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Weighted Statistical Approximation Properties of Univariate and Bivariate λ-Kantorovich Operators

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Abstract. In this study, we consider statistical approximation properties of univariate and bivariate λ-Kantorovich operators. We estimate rate of weighted A-statistical convergence and prove a Voronovskaja-type approximation theorem by a family of linear operators using the notion of weighted A-statistical convergence. We give some estimates for differences of λ-Bernstein and λ-Durrmeyer, and λ-Bernstein and λ-Kantorovich operators. We establish a Voronovskaja-type approximation theorem by weighted A-statistical convergence for the bivariate case.

1. Introduction

Statistical convergence was first introduced by Fast [8] and Steinhaus [10]. An extended definition of statistical convergence with the help of nonnegative regular matrix $A = (a_{n,k})$, called A-statistical convergence, was introduced by Kolk in [6]. Weighted statistical convergence was defined and studied by Karakaya et al. in [24] and also modified by Mursaleen et al. in [12]. For further information about statistical convergence we refer to [7, 21, 22].

Weighted mean matrix method is used to present some statistical approximation properties in terms of Korovkin-type statistical approximation theorem. An extended form of A-statistical convergence has been introduced by Mohiuddine et al. [19] and Mohiuddine [20], namely, weighted A-statistical convergence using a non-negative weighted regular matrix. A new characterization in terms of weighted regular matrix has been given and a Korovkin type approximation theorem through statistically weighted A-summable sequences of real or complex numbers has been proved, too.

Approximation theory has become a powerfull tool to obtain prominent results in many fields of mathematics such as differential equations, orthogonal polynomials and computer-aided geometric design. Bernstein used famous polynomials nowadays called Bernstein polynomials, in 1912, to obtain an alternative proof of Weierstrass's fundamental theorem [5]. Approximation properties of Bernstein operators and their applications in Computer Aided Geometric Design and Computer Graphics have been extensively studied in many articles.

A new type λ-Bernstein operators have been introduced by Cai et al. in [15]. Bernstein operators were modified to create known Kantorovich operators in [14]. These kind of operators have been widely studied

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by many researchers. λ-Bernstein operators were also modified to define λ-Kantorovich operators by Acu et al. [1] as

$$K_{n,\lambda}(f;x) = (n+1)\sum_{i=0}^{n} \tilde{b}_{n,i}(\lambda;x)\int_{i/(n+1)}^{(i+1)/(n+1)} f(t)dt \tag{1}$$

with Bézier bases $\tilde{b}_{n,i}(\lambda;x)$ [25]:

$$\begin{aligned}
\tilde{b}_{n,0}(\lambda;x) &= b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x),\\
\tilde{b}_{n,i}(\lambda;x) &= b_{n,i}(x) + \lambda\left(\frac{n-2i+1}{n^2-1} b_{n+1,i}(x) - \frac{n-2i-1}{n^2-1} b_{n+1,i+1}(x)\right),\ \ i = 1,2\ldots,n-1,\\
\tilde{b}_{n,n}(\lambda;x) &= b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x),
\end{aligned} \tag{2}$$

where shape parameters $\lambda \in [-1,1]$ and the Bernstein basis functions $b_{n,i}(x)$ are defined by

$$b_{n,i}(x) = \binom{n}{i} x^i\,(1-x)^{n-i} \qquad (i = 0,\ldots,n).$$

They proved a quantitative Voronovskaja type theorem by means of DitzianTotik modulus of smoothness and a Grüss–Voronovskaja type theorem for λ-Kantorovich operators.

In [16], Cai et al. have introduced Bzier variant of Kantorovich type λ-Bernstein operators. A global approximation theorem in terms of second order modulus of continuity and a direct approximation theorem by means of the DitzianTotik modulus of smoothness were established. BojanicCheng decomposition method were combined with some analysis techniques to derive an asymptotic estimate on the rate of convergence for some absolutely continuous functions.

In [17], Cai et al. have introduced a family of GBS operators of bivariate tensor product of λ–Kantorovich type. They have given an estimate for the rate of convergence of such operators for B-continuous and B-differentiable functions using the mixed modulus of smoothness. They have also established a Voronovskaja type asymptotic formula for the bivariate λ–Bernstein–Kantorovich operators.

Very recently, Srivastava et al. constructed Stancu-type Bernstein operators based on Bézier bases with shape parameter $\lambda \in [-1,1]$ and calculated their moments. They established uniform convergence of the operators and global approximation result by means of Ditzian-Totik modulus of smoothness. they also constructed the bivariate case of Stancu-type λ-Bernstein operators and studied their approximation behaviors [9].

This paper is divided into five main sections. In Section 1, we give a local direct estimate of the rate of convergence with the help of Lipschitz-type function involving two parameters. In Section 2, we give some estimates for differences of λ-Bernstein and λ-Durrmeyer, and λ-Bernstein and λ-Kantorovich operators. In Section 3, we study statistical approximation properties and estimate rate of weighted A-statistical convergence. In Section 4, we prove a Voronovskaja-type approximation theorem by $\mathring{K}_{n,\lambda}(f;x)$ family of linear operators using the notion of weighted A-statistical convergence. In the final section of the paper, we establish a Voronovskaja-type approximation theorem by weighted A-statistical convergence for bivariate case. We compute rate of convergence with the help of Lipschitz-type function and modulus of continuity for bivariate case.

We need the following results throughout the paper.

Lemma 1.1. *[1, Lemma 2.1] We have following equalities for λ-Kantorovich operators:*

$$\begin{aligned}
K_{n,\lambda}(1;x) &= 1;\\
K_{n,\lambda}(t;x) &= x + \frac{1-2x}{2(n+1)} + \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n^2-1}\lambda;\\
K_{n,\lambda}(t^2;x) &= x^2 + \frac{3nx(2-3x)-3x^2+1}{3(n+1)^2} + \frac{x^{n+1}-x+n(x^{n+1}+x-2x^2)}{(n-1)(n+1)^2}2\lambda.
\end{aligned}$$

First we give a local direct estimate of the rate of convergence with the help of Lipschitz-type function involving two parameters for operators (1). We write

$$Lip_M^{(k_1,k_2)}(\eta) := \left\{ f \in C[0,1] : |f(t) - f(x)| \le M \frac{|t-x|^\eta}{(k_1x^2 + k_2x + t)^{\frac{\eta}{2}}};\ x \in (0,1], t \in [0,1] \right\}$$

for $k_1 \ge 0, k_2 > 0$, where $\eta \in (0,1]$ and M is a positive constant (see [13]).

Theorem 1.2. *If $f \in Lip_M^{(k_1,k_2)}(\eta)$, then we have*

$$|K_{n,\lambda}(f;x) - f(x)| \le M \left[\frac{3n+4}{12(k_1x^2 + k_2x)(n+1)^2} + \frac{|\lambda|}{2(k_1x^2 + k_2x)(n^2-1)} \right]^{\frac{\eta}{2}}$$

for all $\lambda \in [-1,1]$, $x \in (0,1]$ and $\eta \in (0,1]$.

Proof. Let $f \in Lip_M^{(k_1,k_2)}(\eta)$ and $\eta \in (0,1]$. First we show the statement is true for $\eta = 1$. We have

$$\begin{aligned} |K_{n,\lambda}(f;x) - f(x)| &\le |K_{n,\lambda}(|f(t) - f(x)|;x)| + f(x)\, |K_{n,\lambda}(1;x) - 1| \\ &\le \sum_{i=0}^{n} \left| f\left(\frac{i}{n}\right) - f(x) \right| \tilde{b}_{n,i}(\lambda;x) \\ &\le M \sum_{i=0}^{n} \frac{|\frac{i}{n} - x|}{(k_1x^2 + k_2x + t)^{\frac{1}{2}}}\ \tilde{b}_{n,i}(\lambda;x) \end{aligned}$$

for $f \in Lip_M^{(k_1,k_2)}(1)$. By $(k_1x^2 + k_2x + t)^{-1/2} \le (k_1x^2 + k_2x)^{-1/2}$ for $k_1 \ge 0, k_2 > 0$ and applying Cauchy-Schwarz inequality, we have

$$\begin{aligned} |K_{n,\lambda}(f;x) - f(x)| &\le M(k_1x^2 + k_2x)^{-1/2} \sum_{i=0}^{n} \left| \frac{i}{n} - x \right| \tilde{b}_{n,i}(\lambda;x) \\ &= M(k_1x^2 + k_2x)^{-1/2} |K_{n,\lambda}(t-x;x)| \\ &\le M|\alpha_n(x)|^{1/2}(k_1x^2 + k_2x)^{-1/2}. \end{aligned}$$

Hence the statement is true for $\eta = 1$. By monotonicity of operators $K_{n,\lambda}(f;x)$ and applying Hölder's inequality two times with $a = 2/\eta$, $b = 2/(2-\eta)$ we show the statement is true for $\eta \in (0,1]$:

$$\begin{aligned} |K_{n,\lambda}(f;x) - f(x)| &\le \sum_{i=0}^{n} \left| f\left(\frac{i}{n}\right) - f(x) \right| \tilde{b}_{n,i}(\lambda;x) \\ &\le \left(\sum_{i=0}^{n} \left| f\left(\frac{i}{n}\right) - f(x) \right|^{\frac{2}{\eta}} \tilde{b}_{n,i}(\lambda;x) \right)^{\frac{\eta}{2}} \left(\sum_{i=0}^{n} \tilde{b}_{n,i}(\lambda;x) \right)^{\frac{2-\eta}{2}} \\ &\le M \left(\sum_{i=0}^{n} \frac{\left(\frac{i}{n} - x\right)^2 \tilde{b}_{n,i}(\lambda;x)}{\frac{i}{n} + k_1x^2 + k_2x} \right)^{\frac{\eta}{2}} \\ &\le M(k_1x^2 + k_2x)^{-\eta/2} \left\{ \sum_{i=0}^{n} \left(\frac{i}{n} - x\right)^2 \tilde{b}_{n,i}\lambda;x) \right\}^{\frac{\eta}{2}} \\ &\le M(k_1x^2 + k_2x + t)^{-\eta/2} K_{n,\lambda}^{\frac{\eta}{2}}((t-x)^2;x) \\ &\le M \left[\frac{3n+4}{12(k_1x^2 + k_2x)(n+1)^2} + \frac{|\lambda|}{2(k_1x^2 + k_2x)(n^2-1)} \right]^{\frac{\eta}{2}}. \end{aligned}$$

This completes the proof. □

2. Estimates for differences between λ-Bernstein type operators

In this part, we shall give some estimates for differences of λ-Bernstein and λ-Durrmeyer, and λ-Bernstein and λ-Kantorovich operators.

There are two approaches to find estimates for differences of positive linear operators and their derivatives. We refer to [2] for details of differences of operators.

Consider λ-Kantorovich operators in (1), λ-Bernstein operators defined in [15] and λ-Durrmeyer operators defined in [3]. We write λ-Bernstein, λ-Kantorovich and λ-Durrmeyer operators as

$$B_{n,\lambda}(f;x)=\sum_{i=0}^{n}\tilde{b}_{n,i}(\lambda;x)B_{i,n}(f);\quad B_{i,n}(f)=f\left(\frac{i}{n}\right);$$

$$K_{n,\lambda}(f;x)=\sum_{i=0}^{n}\tilde{b}_{n,i}(\lambda;x)K_{i,n}(f);\quad K_{i,n}(f)=(n+1)\int_{i/(n+1)}^{(i+1)/(n+1)}f(t)dt;$$

$$D_{n,\lambda}(f;x)=\sum_{i=0}^{n}\tilde{b}_{n,i}(\lambda;x)D_{i,n}(f);\quad D_{i,n}(f)=(n+1)\int_{0}^{1}b_{n,i}(t)f(t)dt.$$

Remark 2.1. *[2] Let $F:C(I)\to\mathbb{R}$ be a positive linear functional such that $F(1)=1$. If we denote $b^F=F(x)$ and*

$$\mu_i^F=\frac{1}{i}F(e_1-b^Fe_0)^i,$$

then we have $\mu_0^F=1$, $\mu_1^F=0$ and $\mu_2^F=\frac{1}{2}[F(e_2)-(b^F)^2]$.

Theorem 2.2. *Let $f,f''\in C[0,1]$. We have the following estimates for difference of λ-Bernstein and λ-Kantorovich operators:*

(i) $\left|B_{n,\lambda}(f;x)-K_{n,\lambda}(f;x)\right|\le\left\|f''\right\|\alpha_n(x,\lambda)+\omega_1\left(f,1/(2n+2)\right);$

(ii) $\left|B_{n,\lambda}(f;x)-K_{n,\lambda}(f;x)\right|\le 3\omega_2\left(f,1/(2\sqrt{6}n+2\sqrt{6})\right)+5\sqrt{6}\omega_1\left(f,1/(2\sqrt{6}n+2\sqrt{6})\right).$

Proof. We have

$$b^{B_{i,n}}=B_{i,n}(e_1)=\frac{i}{n}\text{ and }b^{K_{i,n}}=K_{i,n}(e_1)=\frac{2i+1}{2(n+1)}$$

for λ-Bernstein and λ-Kantorovich operators. We also have

$$\max_{0\le i\le n}\left|b^{B_{i,n}}-b^{K_{i,n}}\right|=\max_{0\le i\le n}\frac{|n-2i|}{2n(n+1)}=\frac{1}{2(n+1)}.$$

Then the following equalities hold:

$$\mu_2^{B_{i,n}}=\frac{1}{2!}B_{i,n}\left(e_1-b^{B_{i,n}}e_0\right)^2=0;$$

$$\mu_2^{K_{i,n}}=\frac{1}{2!}K_i\left(e_1-b^{K_{i,n}}e_0\right)^2=\frac{1}{24(n+1)^2}.$$

Hence we have

$$\alpha_n(x,\lambda)=\sum_{i=0}^{n}\tilde{b}_{n,i}(\lambda,x)\left(\mu_2^{B_{i,n}}+\mu_2^{K_{i,n}}\right)=\frac{1}{24(n+1)^2}\sum_{i=0}^{n}\tilde{b}_{n,i}(\lambda,x)=\frac{1}{24(n+1)^2}.$$

We prove the theorem if we apply [2, Theorem 3, Theorem 5]. □

Theorem 2.3. *Let $f, f'' \in C[0,1]$. We have the following estimates for difference of λ-Bernstein and λ-Durrmeyer operators:*

(i) $\left|B_{n,\lambda}(f;x) - D_{n,\lambda}(f;x)\right| \leq \left\|f''\right\| \beta_n(x,\lambda) + \omega_1\left(f, 1/(n+2)\right);$

(ii) $\left|B_{n,\lambda}(f;x) - D_{n,\lambda}(f;x)\right| \leq 3\omega_2\left(f, \beta_n^{1/2}(x,\lambda)\right) + 5(n+2)\beta_n^{-1/2}(x,\lambda)\omega_1\left(f, \beta_n^{1/2}(x,\lambda)\right).$

Proof. We have $b^{B_{i,n}} = B_{i,n}(e_1) = \frac{i}{n}$ and $b^{D_{i,n}} = D_{i,n}(e_1) = \frac{i+1}{n+2}$ for λ-Bernstein and λ-Kantorovich operators. We also have $\max_i \left|b^{B_{i,n}} - b^{D_{i,n}}\right| = \frac{1}{n+2}$ since $\left|b^{B_{i,n}} - b^{D_{i,n}}\right| = \frac{|n-2i|}{n(n+2)}$.

Then the following equalities hold:

$$\mu_2^{B_{i,n}} = \frac{1}{2!} B_{i,n}\left(e_1 - b^{B_{i,n}} e_0\right)^2 = 0;$$

$$\mu_2^{D_{i,n}} = \frac{1}{2!} D_{i,n}\left(e_1 - b^{D_{i,n}} e_0\right)^2 = \frac{(i+1)(n+1-i)}{2(n+3)(n+2)^2}.$$

Using all these relations we have

$$\begin{aligned}
\beta_n(x,\lambda) &= \sum_{i=0}^{n} \tilde{b}_{n,i}(\lambda,x)\left(\mu_2^{B_{i,n}} + \mu_2^{D_{i,n}}\right) \\
&= \frac{1}{2(n+3)(n+2)^2} \sum_{i=0}^{n} \tilde{b}_{n,i}(\lambda,x)(i+1)(n+1-i) \\
&= \frac{1}{2(n+3)(n+2)^2} \sum_{i=0}^{n} \tilde{b}_{n,i}(\lambda,x)(1-i^2) + \frac{n}{2(n+3)(n+2)^2} \sum_{i=0}^{n} \tilde{b}_{n,i}(\lambda,x)(1+i) \\
&= \frac{n+1}{2(n+3)(n+2)^2} + \frac{n^2}{2(n+3)(n+2)^2}\left[x(1-x) - \frac{x(1-x)}{n}\right. \\
&\quad \left. + \frac{1-4x+4x^2-x^{n+1}-(1-x)^{n+1}}{n(n-1)}\lambda + \frac{1-x^{n+1}-(1-x)^{n+1}}{n^2(n-1)}\lambda\right].
\end{aligned}$$

We prove the theorem if we apply [2, Theorem 3, Theorem 5]. □

3. Statistical approximation properties of univariate λ-Kantorovich operators

This section is devoted to establish statistical approximation properties of univariate λ-Kantorovich operators and estimate the corresponding rate of convergence by weighted A-statistical convergence. First we give the needed notions and notations.

Natural density of K_n is denoted by $\delta(K) = \lim_n \frac{1}{n}|K_n|$ provided that limit exists, where $K_n = \{k \leq n : k \in K\}$, $K \subseteq \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and vertical bars denote cardinality of the enclosed set. A sequence $x = (x_n)$ of numbers is called statistically convergent to a number L, denoted by $st\text{-}\lim_n x = L$, if, for each $\epsilon > 0$,

$$\delta\{n : n \in \mathbb{N} \text{ and } |x_n - L| \geqq \epsilon\} = 0.$$

A-transform of x denoted by $Ax := \{(Ax)_n\}$ is defined as $(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k$ for a given non-negative infinite summability matrix $A = (a_{nk}), n, k \in \mathbb{N}$. It is provided defined series converges for every $n \in \mathbb{N}_0$. If $\lim_n (Ax)_n = L$ whenever $\lim_n x_n = L$, we say that A is a regular method. Then sequence $x = (x_n)$ is said to be A-statistically convergent to L, denoted by $\mathrm{st}_A\text{-}\lim x = L$, provided that for each $\epsilon > 0$,

$$\sum_{k:|x_k - L| \geqq \epsilon} a_{nk} = 0 \qquad (n \to \infty).$$

A-statistical convergence becomes ordinary statistical convergence which was introduced in [4] if we take $A = (C_1)$, the Cesaro matrix of order one, and it becomes classical convergence if we take $A = I$, the identity matrix. We know that every convergent sequence is statistically convergent to the same limit but not conversely.

Assume that $q = (q_n)$ is a sequence of non-negative numbers so that $q_0 > 0$ and $Q_n = \sum_{k=0}^{n} q_k \to \infty$ as $n \to \infty$. Then $x = (x_n)$ is called weighted A-statistically convergent to L, if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{Q_n} \sum_{k=0}^{n} q_k \sum_{l:|x-L|\geqq\epsilon} a_{kl} = 0.$$

This relation is denoted by $S_A^{\widetilde{N}} - \lim x = L$ in this case. It is clear that weighted A-statistical convergence generalizes A-statistical convergence, which we recover by putting $q_n = 1$ for all $n \in \mathbb{N}$.

We now give main results related to statistical approximation of operators in (1).

Theorem 3.1. *Let $A = (a_{nk})$ be a weighted non-negative regular summability matrix for $n, k \in \mathbb{N}$ and $q = (q_n)$ be a sequence of non-negative numbers such that $q_0 > 0$ and $Q_n = \sum_{k=0}^{n} q_k \to \infty$ as $n \to \infty$. For any $f \in C[0,1]$, we have*

$$S_A^{\widetilde{N}} - \lim_{n\to\infty} \|K_{n,\lambda}(f;x) - f(x)\|_{C[0,1]} = 0.$$

Proof. Consider sequence of functions $e_j(x) = x^j$, where $j \in \{0,1,2\}$ and $x \in [0,1]$. It is sufficient to satisfy

$$S_A^{\widetilde{N}} - \lim_{n\to\infty} \|K_{n,\lambda}(e_j;x) - e_j\|_{C[0,1]} = 0, \qquad j = 0,1,2.$$

From Lemma 1.1, it is clear that

$$S_A^{\widetilde{N}} - \lim_{n\to\infty} \|K_{n,\lambda}(e_0;x) - e_0\|_{C[0,1]} = 0. \tag{3}$$

We also have

$$\begin{aligned} \|K_{n,\lambda}(e_1;x) - e_1\|_{C[0,1]} &= \sup_{x\in[0,1]} \left| \frac{1-2x}{2(n+1)} + \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n^2-1}\lambda \right| \\ &\leq \frac{1}{2(n+1)} + \frac{|\lambda|}{2(n^2-1)} := A(n,\lambda). \end{aligned}$$

We define following sets

$$\begin{aligned} &\Omega := \{n \in \mathbb{N} : \|K_{n,\lambda}(e_1;x) - e_1\|_{C[0,1]} \geqq \bar{\epsilon}\}, \\ &\Omega_1 := \{n \in \mathbb{N} : A(n,\lambda) \geqq \epsilon - \bar{\epsilon}\} \end{aligned}$$

choosing a number $\epsilon > 0$ for a given $\bar{\epsilon} > 0$ such that $\epsilon < \bar{\epsilon}$. Then we see the inclusion $\Omega \subset \Omega_1$ is satisfied and

$$\frac{1}{Q_n} \sum_{k=0}^{n} q_k \sum_{l\in\Omega} a_{kl} \leq \frac{1}{Q_n} \sum_{k=0}^{n} q_k \sum_{l\in\Omega_1} a_{kl} \qquad . \tag{4}$$

for all $n \in \mathbb{N}$. Passing limit as $n \to \infty$ in (4) we have

$$S_A^{\widetilde{N}} - \lim_{n\to\infty} \|K_{n,\lambda}(e_1;x) - e_1\|_{C[0,1]} = 0. \tag{5}$$

We also have

$$\begin{aligned} \|K_{n,\lambda}(e_2;x) - e_2\|_{C[0,1]} &= \sup_{x\in[0,1]} \left| \frac{3nx(2-3x) - 3x^2 + 1}{3(n+1)^2} + \frac{x^{n+1} - x + n(x^{n+1} + x - 2x^2)}{(n-1)(n+1)^2} 2\lambda \right| \\ &\leq \frac{15n+4}{(n+1)^2} + \frac{8|\lambda|}{n^2-1} := B(n,\lambda). \end{aligned}$$

Since $S_A^{\widetilde{N}} - \lim_n B(n,\lambda) = 0$, we get

$$S_A^{\widetilde{N}} - \lim_{n\to\infty} \|K_{n,\lambda}(e_2;x) - e_2\|_{C[0,1]} = 0. \tag{6}$$

We get desired result combining (3), (5) and (6). □

We now estimate rate of weighted A-statistical convergence of operators $K_{n,\lambda}(f;x)$.

Let $A = (a_{nk})$ be a weighted non-negative regular summability matrix and let $q = (q_n)$ be a sequence of non-negative numbers such that $q_0 > 0$ and $Q_n = \sum_{k=0}^{n} q_k \to \infty$ as $n \to \infty$. Also let (u_n) be a positive non-decreasing sequence. We say that a sequence $x = (x_n)$ is weighted A-statistically convergent to L with the rate $o(u_n)$ if

$$\lim_{n\to\infty} \frac{1}{u_n Q_n} \sum_{k=0}^{n} q_k \sum_{l:|x_l - L|\geqq\epsilon} a_{kl} = 0.$$

In this case, we write

$$[stat_A, q_n] - o\,(u_n) = x_n - L.$$

Theorem 3.2. *Let $A = (a_{nk})$ be a weighted non-negative regular summability matrix. Assume that following condition yields:*

$$w(f,\varphi_n) = [stat_A, q_n] - o\,(u_n) \text{ on } [0,1], \text{ where } \varphi_n = \sqrt{\|K_{n,\lambda}((s-x)^2;x)\|_{C[0,1]}}.$$

Then for every bounded $f \in C[0,1]$ we have

$$\|K_{n,\lambda}(f;x) - f(x)\|_{C[0,1]} = [stat_A, q_n] - o\,(u_n).$$

Proof. Let $f(x,y) \in C[0,1]$, then we have

$$\begin{aligned}|K_{n,\lambda}(f;x) - f(x)| &\le |K_{n,\lambda}\Big(|f(t) - f(x)|;x\Big) + \psi\,|K_{n,\lambda}(1;x) - 1| \\ &\le \omega(f,\delta)K_{n,\lambda}\Big(\frac{|t-x|}{\delta} + 1;x\Big) \\ &= \omega(f,\delta)K_{n,\lambda}(1;x) + \omega(f,\delta)\frac{1}{\delta^2}K_{n,\lambda}\left((t-x)^2;x\right)\end{aligned}$$

for any $x, s \in [0,1]$, where $\psi = \sup_{x\in[0,1]} |f(x)|$. Let $\delta := \varphi_n$ for all $n \in \mathbb{N}$. Taking supremum over $x \in [0,\infty)$ on both sides, we obtain

$$\|K_{n,\lambda}(f;x) - f(x)\|_{C[0,1]} \le \omega(f,\varphi_n) + \omega(f,\varphi_n)\frac{1}{\varphi_n^2}\|K_{n,\lambda}((t-x)^2;x)\|_{C[0,1]} = 2\omega(f,\varphi_n).$$

Consider following sets for a given $\epsilon > 0$:

$$\begin{aligned}\mathcal{B} &= \{n : \|K_{n,\lambda}(f;x) - f(x)\|_{C[0,1]} \ge \epsilon\}, \\ \mathcal{W} &= \left\{n : \omega(f,\varphi_n) \ge \frac{\epsilon}{2}\right\}.\end{aligned}$$

The following inequality is clear satisfied:

$$\frac{1}{u_n Q_n} \sum_{k=0}^{n} \sum_{l\in\mathcal{B}} q_k a_{kl} \le \frac{1}{u_n Q_n} \sum_{k=0}^{n} \sum_{l\in\mathcal{W}} q_k a_{kl}.$$

We are led to the following fact by the hypothesis that

$$\|K_{n,\lambda}(f;x) - f(x)\|_{C[0,1]} = [stat_A, q_n] - o\,(u_n)$$

as asserted by Theorem 3.2. □

4. A Voronovskaja-type approximation theorem by weighted A-statistical convergence

We shall prove a Voronovskaja-type approximation theorem by $\mathring{K}_{n,\lambda}(f;x)$ family of linear operators using the notion of weighted A-statistical convergence.

Theorem 4.1. *Let $A = (a_{nk})$ be a weighted non-negative regular summability matrix and let (x_n) be a sequence of real numbers such that $S_A^{\widetilde{N}} - \lim x_n = 0$. Also let $\mathring{K}_{n,\lambda}(f;x)$ be a sequence of positive linear operators acting from $C_B[0,1]$ into $C[0,1]$ defined by*

$$\mathring{K}_{n,\lambda}(f;x) = (1+x_n)K_{n,\lambda}(f;x).$$

Then for every $f(x,y) \in C_B[0,1]$, and $f', f'' \in C_B[0,1]$ we have

$$S_A^{\widetilde{N}} - \lim_{n\to\infty} n\Big\{\mathring{K}_{n,\lambda}(f;x) - f(x)\Big\} = \frac{f'(x)}{2}(1-2x) + \frac{f''(x)}{2}x(1-x).$$

Proof. Let $x \in [0,1]$ and $f', f'' \in C_B[0,1]$. Applying $\mathring{K}_{n,\lambda}(f;x)$ to both sides of Taylor's expansion theorem, we have

$$\mathring{K}_{n,\lambda}(f;x) - f(x) = f'(x)\mathring{K}_{n,\lambda}(t-x;x) + \frac{f''(x)}{2}\mathring{K}_{n,\lambda}((t-x)^2;x) + \mathring{K}_{n,\lambda}((t-x)^2\epsilon(x,t);x),$$

which yields to

$$\begin{aligned} n\Big\{\mathring{K}_{n,\lambda}(f;x) - f(x)\Big\} &= nf'(x)(1+x_n)K_{n,\lambda}(t-x;x) + \frac{n}{2}f''(x)(1+x_n)K_{n,\lambda}((t-x)^2;x) \\ &\quad + n(1+x_n)K_{n,\lambda}((t-x)^2\epsilon(x,t);x). \end{aligned}$$

We also have the following relations

$$\begin{aligned} &\Bigg| n\Big\{\mathring{K}_{n,\lambda}(f;x) - f(x)\Big\} - \Bigg\{ f'(x)\left(\frac{n(1-2x)}{2(n+1)} + \frac{n(1-2x+x^{n+1}-(1-x)^{n+1})}{n^2-1}\right) \\ &\quad + \frac{f''(x)}{2}\left(\frac{nx(1-x)}{n+1} + \frac{n(1-6x+6x^2)}{3(n+1)^2} + \frac{2n[x^{n+1}(1x) + x(1x)^{n+1}]}{n^2 1}\right)\Bigg\}\Bigg| \\ &= nf'(x)x_nK_{n,\lambda}(t-x;x) + \frac{n}{2}f''(x)x_nK_{n,\lambda}((t-x)^2;x) + n(1+x_n)K_{n,\lambda}((t-x)^2\epsilon(x,t);x) \\ &\le x_n\left\{nf'(x)A(n,\lambda) + n\frac{f''(x)}{2}C(n,\lambda)\right\} + n(1+x_n)K_{n,\lambda}((t-x)^2\epsilon(x,t);x) \\ &\le x_n\left\{n\Delta_1 A(n,\lambda) + n\Delta_2 C(n,\lambda)\right\} + n(1+x_n)K_{n,\lambda}((t-x)^2\epsilon(x,t);x), \end{aligned}$$

where

$$\begin{aligned} K_{n,\lambda}(t-x;x) &= \frac{1-2x}{2(n+1)} + \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n^2-1} \le \frac{1}{2(n+1)} + \frac{|\lambda|}{n^2-1}; \\ K_{n,\lambda}((t-x)^2;x) &= \frac{nx(1-x)}{(n+1)^2} + \frac{1-3x+3x^2}{3(n+1)^2} + \frac{2\lambda[x^{n+1}(1x) + x(1x)^{n+1}]}{n^2 1} + \frac{4\lambda x(1-x)}{(n+1)^2(n1)} \\ &\le \frac{3n+4}{12(n+1)^2} + \frac{|\lambda|}{n^2 1} := C(n,\lambda) \end{aligned}$$

by [1, Lemma 2.3] and

$$\Delta_1 = \sup_{x\in[0,1]} |f'(x)| \quad \text{and} \quad \Delta_2 = \sup_{x\in[0,1]} |f''(x)|.$$

Moreover, by Theorem 3.1 we obtain

$$S_A^{\widetilde{N}} - \lim_{n\to\infty} n\ \left(K_{n,\lambda}((t-x)^2\epsilon(x,t);x)\right) = 0.$$

Since $S_A^{\widetilde{N}} - \lim x_n = 0$, we get desired result. □

5. Approximation properties of bivariate λ-Kantorovich operators

In this part, we establish a Voronovskaja-type approximation theorem by weighted A-statistical convergence for bivariate case. We compute rate of convergence with the help of Lipschitz-type function and modulus of continuity for bivariate case.

Let $\mathcal{I} = [0,1]\times[0,1]$ and $(x,y)\in\mathcal{I}$, then we consider bivariate λ-Kantorovich operators

$$\bar{K}_{n,m}^{\lambda_1,\lambda_2}(f;x,y) = (n+1)(m+1)\sum_{k_1=0}^{n}\sum_{k_2=0}^{m}\tilde{b}_{n,k_1}(\lambda_1;x)\tilde{b}_{m,k_2}(\lambda_2;y)\int_{\frac{k_1}{n+1}}^{\frac{k_1+1}{n+1}}\int_{\frac{k_2}{m+1}}^{\frac{k_2+1}{m+1}} f(u,v)dudv$$

for $f(x,y)\in C(\mathcal{I})$, where Bézier bases $\tilde{b}_{n,k_1}(\lambda_1;x)$ and $\tilde{b}_{m,k_2}(\lambda_2;x)$ ($\lambda_1,\lambda_2\in[-1,-1]$) are defined in (2).

As an immediate consequence of Lemma 1.1 we have the following lemma:

Lemma 5.1. *The following equalities hold:*

$$\begin{aligned}
&\bar{K}_{n,m}^{\lambda_1,\lambda_2}(1;x,y) = 1;\\
&\bar{K}_{n,m}^{\lambda_1,\lambda_2}(s;x,y) = x + \frac{1-2x}{2(n+1)} + \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n^2-1}\lambda_1;\\
&\bar{K}_{n,m}^{\lambda_1,\lambda_2}(t;x,y) = y + \frac{1-2y}{2(m+1)} + \frac{1-2y+y^{m+1}-(1-y)^{m+1}}{m^2-1}\lambda_2;\\
&\bar{K}_{n,m}^{\lambda_1,\lambda_2}(s^2;x,y) = x^2 + \frac{3nx(2-3x)-3x^2+1}{3(n+1)^2} + \frac{x^{n+1}-x+n(x^{n+1}+x-2x^2)}{(n-1)(n+1)^2}2\lambda_1;\\
&\bar{K}_{n,m}^{\lambda_1,\lambda_2}(t^2;x,y) = y^2 + \frac{3my(2-3y)-3y^2+1}{3(m+1)^2} + \frac{y^{m+1}-y+m(y^{m+1}+y-2y^2)}{(m-1)(m+1)^2}2\lambda_2.
\end{aligned}$$

Theorem 5.2. *The sequence $\bar{K}_{n,m}^{\lambda_1,\lambda_2}(f;x,y)$ of operators convergences uniformly to $f(x,y)$ by weighted A-statistical convergence on $\mathcal{I}$ for each $f(x,y)\in C(\mathcal{I})$, where $C(\mathcal{I})$ is the set of all real valued continuous functions on $\mathcal{I}$ with the norm*

$$\| f \|_{C(\mathcal{I})} = \sup_{(x,y)\in\mathcal{I}} \left| f(x,y)\right|.$$

Proof. It is enough to prove the following condition

$$S_A^{\widetilde{N}} - \lim_{n,m\to\infty} \bar{K}_{n,m}^{\lambda_1,\lambda_2}\left(e_{ij}(x,y);x,y\right) = x^i y^j,\quad (i,j)\in\{(0,0),(1,0),(0,1)\}$$

converges uniformly on $\mathcal{I}$. We clearly have

$$S_A^{\widetilde{N}} - \lim_{m,n\to\infty} \bar{K}_{n,m}^{\lambda_1,\lambda_2}(e_{00}(x,y);x,y) = e_{00}.$$

We have

$$\begin{aligned}
S_A^{\widetilde{N}} - \lim_{n,m\to\infty} \bar{K}_{n,m}^{\lambda_1,\lambda_2}(e_{10}(x,y);x,y) &= S_A^{\widetilde{N}} - \lim_{n\to\infty}\left(x + \frac{(1-2x)(n+1)+2x^{n+1}-2(1-x)^{n+1}}{2(n^2-1)}\lambda_1\right)\\
&= e_{10}(x,y);\\
S_A^{\widetilde{N}} - \lim_{n,m\to\infty} \bar{K}_{n,m}^{\lambda_1,\lambda_2}(e_{01}(x,y);x,y) &= S_A^{\widetilde{N}} - \lim_{m\to\infty}\left(y + \frac{(1-2y)(m+1)+2y^{m+1}-2(1-y)^{m+1}}{2(m^2-1)}\lambda_2\right)\\
&= e_{01}(x,y)
\end{aligned}$$

by Lemma 5.1, and

$$\begin{aligned}
&S_A^{\widetilde{N}} - \lim_{n,m\to\infty} \bar{K}_{n,m}^{\lambda_1,\lambda_2}\left(e_{02}(x,y)+e_{20}(x,y);x,y\right)\\
&= S_A^{\widetilde{N}} - \lim_{n,m\to\infty}\left\{x^2+\frac{3nx(2-3x)-3x^2+1}{3(n+1)^2}+\frac{x^{n+1}-x+n(x^{n+1}+x-2x^2)}{(n-1)(n+1)^2}2\lambda_1\right.\\
&\qquad\left.+\,y^2+\frac{3my(2-3y)-3y^2+1}{3(m+1)^2}+\frac{y^{m+1}-y+m(y^{m+1}+y-2y^2)}{(m-1)(m+1)^2}2\lambda_2\right\}\\
&= e_{02}(x,y)+e_{20}(x,y).
\end{aligned}$$

Bearing in mind the above conditions and Korovkin type theorem established by Volkov [23]

$$S_A^{\widetilde{N}} - \lim_{m,n\to\infty} \bar{K}_{n,m}^{\lambda_1,\lambda_2}\left(e_{ij}(x,y);x,y\right) = x^i y^j$$

converges uniformly. □

5.1. A weighted A-statistical Voronovskaja-type theorem for bivariate case

Lemma 5.3. *Let $A=(a_{nk})$ be a weighted non-negative regular summability matrix and let (x_n) be a sequence of real numbers such that $S_A^{\widetilde{N}} - \lim x_n = 0$. Also let $\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}(f;x,y)$ be a sequence of positive linear operators acting from $C_B(\mathcal{I})$ into $C(\mathcal{I})$ defined by*

$$\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}(f;x,y) = (1+x_n)\bar{K}_{n,n}^{\lambda_1,\lambda_2}(f;x,y).$$

Then, we have

$$\begin{aligned}
&S_A^{\widetilde{N}} - \lim_{n\to\infty} n\,\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}(s-x;x,y) = \frac{1-2x}{2};\\
&S_A^{\widetilde{N}} - \lim_{n\to\infty} n\,\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}(t-y;x,y) = \frac{1-2y}{2};\\
&S_A^{\widetilde{N}} - \lim_{n\to\infty} n\,\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}((s-x)^2;x,y) = x(1-x);\\
&S_A^{\widetilde{N}} - \lim_{n\to\infty} n\,\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}((t-y)^2;x,y) = y(1-y).
\end{aligned}$$

Proof. Since $S_A^{\widetilde{N}} - \lim x_n = 0$ holds, the following relation

$$\begin{aligned}
n\ \mathbb{K}_{n,n}^{\lambda_1,\lambda_2}(s-x;x,y) &= n\ (1+x_n)\bar{K}_{n,m}^{\lambda_1,\lambda_2}(s-x;x,y)\\
&= n\ (1+x_n)\left[\bar{K}_{n,m}^{\lambda_1,\lambda_2}(s;x,y)-x\bar{K}_{n,m}^{\lambda_1,\lambda_2}(1;x,y)\right]\\
&= (1+x_n)\left[\frac{n(1-2x)}{2(n+1)}+\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n^2-1}n\lambda_1\right]
\end{aligned}$$

implies $S_A^{\widetilde{N}} - \lim_{n\to\infty} n\,\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}(s-x;x,y) = (1-2x)/2$. Also the following relation

$$\begin{aligned}
n\ \mathbb{K}_{n,n}^{\lambda_1,\lambda_2}((s-x)^2;x,y) &= n\ (1+x_n)\bar{K}_{n,n}^{\lambda_1,\lambda_2}((s-x)^2;x,y)\\
&= n\ (1+x_n)\left[\bar{K}_{n,n}^{\lambda_1,\lambda_2}(s^2;x,y)-2x\bar{K}_{n,m}^{\lambda_1,\lambda_2}(sx,y)+x^2\bar{K}_{n,m}^{\lambda_1,\lambda_2}(1;x,y)\right]\\
&= (1+x_n)\left[\frac{n^2x(1-x)}{(n+1)^2}+\frac{n(1-3x(1-x))}{3(n+1)^2}\right.\\
&\qquad\left.+\frac{2n\lambda_1(x^{n+1}(1-x)+x(1-x)^{n+1})}{n^2-1}+\frac{4n\lambda_1x(1-x)}{(n+1)^2(n1)}\right]
\end{aligned}$$

implies $S_A^{\widetilde{N}} - \lim_{n\to\infty} n\,\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}((s-x)^2;x,y) = x(1-x)$. □

Theorem 5.4. *Let $f(x,y) \in C^2(\mathcal{I})$, then, we have*

$$S_A^{\widetilde{N}} - \lim_{n\to\infty} n\left[\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}(f;x,y) - f(x,y)\right] = \frac{1-2x}{2}f_x + \frac{1-2y}{2}f_y + \frac{x(1-x)}{2}f_{xx} + \frac{y(1-y)}{2}f_{yy}.$$

Proof. First we write the Taylor's formula of $f(s,t)$

$$\begin{aligned} f(s,t) &= f(x,y) + f_x(s-x) + f_y(t-y) + \frac{1}{2}\left\{f_{xx}(s-x)^2 + 2f_{xy}(s-x)(t-y) + f_{yy}(t-y)^2\right\} \\ &\quad + \varepsilon(s,t)\left((s-x)^2 + (t-y)^2\right) \end{aligned} \tag{7}$$

for $(x,y) \in \mathcal{I}$, where $(s,t) \in \mathcal{I}$ and $\varepsilon(s,t) \longrightarrow 0$ as $(s,t) \longrightarrow (x,y)$. If we apply sequence of operators $\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}(f;x,y)$ on (7) bearing in mind linearity of operator, we have

$$\begin{aligned} \mathbb{K}_{n,n}^{\lambda_1,\lambda_2}(f;s,t) - f(x,y) &= f_x(x,y)\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}((s-x);x,y) + f_y(x,y)\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}((t-y);x,y) \\ &\quad + \frac{1}{2}\Big\{f_{xx}\bar{K}_{n,m}^{\lambda_1,\lambda_2}((s-x)^2;x,y) + 2f_{xy}\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}((s-x)(t-y);x,y) \\ &\quad + f_{yy}\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}((t-y)^2;x,y)\Big\} + \mathbb{K}_{n,n}^{\lambda_1,\lambda_2}\left(\varepsilon(s,t)\left((s-x)^2 + (t-y)^2\right);x,y\right). \end{aligned}$$

Applying weighted A statistical limit to both sides of the last equality as $n \longrightarrow \infty$, we have

$$\begin{aligned} S_A^{\widetilde{N}} - \lim_{n\to\infty} n\left[\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}(f;s,t) - f(x,y)\right] &= S_A^{\widetilde{N}} - \lim_{n\to\infty} n\left\{f_x(x,y)\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}((s-x);x,y) + f_y(x,y)\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}((t-y);x,y)\right\} \\ &\quad + S_A^{\widetilde{N}} - \lim_{n\to\infty} \frac{n}{2}\Big\{f_{xx}\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}((s-x)^2;x,y) \\ &\quad + 2f_{xy}\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}((s-x)(t-y);x,y) + f_{yy}\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}((t-y)^2;x,y)\Big\} \\ &\quad + S_A^{\widetilde{N}} - \lim_{n\to\infty} n\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}\left(\varepsilon(s,t)\left((s-x)^2 + (t-y)^2\right);x,y\right). \end{aligned}$$

If we apply Hölder inequality to the last term of previous equality, we have

$$\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}\left(\varepsilon(s,t)\left((s-x)^2 + (t-y)^2\right);x,y\right) \leq \sqrt{2}\sqrt{\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}\left(\varepsilon^2(s,t);x,y\right)}\sqrt{\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}\left(\varepsilon(s,t)\left((s-x)^4 + (t-y)^4\right);x,y\right)}.$$

Since $S_A^{\widetilde{N}} - \lim_n n^2\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}((s-x)^4;x,y)$ and $S_A^{\widetilde{N}} - \lim_n n^2\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}((t-y)^4;x,y)$ are finite and $\lim_n \mathbb{K}_{n,n}^{\lambda_1,\lambda_2}(\varepsilon^2(s,t);x,y) = \varepsilon^2(x,y) = 0$, we have

$$\lim_{n\to\infty} n^2\, \mathbb{K}_{n,n}^{\lambda_1,\lambda_2}\left(\varepsilon(s,t)\left((s-x)^4 + (t-y)^4\right);x,y\right) = 0.$$

By Lemma 5.3 and $S_A^{\widetilde{N}} - \lim_{n\to\infty} n\, \mathbb{K}_{n,n}^{\lambda_1,\lambda_2}((s-x)(t-y);x,y) = 0$ we obtain the desired result. □

5.2. Rates of convergence of bivariate operators

Now we compute the rate of convergence of operators $\bar{K}_{n,m}^{\lambda_1,\lambda_2}(f;x,y)$ to $f(x,y)$ by means of the modulus of continuity. We first give the needed definitions.

Complete modulus of continuity for a bivariate case is defined as follows:

$$\omega(f,\delta) = \sup\left\{|f(s,t) - f(x,y)| : \sqrt{(s-x)^2 + (t-y)^2} \leq \delta\right\}$$

for $f \in C(\mathcal{I}_{ab})$ and for every $(s,t),(x,y) \in \mathcal{I}_{ab} = [0,a]\times[0,b]$. Partial moduli of continuity with respect to x and y are defined as

$$\begin{aligned} \omega_1(f,\delta) &= \sup\left\{|f(x_1,y) - f(x_2,y)| : y \in [0,b] \text{ and } |x_1 - x_2| \leq \delta\right\}, \\ \omega_2(f,\delta) &= \sup\left\{|f(x,y_1) - f(x,y_2)| : x \in [0,a] \text{ and } |y_1 - y_2| \leq \delta\right\}. \end{aligned}$$

Peetre's K-functional is given by

$$K(f,\delta)=\inf_{g\in C^{2}(\mathcal{I}_{ab})}\left\{\|f-g\|_{C(\mathcal{I}_{ab})}+\delta\|g\|_{C^{2}(\mathcal{I}_{ab})}\right\}$$

for $\delta>0$, where $C^2(\mathcal{I}_{ab})$ is the space of functions of f such that f, $\frac{\partial^j f}{\partial x^j}$ and $\frac{\partial^j f}{\partial y^j}$ $(j=1,2)$ in $C(\mathcal{I}_{ab})$ [11]. We now give an estimate of the rates of convergence of operators $\bar{K}^{\lambda_1,\lambda_2}_{n,m}(f;x,y)$.

Theorem 5.5. *Let $f(x,y)\in C(\mathcal{I})$, then we have*

$$\left|\bar{K}^{\lambda_1,\lambda_2}_{n,m}(f;x,y)-f(x,y)\right|\leq 4\omega\left(f;\sqrt{C(n,\lambda_1)},\sqrt{C(m,\lambda_2)}\right)$$

for all $x\in\mathcal{I}$, where $C(n,\lambda_1)$ and $C(m,\lambda_2)$ are defined in Theorem 4.1.

Proof. The following inequalities are satisfied

$$\begin{aligned}&|\bar{K}^{\lambda_1,\lambda_2}_{n,m}(f;x,y)-f(x,y)|\\&\leq \bar{K}^{\lambda_1,\lambda_2}_{n,m}(|f(s,t)-f(x,y)|;x,y)\\&\leq \bar{K}^{\lambda_1,\lambda_2}_{n,m}\left(\omega\left(f;\sqrt{(s-x)^2+(t-y)^2}\right);x,y\right)\\&\leq \omega\left(f;\sqrt{C(n,\lambda_1)},\sqrt{C(m,\lambda_2)}\right)\left[\frac{1}{\sqrt{C(n,\lambda_1)C(m,\lambda_2)}}\bar{K}^{\lambda_1,\lambda_2}_{n,m}\left(\sqrt{(s-x)^2+(t-y)^2};x,y\right)\right]\end{aligned}$$

because defined bivariate λ-Kantorovich operators are linear and positive by definition of operators and complete modulus of continuity of $f(x,y)$. We also have

$$\begin{aligned}|\bar{K}^{\lambda_1,\lambda_2}_{n,m}(f;x,y)-f(x,y)| \leq\ & \omega\left(f;\sqrt{C(n,\lambda_1)},\sqrt{C(m,\lambda_2)}\right)\\&\times\left[1+\frac{1}{\sqrt{C(n,\lambda_1)C(m,\lambda_2)}}\left\{\bar{K}^{\lambda_1,\lambda_2}_{n,m}\left((s-x)^2;x,y\right)\bar{K}^{\lambda_1,\lambda_2}_{n,m}\left((t-y)^2;x,y\right)\right\}^{1/2}\right.\\&\left.+\frac{\sqrt{\bar{K}^{\lambda_1,\lambda_2}_{n,m}((s-x)^2;x,y)}}{\sqrt{C(n,\lambda_1)}}+\frac{\sqrt{\bar{K}^{\lambda_1,\lambda_2}_{n,m}((t-y)^2;x,y)}}{\sqrt{C(m,\lambda_2)}}\right]\end{aligned}$$

by Cauchy-Schwartz inequality. Choosing $C(n,\lambda_1)=\bar{K}^{\lambda_1,\lambda_2}_{n,m}((s-x)^2;x,y)$ and $C(m,\lambda_2)=\bar{K}^{\lambda_1,\lambda_2}_{n,m}((t-y)^2;x,y)$ for all $(x,y)\in\mathcal{I}$ we complete the proof. □

Theorem 5.6. *Let $f(x,y)\in C(I)$, then the following inequality holds*

$$\left|\bar{K}^{\lambda_1,\lambda_2}_{n,m}(f;x,y)-f(x,y)\right|\leq 2\left[\omega_1\left(f;C^{1/2}(n,\lambda_1)\right)+\omega_2\left(f;C^{1/2}(m,\lambda_2)\right)\right],$$

where $C(n,\lambda_1)$ and $C(m,\lambda_2)$ are defined in Theorem 4.1.

Proof. By definition of partial modulus of continuity of $f(x,y)$ and Cauchy-Schwartz inequality, we have

$$\begin{aligned}|\bar{K}^{\lambda_1,\lambda_2}_{n,m}(f;x,y)-f(x,y)| &\leq \bar{K}^{\lambda_1,\lambda_2}_{n,m}(|f(s,t)-f(x,y)|;x,y)\\&\leq \bar{K}^{\lambda_1,\lambda_2}_{n,m}(|f(s,t)-f(x,t)|;x,y)+\bar{K}^{\lambda_1,\lambda_2}_{n,m}(|f(x,t)-f(x,y)|;x,y)\\&\leq \bar{K}^{\lambda_1,\lambda_2}_{n,m}(|\omega_1(f;|s-x|)|;x,y)+\bar{K}^{\lambda_1,\lambda_2}_{n,m}(|\omega_2(f;|t-y|)|;x,y)\end{aligned}$$

$$\leq \omega_1(f, C(n,\lambda_1))\left[1+\frac{1}{C(n,\lambda_1)}\bar{K}_{n,m}^{\lambda_1,\lambda_2}(|s-x|;x,y)\right]$$
$$+\omega_2(f, C(m,\lambda_2))\left[1+\frac{1}{C(m,\lambda_2)}\bar{K}_{n,m}^{\lambda_1,\lambda_2}(|t-y|;x,y)\right]$$
$$\leq \omega_1(f, C^{1/2}(n,\lambda_1))\left[1+\frac{1}{C^{1/2}(n,\lambda_1)}\left(\bar{K}_{n,m}^{\lambda_1,\lambda_2}\left((s-x)^2;x,y\right)\right)^{1/2}\right]$$
$$+\omega_2(f, C^{1/2}(m,\lambda_2))\left[1+\frac{1}{C^{1/2}(m,\lambda_2)}\left(\bar{K}_{n,m}^{\lambda_1,\lambda_2}\left((t-y)^2;x,y\right)\right)^{1/2}\right].$$

Choosing $C(n,\lambda_1)$ and $C(m,\lambda_2)$ as defined in Theorem 4.1, we complete the proof. □

We define the Lipschitz class $LipM(\widehat{\beta_1},\widehat{\beta_2})$ for the bivariate case as follows:

$$\left|f(s,t)-f(x,y)\right| \leq M\left|s-x\right|^{\widehat{\beta_1}}\left|t-y\right|^{\widehat{\beta_2}}$$

for $\widehat{\beta_1},\widehat{\beta_2}\in(0,1]$ and $(s,t),(x,y)\in\mathcal{I}_{ab}$.

Theorem 5.7. *Let $f\in Lip_M(\widehat{\beta_1},\widehat{\beta_2})$. Then, for all $(x,y)\in\mathcal{I}_{ab}$, we have*

$$|\bar{K}_{n,m}^{\lambda_1,\lambda_2}(f;x,y)-f(x,y)| \leq MC^{\widehat{\beta_1}/2}(n,\lambda_1)C^{\widehat{\beta_2}/2}(m,\lambda_2),$$

where $C(n,\lambda_1)$ and $C(m,\lambda_2)$ are defined in Theorem 4.1.

Proof. We have

$$\begin{aligned}|\bar{K}_{n,m}^{\lambda_1,\lambda_2}(f;x,y)-f(x,y)| &\leq \bar{K}_{n,m}^{\lambda_1,\lambda_2}(|f(s,t)-f(x,y)|;x,y)\\ &\leq M\bar{K}_{n,m}^{\lambda_1,\lambda_2}(|s-x|^{\widehat{\beta_1}}|t-y|^{\widehat{\beta_2}};x,y)\\ &= M\bar{K}_{n,m}^{\lambda_1,\lambda_2}(|s-x|^{\widehat{\beta_1}}|;x,y)\bar{K}_{n,m}^{\lambda_1,\lambda_2}(|t-y|^{\widehat{\beta_2}};x,y),\end{aligned}$$

since $f\in Lip_M(\widehat{\beta_1},\widehat{\beta_2})$. Then we have

$$\begin{aligned}&|\bar{K}_{n,m}^{\lambda_1,\lambda_2}(f;x,y)-f(x,y)|\\ &\leq M\{\bar{K}_{n,m}^{\lambda_1,\lambda_2}(|s-x|^2;x,y)\}^{\widehat{\beta_1}/2}\{\bar{K}_{n,m}^{\lambda_1,\lambda_2}(1;x,y)\}^{\widehat{\beta_1}/2}\{\bar{K}_{n,m}^{\lambda_1,\lambda_2}(|t-y|^2;x,y)\}^{\widehat{\beta_2}/2}\{\bar{K}_{n,m}^{\lambda_1,\lambda_2}(1;x,y)\}^{\widehat{\beta_2}/2}\\ &= MC^{\widehat{\beta_1}/2}(n,\lambda_1)C^{\widehat{\beta_2}/2}(m,\lambda_2)\end{aligned}$$

by applying the Hölder's inequality for $\widehat{p_1}=\frac{2}{\widehat{\beta_1}},\widehat{q_1}=\frac{2}{2-\widehat{\beta_1}}$ and $\widehat{p_2}=\frac{1}{\widehat{\beta_2}},\widehat{q_2}=\frac{2}{2-\widehat{\beta_2}}$. □

Theorem 5.8. *Let $f(x,y)\in C^1(\mathcal{I}_{ab})$, then we have*

$$|\bar{K}_{n,m}^{\lambda_1,\lambda_2}(f;x,y)-f(x,y)| \leq C^{1/2}(n,\lambda_1)\parallel f_x(x,y)\parallel_{C(\mathcal{I}_{ab})}+C^{1/2}(m,\lambda_2)\parallel f_y(x,y)\parallel_{C(\mathcal{I}_{ab})},$$

where $C(n,\lambda_1)$ and $C(m,\lambda_2)$ are defined in Theorem 4.1.

Proof. The following equality holds

$$f(t)-f(s)=\int_x^t f_u(u,s)du+\int_y^s f_v(x,v)du$$

for $(s,t)\in\mathcal{I}_{ab}$. Applying defined operators on both sides of the last equality, we have

$$|\bar{K}_{n,m}^{\lambda_1,\lambda_2}(f;x,y)-f(x,y)| \leq \bar{K}_{n,m}^{\lambda_1,\lambda_2}\left(\left|\int_x^t f_u(u,s)du\right|;x,y\right)+\bar{K}_{n,m}^{\lambda_1,\lambda_2}\left(\left|\int_y^s f_v(x,v)du\right|;x,y\right).$$

By the help of following relations

$$\begin{aligned}\left|\int_x^t f_u(u,s)du\right| &\leq \| f_x(x,y) \|_{C(\mathcal{I}_{ab})} |s-x|;\\ \left|\int_y^s f_v(x,v)du\right| &\leq \| f_y(x,y) \|_{C(\mathcal{I}_{ab})} |t-y|,\end{aligned}$$

we have

$$\begin{aligned}&|\bar{K}_{n,m}^{\lambda_1,\lambda_2}(f;x,y)-f(x,y)|\\ &\leq \| f_x(x,y) \|_{C(\mathcal{I}_{ab})} \bar{K}_{n,m}^{\lambda_1,\lambda_2}(|s-x|;x,y) + \| f_y(x,y) \|_{C(\mathcal{I}_{ab})} \bar{K}_{n,m}^{\lambda_1,\lambda_2}(|t-y|;x,y).\end{aligned}$$

Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\bar{K}_{n,m}^{\lambda_1,\lambda_2}(f;x,y)-f(x,y) \quad &\leq \| f_x(x,y) \|_{C(\mathcal{I}_{ab})} \{\bar{K}_{n,m}^{\lambda_1,\lambda_2}\left((s-x)^2;x,y\right)\}^{1/2}\{\bar{K}_{n,m}^{\lambda_1,\lambda_2}(1;x,y)\}^{1/2}\\ &+ \| f_y(x,y) \|_{C(\mathcal{I}_{ab})} \{\bar{K}_{n,m}^{\lambda_1,\lambda_2}\left((t-y)^2;x,y\right)\}^{1/2}\{\bar{K}_{n,m}^{\lambda_1,\lambda_2}(1;x,y)\}^{1/2}.\end{aligned}$$

This completes the proof. □

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