left(K_n^{\left[\frac{1}{n}\right]}\left((e_1-x)^2;x\right) + \left(K_n^{\left[\frac{1}{n}\right]}(e_1-x;x)\right)^2\right)\cdot\|g''\|.$$

Using the definition (35) of the operators $\tilde{K}_n^{\left[\frac{1}{n}\right]}$ and Proposition 4.3, it follows

$$\begin{aligned}\left|K_n^{\left[\frac{1}{n}\right]}(f;x) - f(x)\right| &\le \left|\tilde{K}_n^{\left[\frac{1}{n}\right]}(f-g;x)\right| + \left|\tilde{K}_n^{\left[\frac{1}{n}\right]}(g;x) - g(x)\right| + \left|g(x) - f(x)\right| + \left|f\left(\tfrac{2n^2x+n-1}{2n(n-1)}\right) - f(x)\right|\\ &\le 4\left\|f-g\right\| + \gamma_n^2(x)\left\|g''\right\| + \omega(f,\gamma_\omega),\end{aligned}$$

with $\gamma_n^2(x) = K_n^{\left[\frac{1}{n}\right]}\left((e_1-x)^2;x\right) + \left(K_n^{\left[\frac{1}{n}\right]}(e_1-x;x)\right)^2$ and $\gamma_\omega = \left|K_n^{\left[\frac{1}{n}\right]}(e_1-x;x)\right|$.
Now, taking infimum on the right-hand side over all $g \in W_\infty^2$ and using the relation (14), we get

$$\left|K_n^{\left[\frac{1}{n}\right]}(f;x) - f(x)\right| \le 4K_2\left(f, \frac{1}{4}\gamma_n^2(x)\right) + \omega(f,\gamma_\omega) \le N\cdot\omega_2\left(f,\frac{1}{2}\gamma_n(x)\right) + \omega(f,\gamma_\omega),$$

where N is an absolute constant. □

5. Durrmeyer-Baskakov-Lupaş operators based on inverse Pólya-Eggenberger distribution

The third modification of the Baskakov operators called Durrmeyer-Baskakov-Lupaş operators based on the inverse Pólya-Eggenberger distribution is given in introduction for any bounded and integrable function f defined on $[0, +\infty)$, by

$$D_n^{\left[\frac{1}{n}\right]}(f;x) = (n-1)\sum_{k=0}^{\infty} v_{n,k}\left(x,\tfrac{1}{n}\right)\int_0^\infty v_{n,k}\left(t,\tfrac{1}{n}\right)f(t)\,dt. \tag{36}$$

As we announced in the aim of this paper, we give only the definition of the Durrmeyer-Baskakov-Lupaş operators and we let an open gate for further research.

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