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Chen's Type Inequality for Warped Product Pseudo-slant Submanifolds of Kenmotsu f-manifolds

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Abstract. In the present paper, we consider non-trivial warped product pseudo slant submanifolds of type $M_{\perp} \times_f M_{\theta}$ and $M_{\theta} \times_f M_{\perp}$ of Kenmotsu f-manifold $\overline{M}$. Firstly, we get some basic properties of these type warped product submanifolds. Then, we prove the general sharp inequalities for mixed totally geodesic warped product pseudo slant submanifolds and also we consider equality cases. Also generalizes some previous inequalities as well.

1. Introduction

The notion of warped product which is a natural generalization of Riemannian product was introduced to construct the manifolds with negative curvature by Bishop and O'Neill in 1969 [8]. They gave the definition of these manifolds as follows:

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and let f be a positive differentiable function on M_1. Consider the product manifold $M_1 \times M_2$ with its projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$. Then their warped product manifold $M = M_1 \times_f M_2$ is the Riemannian manifold $(M_1 \times M_2, g)$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_{1*}X, \pi_{1*}X) + (f \circ \pi_1)^2 g_2(\pi_{2*}X, \pi_{2*}X)$$

for any vector fields X and Y tangent to M, where $*$ denotes tangent maps. Furthermore, a warped product manifold $M = M_1 \times_f M_2$ is a trivial or simply Riemannian product manifold if the warping function f is constant.

Then many authors make good jobs using this new notation. For example, Kenmotsu proved the existence of almost contact structure on special warped product manifold and also he showed that it has negative sectional curvature -1 [20].

On the other hand the notion of CR-warped product submanifold in a Kähler manifold was introduced by Chen in 2001 [10]. He obtained inequalities for the second fundamental form in terms of warping functions. Then many authors studied the geometric inequalities of warped product submanifolds in different ambient spaces at the series of articles [see [1–6, 21, 22, 24, 30]]. Recently, Şahin [28] constructed a general inequality for warped product pseudo slant isometrically immersed in a Kähler manifold.

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In the present paper, we consider these studies on Kenmotsu f-manifolds and we compute some geometric inequalities of non-trivial warped product pseudo slant submanifolds. The warped product pseudo slant submanifolds are natural extensions of CR-warped product submanifold.

2. Preliminaries

Let $\overline{M}$ be $(2n+s)$-dimensional manifold and φ is a non-null $(1, 1)$ tensor field on M. If φ satisfies

$$\varphi^3 + \varphi = 0, \tag{1}$$

then φ is called an f-structure and $\overline{M}$ is called f-manifold [36]. If $rank\varphi = 2n$, namely $s = 0$, φ is called almost complex structure and if $rank\varphi = 2n+1$, namely $s = 1$, then φ reduces an almost contact structure [16]. $rank\varphi$ is always constant [27].

On an f-manifold $\overline{M}$, P_1 and P_2 operators are defined by

$$P_1 = -\varphi^2, \quad P_2 = \varphi^2 + I, \tag{2}$$

which satisfy

$$\begin{aligned} P_1 + P_2 = I, \qquad & P_1^2 = P_1, \qquad P_2^2 = P_2, \\ \varphi P_1 = P_1\varphi = \varphi, \quad & P_2\varphi = \varphi P_2 = 0. \end{aligned} \tag{3}$$

These properties show that P_1 and P_2 are complement projection operators. There are D and $D^{\perp}$ distributions with respect to P_1 and P_2 operators, respectively [37]. Also, $\dim(D) = 2n$ and $\dim(D^{\perp}) = s$.

Let $\overline{M}$ be $(2n+s)$-dimensional f-manifold and φ is a $(1, 1)$ tensor field, ξ_i is vector field and η^i is 1-form for each $1 \le i \le s$ on M, respectively. If $\left(\varphi, \xi_i, \eta^i\right)$ satisfy

$$\eta^j(\xi_i) = \delta_i^j, \tag{4}$$

$$\varphi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \tag{5}$$

then $\left(\varphi, \xi_i, \eta^i\right)$ is called globally framed f-structure or simply framed f-structure and $\overline{M}$ is called globally framed f-manifold or simply framed f-manifold [25]. For a framed f-manifold $\overline{M}$, the following properties are satisfied [25]:

$$\varphi\xi_i = 0, \tag{6}$$

$$\eta^i \circ \varphi = 0. \tag{7}$$

If on a framed f-manifold $\overline{M}$, there exists a Riemannian metric which satisfies

$$\eta^i(X) = g(X, \xi_i), \tag{8}$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^{s} \eta^i(X)\,\eta^i(Y), \tag{9}$$

for all vector fields X, Y on $\overline{M}$, then $\overline{M}$ is called framed metric f-manifold [17]. On a framed metric f-manifold, fundamental 2-form Φ defined by

$$\Phi(X, Y) = g(X, \varphi Y), \tag{10}$$

for all vector fields $X,\ Y \in \chi\left(\overline{M}\right)$ [17]. For a framed metric f-manifold,

$$N_{\varphi} + 2\sum_{i=1}^{s} d\eta^{i} \otimes \xi_{i}, \tag{11}$$

is satisfied, $\overline{M}$ is called normal framed metric f-manifold, where N_{φ} denotes the Nijenhuis torsion tensor of φ [19].

A globally framed metric f-manifold $\overline{M}$ is called Kenmotsu f-manifold if it satisfies

$$\left(\overline{\nabla}_{X}\varphi\right)Y = \sum_{k=1}^{s}\left\{g\left(\varphi X,\ Y\right)\xi_{k} - \eta^{k}\left(Y\right)\varphi X\right\}, \tag{12}$$

for all vector fields $X,\ Y \in \chi\left(\overline{M}\right)$ [26].

Now, let M be a submanifold immersed in $\overline{M}$. We also denote by g the induced metric on M. Let TM be the Lie algebra of vector fields in M and $T^{\perp}M$ the set of all vector fields normal to M. Denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections of M and $\overline{M}$, respectively. Then, the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_{X}Y = \nabla_{X}Y + h\left(X,\ Y\right) \tag{13}$$

and

$$\overline{\nabla}_{X}V = -A_{V}X + \nabla^{\perp}_{X}V \tag{14}$$

respectively, for any $X,\ Y \in TM$ and any $V \in T^{\perp}M$. Here, $\nabla^{\perp}$ is normal connection in the normal bundle, h is second fundamental form of M and A_{V} is the Weingarten endomorphism associated with V [9]. On the other hand, there is a relation between A_{V} and h such that [9]

$$g\left(A_{V}X,\ Y\right) = g\left(h\left(X,\ Y\right),\ V.\right) \tag{15}$$

The mean curvature vector H is defined by $H = \dfrac{1}{m} traceh$, where m is the dimension of M. M is said to be minimal, totally geodesic and totally umbilical if H vanishes identically and $h = 0$,

$$h\left(X,\ Y\right) = g\left(X,\ Y\right)H, \tag{16}$$

respectively [9]. Furthermore, the second fundamental form h satisfies [9]

$$\left(\overline{\nabla}_{X}h\right)\left(Y,\ Z\right) = \nabla^{\perp}_{X}h\left(Y,\ Z\right) - h\left(\nabla_{X}Y,\ Z\right) - h\left(Y,\ \nabla_{X}Z\right). \tag{17}$$

3. Submanifolds of Globally Framed Metric f-manifolds

In this section, we recall some basic properties of submanifolds of globally framed metric f-manifolds from [7].

Definition 3.1. *Let $\overline{M}$ be a globally framed metric f-manifold and M is a submanifold of $\overline{M}$. For all $X \in \Gamma\left(TM\right)$, we can write*

$$\varphi X = TX + NX, \tag{18}$$

where TX and NX are called tangent and normal component of φX, respectively. Similarly, for each $V \in \Gamma\left(T^{\perp}M\right)$, we have

$$\varphi V = tV + nV. \tag{19}$$

Here, tV is tangent component and nV is normal component of φV.

Corollary 3.1. *Let $\overline{M}$ be a globally framed metric f-manifold and M is a submanifold of $\overline{M}$. Then the following identities hold:*

$$T^2 = -I + \sum_{k=1}^{s} \eta^k \otimes \xi_k - tN, \quad NT + nN = 0, \tag{20}$$

$$Tt + tn = 0, \quad Nt + n^2 = -I, \tag{21}$$

where I denotes the identity transformation.

Proposition 3.1. *Let $\overline{M}$ be a globally framed metric f-manifold and M is a submanifold of $\overline{M}$. Then, T and n are skew-symmetric tensor fields.*

Proposition 3.2. *Let $\overline{M}$ be a globally framed metric f-manifold and M is a submanifold of $\overline{M}$. Then, for $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we have*

$$g(NX, V) = -g(X, tV), \tag{22}$$

which gives the relation between N and t.

Proposition 3.3. *Let $\overline{M}$ be a globally framed metric f-manifold and M is a submanifold of $\overline{M}$. Then, for $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, the following identities hold:*

$$\left(\overline{\nabla}_X \varphi\right) Y = \overline{\nabla}_X \varphi Y - \varphi \overline{\nabla}_X Y \tag{23}$$

$$\left(\overline{\nabla}_X T\right) Y = \overline{\nabla}_X TY - T\overline{\nabla}_X Y, \tag{24}$$

$$\left(\overline{\nabla}_X N\right) Y = \overline{\nabla}_X^{\perp} NY - N\overline{\nabla}_X Y, \tag{25}$$

$$\left(\overline{\nabla}_X t\right) V = \overline{\nabla}_X tV - t\overline{\nabla}_X^{\perp} V, \tag{26}$$

$$\left(\overline{\nabla}_X n\right) V = \overline{\nabla}_X^{\perp} nV - n\overline{\nabla}_X^{\perp} V, \tag{27}$$

$$\left(\overline{\nabla}_X T\right) Y + \left(\overline{\nabla}_Y T\right) X = A_{NX} Y + A_{NY} X + 2th(X, Y), \tag{28}$$

$$\left(\overline{\nabla}_X N\right) Y + \left(\overline{\nabla}_Y N\right) X = 2nh(X, Y) - h(X, TY) - h(Y, TX), \tag{29}$$

$$\left(\overline{\nabla}_X t\right) V = A_{nV} X - TA_V X, \tag{30}$$

$$\left(\overline{\nabla}_X n\right) V = -h(tV, X) - NA_V X, \tag{31}$$

where h is the second fundamental form, ∇ is the Levi-Civita connection and A_V denotes the shape operator corresponding to the normal vector field V.

Definition 3.2. *Let $\overline{M}$ be a globally framed metric f-manifold and M is a submanifold of $\overline{M}$. Then, the TM tangent bundle of M can be decomposed as*

$$TM = \sum_{k=1}^{s} D_\theta \oplus \xi_k, \tag{32}$$

where for each $1 \leq k \leq s$ the ξ_k denotes the distributions spanned by the structure vector fields ξ_k and D_θ is complementary of distributions ξ_k in TM, known as the slant distribution on M.

Theorem 3.1. *Let $\overline{M}$ be a globally framed metric f-manifold and M is a submanifold of $\overline{M}$. Then, M is a slant submanifold if and only if there exists a constant $\mu \in [0,\ 1]$ such that*

$$T^2 = -\mu\left(I - \sum_{k=1}^{s} \eta^k \otimes \xi_k\right). \tag{33}$$

Moreover, if θ is the slant angle of M, then $\mu = \cos^2\theta$.

Corollary 3.2. *Let M be a slant submanifold of a globally framed metric f-manifold $\overline{M}$ with slant angle θ. Then for any vector fields $X,\ Y \in \Gamma(TM)$, we find*

$$g(TX,\ TY) = \cos^2\theta\left\{g(X,\ Y) - \sum_{k=1}^{s} \eta^k(X)\,\eta^k(Y)\right\} \tag{34}$$

and

$$g(NX,\ NY) = \sin^2\theta\left\{g(X,\ Y) - \sum_{k=1}^{s} \eta^k(X)\,\eta^k(Y)\right\}. \tag{35}$$

Definition 3.3. *Let M be a submanifold of a globally framed metric f-manifold $\overline{M}$ and let M be tangent to the structure vector fields ξ_k for each $1 \le k \le s$. For each nonzero vector X tangent to M at p, we denote by $0 \le \theta(X) \le \frac{\pi}{2}$, the angle between φX and T_pM, known as the Wirtinger angle of X. If the $\theta(X)$ is constant, that is, independent of the choice of $p \in M$ and $X \in T_pM - \{\xi_k\}$, for each $1 \le k \le s$, then M is said to be a slant submanifold and the constant angle θ is called slant angle of the slant submanifold*

Here, if $\theta = 0$, M is invariant submanifold and if $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold. A slant submanifold is proper slant if it is neither invariant nor anti-invariant submanifold.

Definition 3.4. *Let M be a submanifold of a a globally framed metric f-manifold $\overline{M}$ We say that M is a pseudo-slant submanifold if there exist two orthogonal distributions D_θ and $D^\perp$ such that*

1) *The TM tangent bundle of M admits the orthogonal direct decomposition $TM = D^\perp \oplus D_\theta$, where for each $1 \le k \le s$ $\xi_k \in \Gamma(D_\theta)$.*

2) *The distribution $D^\perp$ is anti-invariant i. e., $\varphi(D^\perp) \subset (T^\perp M)$.*

3) *The distribution D_θ is slant with angle $\theta \neq \frac{\pi}{2}$, that is, the angle between D_θ and $\varphi(D_\theta)$ is a constant.*

A pseudo-slant submanifold of a globally framed metric f-manifold is called mixed totally geodesic if $h(X,\ Z) = 0$ for all $X \in \Gamma(D^\perp)$ and $Z \in \Gamma(D_\theta)$. Now let $\{e_1,\ \ldots,\ e_n\}$ be an orthonormal basis of the tangent space TM and $e_r$belongs to the orthonormal basis $\{e_{n+1},\ \ldots,\ e_m\}$ of a normal bundle $T^\perp M$, then we define

$$h^r_{ij} = g\left(h\left(e_i,\ e_j\right),\ e_r\right) \quad \text{and} \quad \|h\|^2 = \textstyle\sum_{i,\ j=1}^{n} g\left(h\left(e_i,\ e_j\right),\ h\left(e_i,\ e_j\right)\right), \tag{36}$$

On the other hand, for a differentiable function λ on M, we have

$$\|\nabla\lambda\|^2 = \sum_{i=1}^{n} \left(e_i(\lambda)\right)^2, \tag{37}$$

where the gradient $\nabla grad\lambda$ is defined by $g(\nabla\lambda,\ X) = X\lambda$, for any vector field $X \in \Gamma(TM)$.

Theorem 3.2. *Let M be a proper slant submanifold of a globally framed metric f-manifold $\overline{M}$, such that $\xi_k \in TM$. Then we have*

$$i)\ tNX = \sin^2\theta\left\{-X + \textstyle\sum_{k=1}^{s} \eta^k(X)\,\xi_k\right\}, \qquad ii) nNX = -NTX,$$

for all vector $X \in \Gamma(TM)$.

Proof. By applying φ both sides of (18), it means that

$$\varphi^2 X = \varphi TX + \varphi NX,$$

for any $X \in \Gamma(TM)$. By using (5) and again form (18) and (19), we get

$$-X + \sum_{k=1}^{s} \eta^k(X)\,\xi_k = T^2X + NTX + tNX + nNX.$$

Then by virtue of Corollary 1 and considering the tangential and normal components of the last equation, we get the results. □

4. Pseudo-Slant Submanifolds of Kenmotsu f-manifolds

In this section, we get some useful lemmas to compute main results in the next part.

Lemma 4.1. *Let M be a proper pseudo slant submanifold of Kenmotsu f-manifold $\overline{M}$. Then the following holds*

$$\cos^2 g(\nabla_X Y,\, Z) = g\left(A_{\varphi Z}TY - A_{NTY}Z,\, X\right)$$

for any $X,\ Y \in \sum_{k=1}^{s} D_\theta \oplus \xi_k$ and $Z \in D^\perp$.

Proof. For all vector fields $X,\ Y \in \sum_{k=1}^{s} D_\theta \oplus \xi_k$ and $Z \in D^\perp$, it follows that

$$g(\nabla_X Y,\, Z) = g\left(\overline{\nabla}_X Y,\, Z\right) = g\left(\varphi\overline{\nabla}_X Y,\, \varphi Z\right).$$

By using (23), we get

$$g(\nabla_X Y,\, Z) = g\left(\overline{\nabla}_X \varphi Y,\, \varphi Z\right) - g\left(\left(\overline{\nabla}_X \varphi\right) Y,\, \varphi Z\right).$$

Then by virtue of (12) and (18), we deduce that

$$\begin{aligned} g(\nabla_X Y,\, Z) &= g\left(\overline{\nabla}_X TY,\, \varphi Z\right) + g\left(\overline{\nabla}_X NY,\, \varphi Z\right) \\ &= g(h(X,\, TY),\, \varphi Z) - g\left(\overline{\nabla}_X \varphi NY,\, Z\right) \\ &\quad + g\left(\left(\overline{\nabla}_X \varphi\right) NY,\, Z\right). \end{aligned}$$

From (12) and (19), it yields

$$g(\nabla_X Y,\, Z) = g\left(A_{\varphi Z}TY,\, X\right) - g\left(\overline{\nabla}_X tNY,\, Z\right) - g\left(\overline{\nabla}_X tNY,\, Z\right).$$

By using Theorem 2. we have

$$\begin{aligned} g(\nabla_X Y,\, Z) &= g\left(A_{\varphi Z}TY,\, X\right) + \sin^2\theta\, g\left(\overline{\nabla}_X Y,\, Z\right) \\ &\quad - \sin^2\theta \sum_{k=1}^{s} \eta^k(Y)\, g\left(\overline{\nabla}_X \xi_k,\, Z\right) + g\left(\overline{\nabla}_X NTY,\, Z\right). \end{aligned}$$

Thus using (12), (13) and (14) in the last equation give us the desired result. □

Lemma 4.2. *Let M be a pseudo slant submanifold of Kenmotsu f-manifold $\overline{M}$. Then the following holds*

$$A_{\varphi Z}W - A_{\varphi W}Z = 0, \tag{38}$$

for any $Z,\ W \in D^\perp$.

Proof. It has a similar calculation to Kenmotsu one in [24] and so we omit it. □

Theorem 4.1. *Let M be a proper pseudo slant submanifold of Kenmotsu f-manifold $\overline{M}$. Then the anti-invariant distribution $D^{\perp}$ is integrable.*

Proof. For all vector fields $Z,\ W \in D^{\perp}$ and $X \in \sum_{k=1}^{s} D_{\theta} \oplus \xi_k$, then it follows

$$\begin{aligned} g([Z,\ W],\ X) &= g\left(\overline{\nabla}_Z W,\ X\right) - g\left(\overline{\nabla}_W Z,\ X\right) \\ &= g\left(\varphi\overline{\nabla}_Z W,\ \varphi X\right) - g\left(\varphi\overline{\nabla}_W Z,\ \varphi X\right) + \sum_{k=1}^{s} \eta^k(X)\left\{g\left(\overline{\nabla}_Z W,\ \xi_k\right) - g\left(\overline{\nabla}_W Z,\ \xi_k\right)\right\} \\ &= g\left(\overline{\nabla}_Z \varphi W,\ \varphi X\right) + g\left(\left(\overline{\nabla}_Z \varphi\right) W,\ \varphi X\right) - g\left(\overline{\nabla}_W \varphi Z,\ \varphi X\right) - g\left(\left(\overline{\nabla}_W \varphi\right) Z,\ \varphi X\right) \\ &\quad - \sum_{k=1}^{s} \eta^k(X)\left\{g\left(W,\ \overline{\nabla}_Z \xi_k\right) - g\left(Z,\ \overline{\nabla}_W \xi_k\right)\right\}. \end{aligned}$$

By virtue of (12) and (18), we conclude that

$$g([Z,\ W],\ X) = g\left(\overline{\nabla}_Z \varphi W,\ TX\right) + g\left(\overline{\nabla}_Z \varphi W,\ NX\right) - g\left(\overline{\nabla}_W \varphi Z,\ TX\right) - g\left(\overline{\nabla}_W \varphi Z,\ NX\right).$$

Then by using (13) and (14) and since the vector fields are orthogonal, we deduce

$$g([Z,\ W],\ X) = g\left(A_{\varphi Z} W - A_{\varphi W} Z,\ TX\right) - g\left(\varphi W,\ \overline{\nabla}_Z NX\right) + g\left(\varphi Z,\ \overline{\nabla}_W NX\right).$$

From (38) the first term of the right hand side is identically zero, hence by using (8), (9) and (23) it is said that

$$g([Z,\ W],\ X) = g\left(W,\ \overline{\nabla}_Z \varphi NX\right) - g\left(W,\ \left(\overline{\nabla}_Z \varphi\right) NX\right) - g\left(Z,\ \overline{\nabla}_W \varphi NX\right) + g\left(Z,\ \left(\overline{\nabla}_W \varphi\right) NX\right).$$

Now by using (12) and (19), then we derive

$$g([Z,\ W],\ X) = g\left(W,\ \overline{\nabla}_Z tNX\right) + g\left(W,\ \overline{\nabla}_Z nNX\right) - g\left(Z,\ \overline{\nabla}_W tNX\right) - g\left(Z,\ \overline{\nabla}_W nNX\right).$$

In view of Theorem 2, we get

$$\begin{aligned} g([Z,\ W],\ X) = \sin^2\theta &\left\{g\left(\overline{\nabla}_W Z,\ X\right) - g\left(\overline{\nabla}_Z W,\ X\right) + \sum_{k=1}^{s}\left[g\left(W,\ \overline{\nabla}_Z \xi_k\right) - g\left(Z,\ \overline{\nabla}_W \xi_k\right)\right]\right\} \\ &- g\left(W,\ \overline{\nabla}_Z NTX\right) + g\left(Z,\ \overline{\nabla}_W NTX\right). \end{aligned}$$

By virtue of (12) and (13) and the orthogonality of vector fields, it follows

$$g([Z,\ W],\ X) = \sin^2\theta\left\{g\left(\overline{\nabla}_Z W,\ X\right) - g\left(\overline{\nabla}_W Z,\ X\right)\right\} + g\left(A_{NTX} Z,\ W\right) + g\left(Z,\ A_{NTX} W\right).$$

From the well-known properties, we have

$$g([Z,\ W],\ X) = \sin^2\theta g([Z,\ W],\ X).$$

This implies that $\cos^2\theta g([Z,\ W],\ X) = 0$. By the assumption of the theorem, it can be said that $\cos^2\theta \neq 0$ and thus we have $g([Z,\ W],\ X) = 0$ which means $D^{\perp}$ is integrable. □

5. Warped Product Pseudo-Slant Submanifolds

In this section, we investigate some fundamental properties of warped product pseudo-slant submanifolds of Kenmotsu f-manifolds. Firstly, we give the following lemma from [8] which we use next.

Lemma 5.1. *Let $M = M_1 \times_f M_2$ be a warped product manifold. Then we have*
(i) $\nabla_X Y \in \Gamma(TM_1)$,
(ii) $\nabla_Z X = \nabla_X Z = (X \ln f) Z$,
(iii) $\nabla_Z W = \nabla^\circ_Z W - g(Z, W) \nabla \ln f$,
for all $X, Y \in \Gamma(TM_1)$ *and* $Z, W \in \Gamma(TM_2)$ *,where* ∇ *and* ∇° *denote the Levi-Civita connections on* M_1 *and* M_2*,respectively. Moreover,* $\nabla \ln f$*, the gradient of* $\ln f$*, is defined by* $g(\nabla \ln f, U) = U \ln f$*. A warped product manifold* $M = M_1 \times_f M_2$ *is trivial if the warping function* f *is constant. If* $M = M_1 \times_f M_2$ *is a warped product manifold then it is said to be that* M_1 *is totally geodesic and* M_2 *is totally umbilical submanifold of* M*.*

In the following two examples, we follow a similar method which is used in [24].

Example 5.1. *Let us consider* $\mathbb{R}^{10}$ *with its Cartesian coordinates* $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, t_1, t_2)$ *and the globally framed metric* f*-structure given by*

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial t_k}\right) = 0, \quad 1 \le i,\ j \le 4, \quad t = 1,\ 2.$$

Let $X = \lambda_i \frac{\partial}{\partial x_i} + \mu_j \frac{\partial}{\partial y_j} + \nu_k \frac{\partial}{\partial t_k}$ *be a vector field in* $\mathbb{R}^{10}$*. Then* $\varphi X = -\lambda_i \frac{\partial}{\partial y_i} + \mu_j \frac{\partial}{\partial x_j}$ *and* $\varphi^2 X = -\lambda_i \frac{\partial}{\partial x_i} - \mu_j \frac{\partial}{\partial y_j} = -X + \nu_k \frac{\partial}{\partial t_k}$*. Furthermore,* $g(X, X) = \lambda_i^2 + \mu_j^2 + \nu_k^2$ *and* $g(\varphi X, \varphi X) = \lambda_i^2 + \mu_j^2$*, where* g *is the Euclidean inner product of* $\mathbb{R}^{10}$*. Then we obtain* $g(\varphi X, \varphi X) = g(X, X) - \left[\eta^k(X)\right]^2$*, where* $\eta^k(X) = g(X, \xi_k)$ *and* $\eta^k = dt_k$ *and thus* $\left(\varphi\ \xi_k,\ \eta^k,\ g\right)$ *is a globally framed metric* f*-structure. Now we consider a submanifold* M *of* $\mathbb{R}^{10}$ *defined by the immersion*

$$\chi(u_1, u_2, u_3, u_4, t_1, t_2) = \left(\sqrt{3}u_3,\ 0,\ u_1,\ 0,\ u_2 \sin\theta,\ u_2 \cos\theta,\ 0,\ u_4,\ t_1,\ t_2\right).$$

We set the orthonormal vector fields

$$e_1 = \frac{\partial}{\partial x_3}, \quad e_2 = \sin\theta \frac{\partial}{\partial y_1} + \cos \frac{\partial}{\partial y_2}$$
$$e_3 = \sqrt{3}\frac{\partial}{\partial x_1}, \quad e_4 = \frac{\partial}{\partial y_4}, \quad e_5 = \frac{\partial}{\partial t_1}, \quad e_6 = \frac{\partial}{\partial t_2}.$$

Then it follows that

$$\varphi e_1 = -\frac{\partial}{\partial y_3}, \quad \varphi e_2 = \sin\theta \frac{\partial}{\partial x_1} + \cos \frac{\partial}{\partial x_2}$$
$$\varphi e_3 = -\sqrt{3}\frac{\partial}{\partial y_1}, \quad \varphi e_4 = \frac{\partial}{\partial x_4}, \quad \varphi e_5 = 0, \quad \varphi e_6 = 0.$$

Under these conditions we see that φe_1*and* φe_4 *are orthogonal to* TM*. Hence* M *is a pseudo slant submanifold with anti-invariant distribution* $D^\perp = Span\{e_1, e_4\}$ *and the slant distribution* $D_{\theta_1} = Span\{e_2, e_3\}$ *with slant angle* $\theta_1 = \cos^{-1}(\sin\theta)$ *such that* $\xi_1 = e_5$ *and* $\xi_2 = e_6$ *are tangent to* M*. In fact,* M *is a proper pointwise pseudo-slant submanifold.*

Example 5.2. *Let us consider $\mathbb{R}^8$ with its Cartesian coordinates $(x_1, x_2, x_3, y_1, y_2, y_3, t_1, t_2)$ and the globally framed metric f-structure given by*

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial t_k}\right) = 0, \quad 1 \leq i, j \leq 3, \quad t = 1, 2.$$

We can easily show that $\mathbb{R}^8$ is a globally framed metric f-structure with respect to the Euclidean metric tensor of $\mathbb{R}^8$ in a similar way of Example 1. Let M be a submanifold of $\mathbb{R}^8$ given by the immersion χ as follows

$$\chi(u_1, u_2, u_3, t_1, t_2) = (u_1 \tan u_3, 2u_1 + 2u_2, u_2 \tan u_3, u_2 \cot u_3, 2u_1 - 2u_2, u_1 \cot u_3, t_1, t_2).$$

Then the tangent space of M is spanned by the following vectors

$$Z_1 = \tan u_3 \frac{\partial}{\partial x_1} + 2\frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial y_2} + \cot u_3 \frac{\partial}{\partial y_3},$$

$$Z_2 = \tan u_3 \frac{\partial}{\partial x_3} + 2\frac{\partial}{\partial x_2} - 2\frac{\partial}{\partial y_2} + \cot u_3 \frac{\partial}{\partial y_1},$$

$$Z_3 = -u_1 \sec^2 u_3 \frac{\partial}{\partial x_1} + u_2 \sec^2 u_3 \frac{\partial}{\partial x_3} - u_2 \csc^2 u_3 \frac{\partial}{\partial y_1} - u_1 \csc^2 u_3 \frac{\partial}{\partial y_3},$$

$$Z_4 = \frac{\partial}{\partial t_1}, \quad Z_5 = \frac{\partial}{\partial t_2}.$$

Now, we obtain

$$\varphi Z_1 = -\tan u_3 \frac{\partial}{\partial y_1} - 2\frac{\partial}{\partial y_2} + 2\frac{\partial}{\partial x_2} + \cot u_3 \frac{\partial}{\partial x_3},$$

$$\varphi Z_2 = -\tan u_3 \frac{\partial}{\partial y_3} - 2\frac{\partial}{\partial y_2} - 2\frac{\partial}{\partial x_2} + \cot u_3 \frac{\partial}{\partial x_1},$$

$$\varphi Z_3 = -u_1 \sec^2 u_3 \frac{\partial}{\partial y_1} - u_2 \sec^2 u_3 \frac{\partial}{\partial y_3} - u_2 \csc^2 u_3 \frac{\partial}{\partial x_1} - u_1 \csc^2 u_3 \frac{\partial}{\partial x_1},$$

$$\varphi Z_4 = 0, \quad \varphi Z_5 = 0.$$

Hence we see that φZ_3 is orthogonal to TM and so it is said that the anti-invariant distribution $D^\perp = span\{Z_3\}$ and $D_\theta = span\{Z_1, Z_2\}$ is a proper slant distribution with slant angle $\theta = arccos\left(\frac{\tan^2 u_3 + \cot^2 u_3 - 2}{\tan^2 u_3 + \cot^2 u_3 + 2}\right)$ such that $\xi_1 = \frac{\partial}{\partial t_1}$ and $\xi_2 = \frac{\partial}{\partial t_2}$ are tangent to M which means M is a proper pseudo slant submanifold. It is easy to see that both distributions are integrable. If we denote the integral manifolds of $D^\perp$and D_θ by $M_\perp$ and M_θ, respectively. then the metric tensor g of M is computed as

$$g = 9\left(du_1^2 + du_2^2\right) + \left(u_1^2 + u_2^2\right)\left(\sec^2 u_3 + \csc^2 u_3\right)^2 du_3^2 + dt_1^2 + dt_2^2.$$

Thus M is a warped product pseudo slant submanifold $M = M_\theta \times_f M_\perp$ with the warping function $f = \sqrt{\left(u_1^2 + u_2^2\right)(\sec^2 u_3 + \csc^2 u_3)^2}$.

Now, we prove some lemmas for the next section. We begin with the following.

Lemma 5.2. *Let $M = M_\theta \times_f M_\perp$ be a warped product pseudo slant submanifold of a Kenmotsu f-manifold $\overline{M}$. Then the followings hold.*
(i) $g(h(X, Y), \varphi Z) = g(h(X, Z), NY)$,
(ii) $g(h(X, Z), \varphi W) = g(h(X, W), \varphi Z)$,
for any $X, Y \in \Gamma(TM_\theta)$ and $Z, W \in \Gamma(TM_\perp)$.

Proof. For any $X, Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, we obtain

$$g(h(X, Y), \varphi Z) = g\left(\overline{\nabla}_X Y, \varphi Z\right) = g\left(\left(\overline{\nabla}_X \varphi\right) Y, Z\right) - g\left(\overline{\nabla}_X \varphi Y, Z\right).$$

By using (12) the first term of right hand side is identically zero and in view of the orthogonality of vector fields, then we get

$$g(h(X, Y), \varphi Z) = g\left(\overline{\nabla}_X Z, \varphi Y\right) g(\nabla_X Z, TY) + g\left(\overline{\nabla}_X Z, NY\right).$$

Thus taking into account of (13), (14) and Lemma 3 (ii) in the last equation, then we derive the first identity. To prove second part, let us consider $X \in \Gamma(TM_\theta)$ and $Z, W \in \Gamma(TM_\perp)$ it follows that

$$g(h(X, Z), \varphi W) = g\left(\overline{\nabla}_X Z, \varphi W\right) = -g\left(\varphi \overline{\nabla}_X Z, W\right).$$

By virtue of (23), then we deduce

$$g(h(X, Z), \varphi W) = g\left(\left(\overline{\nabla}_X \varphi\right) Z, W\right) - g\left(\overline{\nabla}_X \varphi Z, W\right).$$

Substituting (12) and (14) in the previous equation, we have (ii) which completes the proof. □

Lemma 5.3. *Let $M = M_\theta \times_f M_\perp$ be a warped product pseudo slant submanifold of a Kenmotsu f-manifold $\overline{M}$. Then we have*
(i) $\sum_{k=1}^{s} \xi_k \ln f = s$
(ii) $g(h(Z, W), NX) = g(h(X, W), \varphi Z) - (TX \ln f)\, g(Z, W)$
(iii) $g(h(Z, W), NTX) = g(h(TX, W), \varphi Z) - \cos^2\theta \left\{(X \ln f) - s\sum_{k=1}^{s} \eta^k(X)\right\} g(Z, W)$
for all $X \in \Gamma(TM_\theta)$ and $Z, W \in \Gamma(TM_\perp)$.

Proof. Let us consider $X \in \Gamma(TM_\theta)$ and $\xi_k \in \Gamma(TM_\perp)$ for each $1 \le k \le s$. Then we have $\sum_{k=1}^{s} \overline{\nabla}_X \xi_k = \sum_{k=1}^{s} (X \ln f)\, \xi_k$ and by taking the inner product with ξ_i, we obtain $(X \ln f) = \sum_{k=1}^{s} g\left(\overline{\nabla}_X \xi_k, \xi_i\right) = 0$ which implies that f is constant. Hence, we consider the structure vector fields $\xi_k' s$ tangent to M_θ and so we can write $\sum_{k=1}^{s} \overline{\nabla}_Z \xi_k = \sum_{k=1}^{s} \{\nabla_Z \xi_k + h(Z, \xi_k)\}$ and moreover by using (12) and Lemma 3 (ii), we have (i). For the second property of the lemma, let us consider any $Z, W \in \Gamma(TM_\perp)$ and $X \in \Gamma(TM_\theta)$, then we derive

$$g(h(Z, W), NX) = g\left(\overline{\nabla}_Z W, \varphi X\right) - g\left(\overline{\nabla}_Z W, TX\right).$$

By virtue of (9) and (23) and in view of the orthogonality of vector fields, we conclude that

$$g(h(Z, W), NX) = g\left(\left(\overline{\nabla}_Z \varphi\right) W, X\right) - g\left(\overline{\nabla}_Z \varphi W, X\right) - g\left(W, \overline{\nabla}_Z TX\right).$$

By taking into account of (12), (13), (14) and Lemma 3 (ii), we obtain (ii) of this lemma. By interchanging X by TX in (ii) and by using Theorem 1 and $\sum_{k=1}^{s} \xi_k \ln f = s$, we have the desired results. □

Now we make the characterization of warped product submanifold of a Kenmotsu f-manifold which is mixed totally geodesic. Firstly let us recall the definition of the mixed totally geodesic.

A warped product submanifold $M = M_1 \times_f M_2$ of a Kenmotsu f-manifold $\overline{M}$ is called mixed totally geodesic, if $h(X, Z) = 0$ for any $X \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$, where M_1 and M_2 are Riemannian submanifolds of $\overline{M}$.

Theorem 5.1. *Let M be a proper pseudo slant submanifold of Kenmotsu f-manifold $\overline{M}$. Then M is locally a mixed totally geodesic warped product submanifold if and only if*

$$A_{\varphi Z}X = 0 \quad and \quad A_{NTX}Z = -\cos^2\theta\left\{(X\mu) - s\sum_{k=1}^{s}\eta^k(X)\right\}Z \tag{39}$$

for any $Z \in D^{\perp}$ and $X \in \sum_{k=1}^{s} D_\theta \oplus \xi_k$ for some smooth function on M such that $W(\mu) = 0$, for all vector fields $W \in D^{\perp}$.

Proof. Let M be a mixed totally geodesic warped product submanifold of a Kenmotsu f-manifold. Then $A_{\varphi Z}X = 0$ holds from Lemma 4 (i). On the other hand, by using Lemma 5 (iii) we get the second part of the lemma.

Conversely, let M be a proper pseudo slant submanifold of Kenmotsu f-manifold $\overline{M}$ with the anti-invariant and slant distributions $D^{\perp}$ and $\sum_{k=1}^{s} D_\theta \oplus \xi_k$ such that (39) holds. Now from Lemma 1, we obtain

$$g(\nabla_X Y, Z) = \sec^2\theta g\left(A_{\varphi Z}TY - A_{NTY}Z, X\right)$$

for any $Z \in D^{\perp}$ and $X, Y \in \sum_{k=1}^{s} D_\theta \oplus \xi_k$. By virtue of (39) and in view of the orthogonality of vector fields, it follows $g(\nabla_X Y, Z) = 0$ which implies that the leaves of the distribution $\sum_{k=1}^{s} D_\theta \oplus \xi_k$ are totally geodesic in M. On the other hand, from Theorem 3 the anti-invariant distribution $D^{\perp}$ is integrable and then if we consider a leaf of $M_{\perp}$ of $D^{\perp}$ in M and if $\widetilde{h}$ is the second fundamental form of $M_{\perp}$ in M, then we deduce

$$g\left(\widetilde{h}(Z, W), X\right) = g(\nabla_Z W, X) = g\left(\overline{\nabla}_Z W, X\right)$$

for all vector fields $Z, W \in D^{\perp}$ and $X \in \sum_{k=1}^{s} D_\theta \oplus \xi_k$. Taking into account of (9) in the last equation, we get

$$\begin{aligned} g\left(\widetilde{h}(Z, W), X\right) &= g\left(\varphi\overline{\nabla}_Z W, \varphi X\right) + \sum_{k=1}^{s}\eta^k(X)\, g\left(\overline{\nabla}_Z W, \xi_k\right) \\ &= g\left(\overline{\nabla}_Z \varphi W, \varphi X\right) - g\left(\left(\overline{\nabla}_Z\varphi\right)W, \varphi X\right) - \sum_{k=1}^{s}\eta^k(X)\, g\left(W, \overline{\nabla}_Z\xi_k\right). \end{aligned}$$

Again by using (12) and the orthogonality of vector fields, we derive

$$\begin{aligned} g\left(\widetilde{h}(Z, W), X\right) &= -g\left(\varphi W, \overline{\nabla}_Z\varphi X\right) - \sum_{k=1}^{s}\eta^k(X)\, g(Z, W) \\ &= -g\left(\varphi W, \overline{\nabla}_Z TX\right) - g\left(\varphi W, \overline{\nabla}_Z NX\right) - \sum_{k=1}^{s}\eta^k(X)\, g(Z, W). \end{aligned}$$

By taking into account of (9), (13), (14) and (23) in the last equation, it yields

$$g\left(\widetilde{h}(Z, W), X\right) = -g(\varphi W, h(Z, TX)) + g\left(W, \overline{\nabla}_Z\varphi NX\right) - g\left(W, \left(\overline{\nabla}_Z\varphi\right)NX\right) - \sum_{k=1}^{s}\eta^k(X)\, g(Z, W).$$

From (12) and (19), we obtain

$$g\left(\widetilde{h}(Z, W), X\right) = g\left(A_{\varphi W}TX, Z\right) + g\left(\overline{\nabla}_Z tNX, W\right) + g\left(\overline{\nabla}_Z nNX, W\right) - \sum_{k=1}^{s}\eta^k(X)\, g(Z, W).$$

By virtue of (39), the first term in the right hand side is identically zero, thus from Theorem 2 we find

$$g\left(\widetilde{h}(Z, W), X\right) = -\sin^2\theta\left\{g\left(\overline{\nabla}_Z X, W\right) - \sum_{k=1}^{s}\eta^k(X)\, g\left(W, \overline{\nabla}_Z\xi_k\right)\right\} - g\left(\overline{\nabla}_Z NTX, W\right) - \sum_{k=1}^{s}\eta^k(X)\, g(Z, W).$$

Now in view of (39), then we have

$$g\left(\widetilde{h}(Z,\ W),\ X\right) = -(X\mu)\,g(Z,\ W).$$

Considering the definition of gradient the above equation gives us

$$\widetilde{h}(Z,\ W) = -\nabla^{\theta}\mu g(Z,\ W)$$

where $\nabla^{\theta}\mu$ is the gradient of the function μ. This implies that $M_{\perp}$ is totally umbilical in M with the mean curvature $\widetilde{H} = -\nabla^{\theta}\mu$. Furthermore, it can be proven that $\widetilde{H}$ is parallel corresponding to the normal connection $\widetilde{D}$ of $M_{\perp}$ in M as a similar way of [21]. Hence it is said that $M_{\perp}$ is an extrinsic sphere in M. Moreover, by virtue of Hiepko [18] we deduce that M is a warped product submanifold which completes the proof. □

6. A Geometric Inequality for a Warped Product Pseudo Slant Submanifold of the form $M_{\theta} \times_f M_{\perp}$

In this section, we obtain a geometric inequality of warped product pseudo slant submanifold in terms of the second fundamental form such that ξ_k is tangent to the invariant submanifold and the mixed totally geodesic submanifold for each $1 \leq k \leq s$.

Now, let $M = M_{\theta}\times_f M_{\perp}$ be $(m+s-1)$-dimensional warped product pseudo slant submanifold of $(2n+s)$-dimensional Kenmotsu f-manifold $\overline{M}$ with M_{θ} of dimension $d_1 = 2p+s$ and $M_{\perp}$ of dimension $d_2 = q$, where M_{θ} and $M_{\perp}$ are the integral manifolds of D_{θ} and $D^{\perp}$, respectively such that $\xi_k' s$ are tangent to M_{θ}, where $M_{\perp}$ and M_{θ} are anti-invariant and proper slant submanifolds of $\overline{M}$. Then we consider $\{e_1, \ldots e_q\}$ and $\{e_{q+1} = e_1^*, \ldots e_{q+p} = e_p^*, e_{q+p+1} = e_{p+1}^* = \sec\theta T e_1^*, \ldots, e_{q+2p} = e_{2p}^* = \sec\theta T e_p^*, e_{m-1} = e_{q+2p+1} = e_{2p+1}^* = \xi_1, \ldots, e_{m+s-1} = e_{q+2p+s} = e_{2p+s}^* = \xi_s\}$ are orthonormal basis of $D^{\perp}$and D_{θ}, respectively. Hence the orthonormal basis of the normal subbundles $\varphi D^{\perp}$, ND_{θ} and ν are $\{e_{m+s} = \bar{e}_1 = \varphi e_1, \ldots, e_{m+s+q} = \bar{e}_q = \varphi e_q\}$, $\{e_{m+s-1+q+1} = \bar{e}_{q+1} = \csc\theta N e_1^*, \ldots, e_{m+s-1+q+p} = \bar{e}_{q+p} = \csc\theta N e_p^*, e_{m+s+q+p} = \bar{e}_{q+p+1} = \csc\theta\sec\theta N T e_1^*, \ldots e_{m+s-1+q+2p} = \bar{e}_{q+2p} = \csc\theta\sec\theta N T e_p^*\}$ $\{e_{2(m+s-1)} = \bar{e}_m, \ldots, e_{2n+s} = \bar{e}_{2(n-m+s)}\}$, respectively. It is clear that the dimensions of the normal subspaces $\varphi D^{\perp}$, ND_{θ} and ν are q, $2p$ and $2(n-m+s)$, respectively.

Theorem 6.1. *Let $M = M_{\theta}\times_f M_{\perp}$ be m-dimensional mixed totally geodesic warped product pseudo slant submanifold of a $(2n+s)$-dimensional Kenmotsu f-manifold $\overline{M}$ such that $\xi_k \in \Gamma(TM_{\theta})$, where $M_{\perp}$ is an anti-invariant submanifold of dimension $d_2 = q$ and M_{θ} is a proper slant submanifold of dimension $d_1 = 2p+s$ of $\overline{M}$. Then we have*

(*i*) *The squared norm of the second fundamental form of M is given by*

$$\|h\|^2 \geq q\cot^2\theta\left(\left\|\nabla^{\theta}\ln f\right\|^2 - s^2\right) \tag{40}$$

where $\nabla^{\theta}\ln f$ is gradient of the function $\ln f$ along M_{θ}.

(*ii*) *The equality holds in (40), if M_{θ} is totally geodesic and $M_{\perp}$ is a totally umbilical submanifold of $\overline{M}$.*

Proof. By virtue of (36), we have

$$\|h\|^2 = \sum_{i,\ j=1}^{m+s-1} g\left(h\left(e_i,\ e_j\right),\ h\left(e_i,\ e_j\right)\right) = \sum_{r=m+s}^{2n+s}\sum_{i,\ j=1}^{m+s-1} g\left(h\left(e_i,\ e_j\right),\ e_r\right)^2.$$

Then in view of established frame above, we derive

$$\|h\|^2 = \sum_{r=m+s}^{2n+s}\sum_{i,\ j=1}^{q} g\left(h\left(e_i,\ e_j\right),\ e_r\right)^2 + 2\sum_{r=m+s}^{2n+s}\sum_{i=1}^{q}\sum_{j=1}^{2p+s} g\left(h\left(e_i,\ e_j^*\right),\ e_r\right)^2 + \sum_{r=m+s}^{2n+s}\sum_{i,\ j=1}^{2p+s} g\left(h\left(e_i^*,\ e_j^*\right),\ e_r\right)^2. \tag{41}$$

Since M is a mixed totally geodesic submanifold hence the second term in the right hand side of (41) vanishes identically. Then we deduce

$$\begin{aligned}
\|h\|^2 &= \sum_{r=m+s}^{m+s+q}\sum_{i,\,j=1}^{q} g\left(h\left(e_i,\,e_j\right),\,e_r\right)^2 + \sum_{r=m+s+q}^{2(m+s-1)-1}\sum_{i,\,j=1}^{q} g\left(h\left(e_i,\,e_j\right),\,e_r\right)^2 \\
&+ \sum_{r=2(m+s-1)}^{2n+s}\sum_{i,\,j=1}^{q} g\left(h\left(e_i,\,e_j\right),\,e_r\right)^2 + \sum_{r=m+s}^{m+s-1+q}\sum_{i,\,j=q+1}^{m+s-1} g\left(h\left(e_i,\,e_j\right),\,e_r\right)^2 \\
&+ \sum_{r=m+s+q}^{2(m+s-1)-1}\sum_{i,\,j=q+1}^{m+s-1} g\left(h\left(e_i,\,e_j\right),\,e_r\right)^2 + \sum_{r=2(m+s-1)}^{2n+s}\sum_{i,\,j=q+1}^{m+s-1} g\left(h\left(e_i,\,e_j\right),\,e_r\right)^2 \qquad (42)\\
&= \sum_{r=1}^{q}\sum_{i,\,j=1}^{q} g\left(h\left(e_i,\,e_j\right),\,\widetilde{e}_r\right)^2 + \sum_{r=q+1}^{q+2p}\sum_{i,\,j=1}^{q} g\left(h\left(e_i,\,e_j\right),\,\widetilde{e}_r\right)^2 \\
&+ \sum_{r=m+s-1}^{2n+2-m}\sum_{i,\,j=1}^{q} g\left(h\left(e_i,\,e_j\right),\,\widetilde{e}_r\right)^2 + \sum_{r=1}^{q}\sum_{i,\,j=q+1}^{m+s+-1} g\left(h\left(e_i,\,e_j\right),\,\widetilde{e}_r\right)^2 \\
&+ \sum_{r=q+1}^{q+2p}\sum_{i,\,j=q+1}^{m+s+-1} g\left(h\left(e_i,\,e_j\right),\,\widetilde{e}_r\right)^2 + \sum_{r=m+s-1}^{2n+2-m}\sum_{i,\,j=q+1}^{m+s+-1} g\left(h\left(e_i,\,e_j\right),\,\widetilde{e}_r\right)^2 .
\end{aligned}$$

Now we calculate only the second term in (42) and we will leave all the positive terms, then we obtain

$$\begin{aligned}
\|h\|^2 &\geq \sum_{r=q+1}^{q+2p}\sum_{i,\,j=1}^{q} g\left(h\left(e_i,\,e_j\right),\,\widetilde{e}_r\right)^2 \\
&= \sum_{r=q+1}^{q+p}\sum_{i,\,j=1}^{q} g\left(h\left(e_i,\,e_j\right),\,\widetilde{e}_r\right)^2 + \sum_{r=q+p+1}^{q+2p}\sum_{i,\,j=1}^{q} g\left(h\left(e_i,\,e_j\right),\,\widetilde{e}_r\right)^2 .
\end{aligned}$$

By using the frame of ND_θ, we derive

$$\|h\|^2 \geq \sum_{r=1}^{p}\sum_{i,\,j=1}^{q} g\left(h\left(e_i,\,e_j\right),\ \csc\theta N e_r^*\right)^2 + \sum_{r=1}^{p}\sum_{i,\,j=1}^{q} g\left(h\left(e_i,\,e_j\right),\ \csc\theta\sec\theta N T e_r^*\right)^2 .$$

From Lemma 5 (ii) and (iii), we conclude that

$$\begin{aligned}
\|h\|^2 &\geq \csc^2\theta\sum_{r=1}^{p}\sum_{i,\,j=1}^{q} (Te_r^*\ln f)^2 g\left(e_i,\,e_j\right)^2 + \cot^2\theta\sum_{r=1}^{p}\sum_{i,\,j=1}^{q} (\overline{\eta}\,(e_r^*) - e_r^*\ln f)^2 g\left(e_i,\,e_j\right)^2 \\
&= q\csc^2\theta\sum_{r=1}^{2p+s} g\left(e_r^*,\ T\nabla^\theta\ln f\right)^2 - q\csc^2\theta\sum_{r=p+1}^{2p} g\left(e_r^*,\ T\nabla^\theta\ln f\right)^2 \\
&\quad -q\csc^2\theta\left(\sum_{l=1}^{s} Te_{2p+l}^*\ln f\right)^2 + q\cot^2\theta\sum_{r=1}^{p}(e_r^*\ln f)^2 ,
\end{aligned}$$

where $\overline{\eta}\,(e_r^*) = \sum_{l=1}^{s}\eta^l\,(e_r^*)$. Since $Te_{2p+l}^* = T\xi_l = 0$ for each $1 \leq l \leq s$, then by using (37), we derive

$$\|h\|^2 \geq q\csc^2\theta\left\|T\nabla^\theta\ln f\right\|^2 - q\csc^2\theta\sum_{r=1}^{2p} g\left(e_{p+r}^*,\ T\nabla^\theta\ln f\right)^2 + q\cot^2\theta\sum_{r=1}^{p}(e_r^*\ln f)^2 .$$

Finally, by taking into account of (34) and Lemma 5 (ii), we arrive at

$$\begin{aligned}\|h\|^2 &\geq q\cot^2\theta\left(\left\|\nabla^\theta \ln f\right\|^2 - s^2\right) - q\csc^2\theta\sum_{r=1}^{p} g\left(\sec\theta T e_r^*,\ T\nabla^\theta \ln f\right)^2 + q\cot^2\theta\sum_{r=1}^{p}(e_r^*\ln f)^2\\ &= q\cot^2\theta\left(\left\|\nabla^\theta \ln f\right\|^2 - s^2\right) - q\cot^2\theta\sum_{r=1}^{p} g\left(e_r^*,\ \nabla^\theta \ln f\right)^2 + q\cot^2\theta\sum_{r=1}^{p}(e_r^*\ln f)^2.\end{aligned}$$

By virtue of definition of gradient, the second term of right hand side in the above equation is negatively equal to the third term and thus (40) holds. If we have equality case in (40), then in view of the mixed geodesic condition, we derive

$$h(D_\theta,\ D^\perp) = 0. \tag{43}$$

Now by using Lemma 4 (i) and (43), it follows that

$$h(D^\perp,\ D^\perp) = 0. \tag{44}$$

Since $M_\perp$ is totally geodesic in M, by virtue of (44), we get $M_\perp$ is totally geodesic in $\overline{M}$. In a similar way, from the leaving fifth and sixth terms in (42), we find

$$h(D^\perp,\ D^\perp) \perp ND^\perp \quad \text{and} \quad h(D^\perp,\ D^\perp) \perp \nu$$

which implies that

$$h(D^\perp,\ D^\perp) \in \varphi D_\theta. \tag{45}$$

Furthermore, taking into account of Lemma 5 (ii) and (43), we deduce that

$$g(h(Z,\ W),\ NX) = (TX\ln f)\, g(Z,\ W).$$

Hence we conclude that M_θ is totally umbilical in $\overline{M}$ by using the fact that $M_\perp$ is totally umbilical in M which complete the proof of the theorem. □

7. Some Applications

In this section we discuss some consequences of our derived results. Theorem 5.1 implies for $s = 0$ and $s = 1$, respectively.

Theorem 7.1. *[28] Let M be a proper pseudo slant submanifold of Kaehler manifold $\overline{M}$. Then M is locally a mixed totally geodesic warped product submanifold of type $M = M_\theta \times M_\perp$ if and only if*

$$A_{JZ}X = 0 \quad \textit{and} \quad A_{NTX}Z = -\cos^2\theta(X\mu)Z \tag{46}$$

for any $Z \in D^\perp$ and $X \in \Gamma(D_\theta)$ for some smooth function on M such that $W(\mu) = 0$, for all vector fields $W \in D^\perp$.

Next result was proved in [24] that

Theorem 7.2. *[24] Let M be a proper pseudo slant submanifold of Kenmotsu manifold $\overline{M}$. Then M is locally a mixed totally geodesic warped product submanifold of type $M = M_\theta \times M_\perp$ if and only if*

$$A_{\varphi Z}X = 0 \quad \textit{and} \quad A_{NTX}Z = -\cos^2\theta\Big(X\mu - \eta(X)\Big)Z, \tag{47}$$

for any $Z \in D^\perp$ and $X \in \Gamma(D_\theta \oplus \xi)$ for some smooth function on M such that $W(\mu) = 0$, for all vector fields $W \in D^\perp$.

On the other hand, Theorem 6.1 implies the following for $s = 0$.

Theorem 7.3. *[28] Let $M = M_\theta \times_f M_\perp$ be m-dimensional mixed totally geodesic warped product pseudo slant submanifold of a $(2n)$-dimensional Kaehler manifold $\overline{M}$, where $M_\perp$ is an anti-invariant submanifold of dimension $d_2 = q$ and M_θ is a proper slant submanifold of dimension $d_1 = 2p$ of $\overline{M}$. Then we have*

(*i*) *The squared norm of the second fundamental form of M is given by*

$$\|h\|^2 \geq q \cot^2 \theta \left\| \nabla^\theta \ln f \right\|^2 \tag{48}$$

where $\nabla^\theta \ln f$ is gradient of the function $\ln f$ along M_θ.

(*ii*) *The equality holds in (40), if M_θ is totally geodesic and $M_\perp$ is a totally umbilical submanifold of $\overline{M}$.*

Similarly, for substitution $s = 1$ in Theorem 6.1, we get the following result which obtained in [24]

Theorem 7.4. *[24] Let $M = M_\theta \times_f M_\perp$ be m-dimensional mixed totally geodesic warped product pseudo slant submanifold of a $(2n + 1)$-dimensional Kenmotsu manifold $\overline{M}$ such that $\xi \in \Gamma(TM_\theta)$, where $M_\perp$ is an anti-invariant submanifold of dimension $d_2 = q$ and M_θ is a proper slant submanifold of dimension $d_1 = 2p + 1$ of $\overline{M}$. Then we have*

(*i*) *The squared norm of the second fundamental form of M is given by*

$$\|h\|^2 \geq q \cot^2 \theta \left(\left\| \nabla^\theta \ln f \right\|^2 - 1 \right), \tag{49}$$

where $\nabla^\theta \ln f$ is gradient of the function $\ln f$ along M_θ.

(*ii*) *The equality holds in (40), if M_θ is totally geodesic and $M_\perp$ is a totally umbilical submanifold of $\overline{M}$.*

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