



Scalarization and Well-Posedness for Set Optimization Using Coradiant Sets

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Abstract. The aim of this paper is to study scalarization and well-posedness for a set-valued optimization problem with order relations induced by a coradiant set. We introduce the notions of the set criterion solution for this problem and obtain some characterizations for these solutions by means of nonlinear scalarization. The scalarization function is a generalization of the scalarization function introduced by Khoshkhabar-amiranloo and Khorram. Moreover, we define the pointwise notions of LP well-posedness, strong DH -well-posedness and strongly B -well-posedness for the set optimization problem and characterize these properties through some scalar optimization problem based on the generalized nonlinear scalarization function respectively.

1. Introduction

In recent years, optimization problems with set-valued objective maps, have received an increasing attention due to its extensive application in many fields such as economics, differential inclusions and optimal control, see [1–3, 7]. There are two types of criteria of solutions for set-valued optimization problems: the vector criterion and the set criterion. The vector criterion, introduced in [4, 8], consists of looking for efficient points of the union of the image sets of the feasible region under the set-valued objective map. In contrast to the vector criterion, the set criterion, also called set optimization, introduced by Kuroiwa [9], bases on an order relation among sets and consists of looking for minimal elements of the family of the image sets of the feasible region under the set-valued objective map. Therefore, the set criterion seems to be more natural and interesting than the vector criterion, whenever one needs to consider preferences over sets.

It is well known that well-posedness plays a crucial role in the stability theory for optimization problems. The notion of well-posedness was first introduced by Tykhonov [10] for scalar optimization problems. Since then, various notions of well-posedness have been introduced and studied for different kinds of optimization problems, see [5] and the references there in. Zhang et al. [11] introduced three kinds of well-posedness for set optimization problems and established the equivalent relations between these well-posedness and the well-posedness of three kinds of scalar optimization problems respectively using a generalized Gerstewitz's function. Gutiérrez et al. [12] generalized some results of [11] on Tykhonov

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well-posed set optimization problems. Long and Peng [13] introduced three kinds of B -well-posedness for set optimization problems and gave some characterizations for these properties. Long et al. [14] introduced the notions of pointwise L -well-posedness and pointwise DH -well-posedness for set optimization problems and obtained some relations among these well-posedness and pointwise B -well-posedness defined in [13]. Han and Huang [15] gave some characterizations for the generalized l - B -well-posedness and the generalized u - B -well-posedness of set optimization problems. Khoshkhabar-amiranloo and Khorram [16] introduced the pointwise notions of LP well-posedness, strongly DH -well-posedness and strongly B -well-posedness and gave some characterizations for these properties using a new scalarization function introduced in [17].

The classical order relations among sets used in the set optimization are preorders induced by a convex cone. However, there are many situations in practice, especially in economics [6, 18], where one has to deal with order relations which are not necessarily preorders. This motivated many authors to consider order relations which are not preorders for optimal solutions of set optimization, see [19–22]. Naturally, it is meaning to study scalarization and well-posedness for these set optimization problems.

In this paper, we study a set-valued optimization problem with order relations induced by a coradiant set. In general, these order relations are not preorders. First, we introduce the notions of the set criterion solution for this problem and show that these concepts generalize the corresponding solution notions for the classical set optimization problem. Then, using a generalization of the nonlinear scalarization function introduced in [17], we give some characterizations for these solutions. Finally, we generalize three kinds of well-posedness introduced in [16] to above set optimization problem and obtain some characterizations of these properties through some scalar optimization problem respectively.

The rest of the paper is organized as follows: In Section 2, we give the main notions, lemmas and propositions. In Section 3, we introduce the notions of the set criterion solution for a set-valued optimization problem using coradiant sets and obtain some relations among these solutions. In Section 4, we give a generalization of the nonlinear scalarization function introduced in [17] and obtain some characterizations of the solutions defined in Section 3 by means of this generalized nonlinear scalarization function. In Section 5, we define the pointwise notions of LP well-posedness, strong DH -well-posedness, strongly B -well-posedness for the set optimization problem using coradiant sets and characterize these properties through some scalar optimization problem based on the generalized nonlinear scalarization function introduced in Section 4 respectively.

2. Preliminaries

Let X and Y be two normed vector spaces, $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y , M be a nonempty subset of X . Given a set $A \subseteq Y$, we denote by clA , $intA$, A^c and $diamA$ the topological closure, the topological interior, the complement and the diameter of A respectively. Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower bounded function. Denote by $\operatorname{argmin}(f, M)$ and $\inf_{x \in M} f$ the set of minimal solutions and the infimum of f on M . Let $F : X \rightrightarrows Y$ be a set-valued map. The effective domain of F is defined as $\operatorname{dom}(F) := \{x \in X | F(x) \neq \emptyset\}$. The distance from a point $y \in Y$ to A is defined as $d(y, A) := \inf_{a \in A} \|y - a\|$, $d(y, \emptyset) = +\infty$, where $\|y\|$ denotes the norm of y in Y . Also, $N_r(y)$ ($\bar{N}_r(y)$) denotes the open (closed) ball with centre y and radius $r > 0$.

Definition 2.1. A set $D \subseteq Y$ is called a coradiant set if $\alpha d \in D, \forall d \in D, \forall \alpha > 1$.

For a coradiant set D in Y , we denote $D(\varepsilon) := \varepsilon D, \forall \varepsilon > 0$ and $D(0) := \bigcup_{\varepsilon > 0} D(\varepsilon) \cup \{0\}$. Clearly, $D(0)$ is a cone. We say that $A \subseteq Y$ is D -closed if $A + D$ is closed and D -bounded if for each neighbourhood U of zero in Y there exists a constant $t > 0$ such that $A \subseteq tU + D$.

Lemma 2.2. [23] Let D be a convex coradiant set. Then

(i) $D(\varepsilon) + D(\delta) \subseteq D(\varepsilon), \forall \varepsilon, \delta > 0$.

(ii) $D(\varepsilon) + D(0) = D(\varepsilon), \forall \varepsilon > 0$.

Remark 2.3. By Lemma 2.2, we can see that a coradiant set D is convex $\Rightarrow D + D(0) = D \Rightarrow D + D \subseteq D$. Moreover, it is easy to check that for a convex coradiant set D , if $0 \in D$, then D is a convex cone such that $D = D(0)$, if $0 \notin D$, then D is an improvement set with respect to the convex cone $D(0)$ (for more details about the improvement set see [24]). Notice that the above conclusions may not hold if the convexity of D is replaced by $D + D \subseteq D$. For example, if let $D := \{(y_1, y_2) \in \mathbb{R}^2 | y_2 \geq 1\} \cup \{0\}$, then D is a coradiant set such that $D + D \subseteq D$ and $0 \in D$, but D is not a convex cone, if let $D := \{(y_1, y_2) \in \mathbb{R}^2 | y_1 \geq 1, y_2 \geq 0, y_1 \geq y_2\} \cup \{(y_1, y_2) \in \mathbb{R}^2 | y_1 \geq 0, y_2 \geq 2, y_1 \leq y_2\}$, then D is a coradiant set such that $D + D \subseteq D$ and $0 \notin D$, but D is not an improvement set with respect to the convex cone $D(0)$.

Proposition 2.4. Let D be a proper ($\emptyset \neq D \neq Y$) coradiant set such that $D + D \subseteq D$. Then $0 \in clD$ iff $D + D = D$.

Proof. For the 'only if' part, it is enough to prove that $D \subseteq D + D$. Suppose that $D \not\subseteq D + D$, then there exists $d \in D$ such that $d \notin D + D$. Since D is a coradiant set, then $(1 + \frac{1}{n})d \in D$ for any $n \in \mathbb{N}$. It follows that $-\frac{1}{n}d \notin D$ for any $n \in \mathbb{N}$, which means $0 \notin clD$, a contradiction.

For the 'if' part, suppose that $0 \notin clD$. Then there exists $r > 0$ such that $N_r(0) \cap clD = \emptyset$. Let $r_{max} := \max\{r > 0, N_r(0) \cap clD = \emptyset\}$, then for any $r' > r_{max}$, $N_{r'}(0) \cap D \neq \emptyset$. It is clear that for any $r > 0$, $(Y \setminus N_r(0)) \cap D(0)$ is a coradiant set. So $D = D + D \subseteq ((Y \setminus N_{r_{max}}(0)) \cap D(0)) + ((Y \setminus N_{r_{max}}(0)) \cap D(0)) = (Y \setminus N_{2r_{max}}(0)) \cap D(0)$, which means $N_{2r_{max}}(0) \cap D = \emptyset$, a contradiction. \square

Definition 2.5. Let D be a closed coradiant set. Then $B \subseteq D$ is called a base of D if

$$D = \{ty | y \in B, t \geq 1\}.$$

Proposition 2.6. Let D be a proper closed coradiant set such that $D + D \subseteq D$. If $0 \in D$ and D has a bounded base B , then $D = D(0)$.

Proof. It is enough to prove that $D(0) \subseteq D$. Suppose that $D(0) \not\subseteq D$, then there exists $0 < \varepsilon < 1$ such that $\forall 0 < \alpha \leq \varepsilon, D(\alpha) \not\subseteq D$, that is to say there exist $0 < \varepsilon < 1$ and $d \in D$ such that $\forall 0 < \alpha \leq \varepsilon, \alpha d \notin D$. Since B is a base of D , then, for $d \in D$, there exist $t \geq 1$ and $b \in B$ such that $d = tb$. Therefore, there exist $0 < \varepsilon < 1$, $t \geq 1$ and $b \in B$ such that $\forall 0 < \alpha \leq \varepsilon, \alpha tb \notin D$. Let $\alpha \rightarrow 0$. Since D is closed and $0 \in D$, then $\alpha tb \notin D$ implies $\|b\| \rightarrow +\infty$. A contradiction with B is bounded. \square

Lemma 2.7. Let D be a proper solid coradiant set such that $D + D \subseteq D$. Then for any $A \subseteq \mathcal{P}(Y)$, $e \in intD$ and $r, \varepsilon > 0$, the following statements hold:

- (i) $cl(A + D) + D \subseteq cl(A + D)$.
- (ii) $(A + D)^c - D \subseteq (A + D)^c$.
- (iii) $(cl(A + D))^c - D \subseteq (cl(A + D))^c$.

Proof. (i) For any $z \in cl(A + D) + D$, there exist $\{a_n\} \subseteq A, \{d_n\} \subseteq D$ and $d' \in D$ such that $a_n + d_n + d' \rightarrow z$. Let $\{d'_n\} := \{d_n + d'\}$. Since $D + D \subseteq D$, then $\{d'_n\} \subseteq D$. Therefore, $z \in cl(A + D)$.

(ii) Suppose that $(A + D)^c - D \not\subseteq (A + D)^c$. Let $z \in ((A + D)^c - D) \setminus (A + D)^c$. So, by $D + D \subseteq D$, there exists $d' \in D$ such that $z + d' \in (A + D)^c \cap (A + D + d') \subseteq (A + D)^c \cap (A + D)$, which is impossible.

(iii) Suppose that $(cl(A + D))^c - D \not\subseteq (cl(A + D))^c$. Let $z \in ((cl(A + D))^c - D) \setminus (cl(A + D))^c$. So there exists $d' \in D$ such that $z + d' \in (cl(A + D))^c \cap (cl(A + D) + d') \subseteq (cl(A + D))^c \cap (cl(A + D))$ by part (i), which is impossible. \square

Lemma 2.8. Let D be a proper solid coradiant set such that $D + D(0) \subseteq D$. Then for any $A \subseteq \mathcal{P}(Y)$, $e \in intD$ and $r, \varepsilon > 0$, the following statements hold:

- (i) $cl(A + D) \subseteq A - re + D$.
- (ii) $A + D \subseteq int(A - re + D)$.
- (iii) $cl(A + N_r(0) + D) \subseteq A + N_{r+\varepsilon}(0) + D$.

Proof. (i) For any $z \in cl(A + D)$, there exist $\{a_n\} \subseteq A$ and $\{d_n\} \subseteq D$ such that $a_n + d_n \rightarrow z$. Since $-re + int(D(0))$ is a neighborhood of zero and $D + D(0) \subseteq D$, then for n large enough, $z \in a_n + d_n - re + int(D(0)) \subseteq A - re + D$.

(ii) As $-re + int(D(0))$ is a neighborhood of zero and $D + D(0) \subseteq D$, then

$$A + D \subseteq A + D - re + int(D(0)) \subseteq int(A + D - re + D(0)) = int(A - re + D).$$

(iii) As D is a proper solid coradiant set, then $0 \notin intD$. It follows that $e \neq 0$ from $e \in intD$. Let $\varepsilon' = \frac{\varepsilon}{2\|e\|}$. By part (i), $cl(A + N_r(0) + D) \subseteq A + N_r(0) - \frac{\varepsilon}{2\|e\|}e + D \subseteq A + N_{r+\varepsilon}(0) + D$. □

3. Optimal solutions of set optimization using coradiant sets

Let $\mathcal{D} := \{D \subseteq Y \mid D + D \subseteq D, D \text{ is a proper, closed, solid, pointed coradiant set}\}$. Throughout this paper, we assume that Y is ordered by a coradiant set $D \in \mathcal{D}$.

Let us recall some definitions in the theory of vector optimization. It is known that the coradiant set D induces the following order relations in Y . For any $y, y' \in Y$, we write $y \leq y'$ if $y' - y \in D$ and $y \ll y'$ if $y' - y \in intD$. Let $a \in A$, we say that a is a minimal (maximal) point of A with respect to D and we write $a \in MinA$ ($a \in MaxA$) if $A \cap (a - D) \subseteq \{a\}$ ($A \cap (a + D) \subseteq \{a\}$). We say that a is a weak minimal (weak maximal) point of A with respect to D and we write $a \in WMinA$ ($a \in WMaxA$) if $A \cap (a - intD) = \emptyset$ ($A \cap (a + intD) = \emptyset$). Clearly, $MinA \subseteq WMinA$ and $MaxA \subseteq WMaxA$.

Assume that $F : X \rightrightarrows Y$ is a set-valued map and $M \subseteq Dom(F)$. The general set-valued optimization problem is defined by:

$$(\mathbf{SOP}) \quad \min_{x \in M} F(x).$$

Let $F(M) := \bigcup_{x \in M} F(x)$, the vector criterion solutions of **(SOP)** are defined as follows. It is said that an element $x_0 \in M$ is a minimal (maximal) solution of **(SOP)**, denoted as $x_0 \in Min(F, M)$ ($x_0 \in Max(F, M)$), if there exists $y_0 \in F(x_0)$ such that $y_0 \in MinF(M)$ ($y_0 \in MaxF(M)$). In the same way, it is said that an element $x_0 \in M$ is a weak minimal (weak maximal) solution of **(SOP)**, denoted as $x_0 \in WMin(F, M)$ ($x_0 \in WMax(F, M)$), if there exists $y_0 \in F(x_0)$ such that $y_0 \in WMinF(M)$ ($y_0 \in WMaxF(M)$).

Let $A, B \subseteq \mathcal{P}(Y)$, we denote by $\leq^l, \ll^l, \leq^u, \ll^u, \sim^l, \sim^u$ the following set order relations on $\mathcal{P}(Y)$.

$$A \leq^l B \Leftrightarrow B \subseteq A + D,$$

$$A \ll^l B \Leftrightarrow B \subseteq A + intD,$$

$$A \leq^u B \Leftrightarrow A \subseteq B - D,$$

$$A \ll^u B \Leftrightarrow A \subseteq B - intD,$$

$$A \sim^l B \Leftrightarrow A \leq^l B \text{ and } B \leq^l A,$$

$$A \sim^u B \Leftrightarrow A \leq^u B \text{ and } B \leq^u A.$$

Remark 3.1. Since a cone is a coradiant set, then for any proper, closed, solid, pointed convex cone $K \subseteq Y$, we have $K \in \mathcal{D}$ by Remark 2.3. Therefore, the above set order relations are generalization of the corresponding set order relations defined in [25].

Remark 3.2. It is easy to prove that if $A \sim^l B$ ($A \sim^u B$), then $A + D = B + D$ ($A - D = B - D$). Notice that the opposite conclusion does not hold in general, for example, let $Y = \mathbb{R}$, $A = [-2, 2]$, $B = \{-2\}$ and $D = [2, +\infty)$. Then $D \in \mathcal{D}$ and $A + D = B + D = [0, +\infty)$. Obviously, $A \not\sim^l B$.

Proposition 3.3. The following statements hold:

- (i) If $A \sim^l B$, then $MinA = MinB$.

(ii) If $A \sim^u B$, then $MaxA = MaxB$.

Proof. (i) If $MinA = MinB = \emptyset$, the conclusion is true. Without loss of generality, suppose that $MinA \neq \emptyset$. Let $a_1 \in MinA \setminus MinB$. Since $B \leq^l A$, then there exists $b_1 \in B$ such that $a_1 \in b_1 + D$. Also, as $A \leq^l B$, there exists $a_2 \in A$ such that $b_1 \in a_2 + D$. In consequence, $a_2 \in b_1 - D \subseteq a_1 - D - D \subseteq a_1 - D$ because $D + D \subseteq D$. If $a_2 \neq a_1$, then $a_1 \notin MinA$, a contradiction. Suppose that $a_2 = a_1$, then $a_1 = b_1$ and from $a_1 \notin MinB$, we deduce that there exists $b_2 \in B \setminus A$ such that $b_2 \in a_1 - D$. Therefore, by $A \leq^l B$ again, there exists $a_3 \in A$ such that $b_2 \in a_3 + D$. So $a_3 \in b_2 - D \subseteq a_1 - D - D \subseteq a_1 - D$, which contradicts with $a_1 \in MinA$ and $a_3 \neq a_1$.

(ii) Similarly to the proof of (i). □

Definition 3.4. A set $A \in \mathcal{P}(Y)$ has the Min (Max) Property if for all $x \in A$ there exists $a \in MinA$ ($a \in MaxA$) such that $a \leq x$ ($x \leq a$).

We denote Ω (Θ) be the family of subsets of Y which has the Min (Max) Property.

Proposition 3.5. Let $A, B \in \Omega$ (Θ). If $MinA = MinB$ ($MaxA = MaxB$), then $A \sim^l B$ ($A \sim^u B$).

Proof. We only prove the situation of \sim^l . Suppose that $A \not\sim^l B$, without loss of generality, $A \not\leq^l B$. Then there exists $b \in B$ such that $b \notin a + D$ for all $a \in A$. In consequence $A \cap (b - D) = \emptyset$. If $b \in MinB = MinA$, we get a contradiction. If $b \notin MinB$. Since B has the Min Property, there exists $b_0 \in MinB$ such that $b \in b_0 + D$ and $b_0 - D \subseteq b - D - D \subseteq b - D$ by $D + D \subseteq D$. Therefore $A \cap (b_0 - D) = \emptyset$, which contradicts with $b_0 \in MinB = MinA$. □

Now, we introduce the following notations of efficient set using the above set order relations, which are generalizations of the notions of efficient set defined in [25] and [26] according to Remark 3.1.

Definition 3.6. Let $\mathcal{A} \subseteq \mathcal{P}(Y)$. It is said that

- (i) $A \in \mathcal{A}$ is an l -minimal set of \mathcal{A} if for any $B \in \mathcal{A}$ such that $B \leq^l A$ implies $A \leq^l B$. The family of l -minimal sets of \mathcal{A} is denoted by $l\text{-Min}\mathcal{A}$.
- (ii) $A \in \mathcal{A}$ is a weak l -minimal set of \mathcal{A} if for any $B \in \mathcal{A}$ such that $B \ll^l A$ implies $A \ll^l B$. The family of weak l -minimal sets of \mathcal{A} is denoted by $l\text{-WMin}\mathcal{A}$.
- (iii) $A \in \mathcal{A}$ is an u -minimal set of \mathcal{A} if for any $B \in \mathcal{A}$ such that $B \leq^u A$ implies $A \leq^u B$. The family of u -minimal sets of \mathcal{A} is denoted by $u\text{-Min}\mathcal{A}$.
- (iv) $A \in \mathcal{A}$ is a weak u -minimal set of \mathcal{A} if for any $B \in \mathcal{A}$ such that $B \ll^u A$ implies $A \ll^u B$. The family of weak u -minimal sets of \mathcal{A} is denoted by $u\text{-WMin}\mathcal{A}$.

Remark 3.7. It is clear that if $A \in \mathcal{A}$ is an l -minimal (u -minimal) set of \mathcal{A} and $B \in \mathcal{A}$ satisfies $A \sim^l B$ ($A \sim^u B$), then B is also an l -minimal (u -minimal) set of \mathcal{A} .

We denote by \mathcal{F} the family of all image sets under F , that is, $\{F(x)\}_{x \in M}$. Using the above efficient sets, the set criterion solutions of (SOP) are denoted as follows. We say that $x_0 \in M$ is an l -minimal (u -minimal) solution of (SOP), denoted as $x_0 \in l\text{-Min}(F, M)$ ($x_0 \in u\text{-Min}(F, M)$), if $F(x_0)$ is an l -minimal (u -minimal) set of \mathcal{F} . In the same way, we say that $x_0 \in M$ is a weak l -minimal (weak u -minimal) solution of (SOP), denoted as $x_0 \in l\text{-WMin}(F, M)$ ($x_0 \in u\text{-WMin}(F, M)$), if $F(x_0)$ is a weak l -minimal (weak u -minimal) set of \mathcal{F} . Moreover, we say that $x_0 \in M$ is a strict l -minimal (strict u -minimal) solution of (SOP) in the sense of [19], denoted as $x_0 \in l\text{-SMin}(F, M)$ ($x_0 \in u\text{-SMin}(F, M)$), if $F(x) \not\leq^l F(x_0), \forall x \in M \setminus \{x_0\}$ ($F(x) \not\leq^u F(x_0), \forall x \in M \setminus \{x_0\}$). Clearly, $l\text{-SMin}(F, M) \subseteq l\text{-Min}(F, M)$ ($u\text{-SMin}(F, M) \subseteq u\text{-Min}(F, M)$).

Lemma 3.8. Let $\mathcal{A} \subseteq \mathcal{P}(Y)$ and $A \in \mathcal{A}$ be a set with $WMinA \neq \emptyset$ ($WMaxA \neq \emptyset$). Then $A \in l\text{-WMin}\mathcal{A}$ ($A \in u\text{-WMin}\mathcal{A}$) iff there is not a set $B \in \mathcal{A}$ such that $B \ll^l A$ ($B \ll^u A$).

Proof. We only proof the situation of weak l -minimal sets. Clearly, it is enough to prove the 'only if' part. Let $a \in WMinA$. Suppose that there exists a set $B \in \mathcal{A}$ such that $B \ll^l A$, then $A \ll^l B$ because $A \in l-WMin\mathcal{A}$. As $D + D \subseteq D$, so $A \subseteq B + intD \subseteq A + intD + intD \subseteq A + int(D + D) \subseteq A + intD$. Since $a \subseteq A$, then there exist $a' \in A$ and $d' \in intD$ such that $a = a' + d'$, it follows that $a' \in A \cap (a - intD)$, a contradiction with $a \in WMinA$. \square

Proposition 3.9. Let $\mathcal{A} \subseteq \mathcal{P}(Y)$. If $D + D(0) = D$, then

(i) $l-Min\mathcal{A} \subseteq l-WMin\mathcal{A}$.

(ii) $u-Min\mathcal{A} \subseteq u-WMin\mathcal{A}$.

Proof. (i) For any $A \in l-Min\mathcal{A}$, suppose that $B \in \mathcal{A}$ and $B \ll^l A$, i.e.,

$$A \subseteq B + intD \tag{1}$$

It is clear that $B \leq^l A$. Since $A \in l-Min\mathcal{A}$, then $A \leq^l B$, i.e.,

$$B \subseteq A + D \tag{2}$$

and $A \sim^l B$. Then, by Remark 3.2,

$$A + D = B + D \tag{3}$$

From (1) and (2), we obtain

$$B \subseteq A + D \subseteq B + intD + D \tag{4}$$

On the other hand, adding $intD$ to equality (3), we have

$$A + D + intD = B + D + intD$$

So, by (4) and $D + D(0) = D$, we conclude that $B \subseteq A + D + intD \subseteq A + D(0) + intD \subseteq A + int(D(0) + D) = A + intD$, i.e., $A \ll^l B$. Therefore $A \in l-WMin\mathcal{A}$.

(ii) Similarly to the proof of (i). \square

Proposition 3.10. If $D + D(0) = D$, then

(i) $WMin(F, M) \subseteq l-WMin(F, M)$.

(ii) $WMax(F, M) \subseteq u - WMin(F, M)$.

Proof. (i) Let $x_0 \in M$ be a weak minimal solution of (SOP), then there exists $y_0 \in F(x_0)$ such that $y_0 \in WMinF(M)$. Suppose that x_0 is not a weak l -minimal solution of (SOP). Then, by Lemma 3.8, there exists $x' \in M$ such that $F(x') \ll F(x_0)$, i.e., $F(x_0) \subseteq F(x') + intD$. In particular $y_0 \in F(x') + intD$, which contradicts with $y_0 \in WMinF(M)$.

(ii) Similarly to the proof of (i). \square

Let Λ be a property of sets in Y , then we say that F is Λ -valued on M if $F(x)$ has the property Λ for each $x \in M$.

Definition 3.11. Let M be a nonempty convex subset of X . A set-valued mapping $G : X \rightrightarrows Y$ is said to be

(i) strictly lower D -convex on M if for any $x_1, x_2 \in M$ with $x_1 \neq x_2$ and for any $t \in (0, 1)$, one has

$$tG(x_1) + (1 - t)G(x_2) \subseteq G(tx_1 + (1 - t)x_2) + intD.$$

(ii) strictly upper D -convex on M if for any $x_1, x_2 \in M$ with $x_1 \neq x_2$ and for any $t \in (0, 1)$, one has

$$G(tx_1 + (1-t)x_2) \subseteq tG(x_1) + (1-t)G(x_2) - \text{int}D.$$

Proposition 3.12. Assume that M is convex, $D + D(0) = D$ and F is strictly lower D -convex on M with nonempty compact values. Then $l\text{-WMin}(F, M) = l\text{-Min}(F, M) = l\text{-SMin}(F, M)$.

Proof. By Proposition 3.9, it is enough to prove $l\text{-WMin}F(M) \subseteq l\text{-SMin}F(M)$. For any $x_0 \in l\text{-WMin}(F, M)$ and any $x' \in M$ such that $F(x') \leq^l F(x_0)$, i.e.,

$$F(x_0) \subseteq F(x') + D. \quad (5)$$

Suppose that $x' \neq x_0$. Since F is strictly lower D -convex on M , then

$$tF(x_0) + (1-t)F(x') \subseteq F(tx_0 + (1-t)x') + \text{int}D, \forall t \in (0, 1). \quad (6)$$

By (5), (6) and $D + D(0) = D$, we have

$$\begin{aligned} F(x_0) &\subseteq tF(x_0) + (1-t)F(x_0) \subseteq tF(x_0) + (1-t)F(x') + (1-t)D \\ &\subseteq F(tx_0 + (1-t)x') + \text{int}D + (1-t)D \subseteq F(tx_0 + (1-t)x') + \text{int}D + D(0) \\ &\subseteq F(tx_0 + (1-t)x') + \text{int}D, \forall t \in (0, 1). \end{aligned}$$

So $F(tx_0 + (1-t)x') \ll^l F(x_0), \forall t \in (0, 1)$. Since $F(x_0)$ is compact and $\text{Min}F(x_0) \subseteq \text{WMin}F(x_0)$, then $\text{WMin}F(x_0) \neq \emptyset$. According to Lemma 3.8, we can get that $x_0 \notin l\text{-WMin}(F, M)$, a contradiction. Therefore, $x' = x_0$ and so $x_0 \in l\text{-SMin}(F, M)$. \square

Proposition 3.13. Assume that M is convex, $D + D(0) = D$ and F is strictly upper D -convex on M with nonempty compact convex values. Then $u\text{-WMin}F(M) = u\text{-Min}F(M) = u\text{-SMin}F(M)$.

Proof. By Proposition 3.9, it is enough to prove $u\text{-WMin}F(M) \subseteq u\text{-SMin}F(M)$. For any $x_0 \in u\text{-WMin}F(M)$ and any $x' \in M$ such that $F(x') \leq^u F(x_0)$, i.e.,

$$F(x') \subseteq F(x_0) - D. \quad (7)$$

Suppose that $x' \neq x_0$. Since F is strictly upper D -convex on M , then

$$tF(x_0) + (1-t)F(x') \subseteq tF(x_0) + (1-t)F(x') - \text{int}D, \forall t \in (0, 1). \quad (8)$$

From (7) and (8), we obtain

$$F(tx_0 + (1-t)x') \subseteq tF(x_0) + (1-t)F(x_0) - (1-t)D - \text{int}D, \forall t \in (0, 1).$$

By the convexity of $F(x_0)$ and $D + D(0) = D$, it follows that

$$F(tx_0 + (1-t)x') \subseteq F(x_0) - D(0) - \text{int}D \subseteq F(x_0) - \text{int}D, \forall t \in (0, 1).$$

So $F(tx_0 + (1-t)x') \ll^u F(x_0), \forall t \in (0, 1)$. Since $F(x_0)$ is compact and $\text{Max}F(x_0) \subseteq \text{WMax}F(x_0)$, then $\text{WMax}F(x_0) \neq \emptyset$. According to Lemma 3.8, we can get that $x_0 \notin u\text{-WMin}F(M)$, a contradiction. Therefore, $x' = x_0$ and so $x_0 \in u\text{-SMin}F(M)$. \square

Remark 3.14. By Remark 2.2 and taking into account a cone is a radiant set, we can see that

- (i) Propositions 3.3 and 3.5 are generalizations of Propositions 8 and 11 in [27] respectively.
- (ii) Lemma 3.8, Propositions 3.9 and 3.10 are generalizations of Lemma 2.6, Propositions 2.7 and 2.10 in [26] respectively.
- (iii) Propositions 3.12 and 3.13 are generalizations of Propositions 2.1 and 2.2 in [15] respectively.

Similar to the proof of Propositions 2.3 and 2.4 in [15], we can get the following proposition.

Proposition 3.15. *Assume that M is closed and $D + D(0) = D$. Then the following statements hold:*

- (i) *If F is u.s.c. on M with nonempty compact values, then $l\text{-WMin}(F, M)$ is closed.*
- (ii) *If F is l.s.c. on M with nonempty compact values, then $u\text{-WMin}F(M)$ is closed.*

From Propositions 3.12, 3.13 and 3.15, we can get the following proposition.

Proposition 3.16. *Assume that M is convex and closed, and $D + D(0) = D$. Then the following statements hold:*

- (i) *If F is u.s.c. and strictly lower D -convex on M with nonempty compact values, then $l\text{-SMin}(F, M)$ is closed.*
- (ii) *If F is l.s.c. and strictly upper D -convex on M with nonempty compact convex values, then $u\text{-SMin}F(M)$ is closed.*

4. Scalarization of set optimization using coradiant sets

In this section, we first give a generalization of a nonlinear scalarization function introduced in [17] and provide some properties of this function.

Let $B \in \mathcal{P}(Y)$. Consider the function $\rho_B^l : \mathcal{P}(Y) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ given by

$$\rho_B^l(A) = \sup_{b \in B} d(b, A + D).$$

Definition 4.1. [28]

- (i) *A function $T : \mathcal{P}(Y) \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be \leq^l increasing iff*

$$A_1, A_2 \in \mathcal{P}(Y), A_1 \leq^l A_2 \Rightarrow T(A_1) \leq T(A_2).$$

- (ii) *A function $T : \mathcal{P}(Y) \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be convex iff*

$$T(\lambda A_1 + (1 - \lambda)A_2) \leq \lambda T(A_1) + (1 - \lambda)T(A_2), \forall A_1, A_2 \in \mathcal{P}(Y), \lambda \in [0, 1].$$

T is said to be concave if $-T$ is convex.

Proposition 4.2. *Let $A, B \in \mathcal{P}(Y)$ and $k \in Y$. The following statements hold:*

- (i) *If $D + D(0) = D$, then*

$$\rho_{B+D}^l(A) \leq \rho_{B+D(0)}^l(A) = \rho_B^l(A) = \rho_{clB}^l(A) = \rho_B^l(clA) = \rho_B^l(A + D(0)) \leq \rho_B^l(A + D).$$

- (ii) $\rho_B^l(A + k) \leq \rho_B^l(A) + \|k\|$.

- (iii) *If $D + D(0) = D$ and B is $D(0)$ -bounded, then $\rho_B^l(A) < +\infty$.*

- (iv) *If A is D -closed, then $A \leq^l B \Leftrightarrow \rho_B^l(A) = 0$.*

- (v) $\rho_B^l(A) \leq r \Leftrightarrow B \subseteq cl(A + D) + \bar{N}_r(0)$.

- (vi) $\rho_B^l(A) < r \Rightarrow A + N_r(0) \leq^l B$.

- (vii) ρ_B^l is \leq^l increasing.

- (viii) *If D is convex, then ρ_B^l is convex.*

Proof. (i) By the definition of $\rho_B^l(A)$ and $D + D(0) = D$, it is easy to see that

$$\rho_B^l(A) = \rho_{clB}^l(A) = \rho_B^l(clA) = \rho_B^l(A + D(0)).$$

Since $0 \in D(0)$ and $D \subseteq D(0)$, then $\rho_{B+D}^l(A) \leq \rho_{B+D(0)}^l(A)$, $\rho_{B+D(0)}^l(A) \geq \rho_B^l(A)$ and $\rho_B^l(A + D(0)) \leq \rho_B^l(A + D)$. Therefore, it is enough to prove $\rho_{B+D(0)}^l(A) \leq \rho_B^l(A)$. Indeed, for each $b \in B$ and $e \in D(0)$, we have

$$\begin{aligned} d(b, A + D) &= \inf_{a \in A, d \in D} \|a + d - b\| = \inf_{a \in A, d \in D} \|a + d + e - (b + e)\| \\ &\geq \inf_{a \in A, d' \in D} \|a + d' - (b + e)\| = d(b + e, A + D). \end{aligned}$$

because $d' = d + e \in D + D(0) = D$. Then

$$\rho_{B+D(0)}^l(A) = \sup_{b \in B, e \in D(0)} d(b + e, A + D) \leq \sup_{b \in B} d(b, A + D) = \rho_B^l(A).$$

(ii)

$$\begin{aligned} \rho_B^l(A + k) &= \sup_{b \in B} d(b, A + k + D) = \sup_{b \in B} \inf_{a \in A, d \in D} \|b - a - d - k\| \\ &\leq \sup_{b \in B} \inf_{a \in A, d \in D} (\|b - a - d\| + \|k\|) \leq \rho_B^l(A) + \|k\|. \end{aligned}$$

(iii) Suppose that B is $D(0)$ -bounded, then there exists $t > 0$ such that $B \subseteq tN_1(0) + D(0)$. Let $a \in A$. By the definition of $\rho_B^l(A)$ and part (i) (ii), we have

$$\rho_B^l(A) \leq \rho_{tN_1(0)+D(0)}^l(A) = \rho_{tN_1(0)}^l(A) \leq \rho_{tN_1(0)}^l(a) \leq t + \|a\| < +\infty.$$

(iv) If $A \leq^l B$, then $B \subseteq A + D$ and so $\rho_B^l(A) = 0$. Conversely, suppose that $A \not\leq^l B$. Then there exists $b \in B$ such that $b \notin A + D = cl(A + D)$ because A is D -closed. Therefore $d(b, A + D) > 0$ and thus $\rho_B^l(A) > 0$. A contradiction with $\rho_B^l(A) = 0$.

(v) It is clear from the definition of $\rho_B^l(A)$.

(vi)

$$\begin{aligned} \rho_B^l(A) < r &\Rightarrow \forall b \in B, d(b, A + D) < r \Rightarrow \forall b \in B, \exists z_b \in A + D, d(b, z_b) < r \\ &\Rightarrow B \subseteq A + D + N_r(0) \Rightarrow A + N_r(0) \leq^l B. \end{aligned}$$

(vii)

$$\begin{aligned} A_1 \leq^l A_2 &\Rightarrow A_2 \subseteq A_1 + D \Rightarrow A_2 + D \subseteq A_1 + D + D \subseteq A_1 + D \\ &\Rightarrow \forall b \in B, d(b, A_1 + D) \leq d(b, A_2 + D) \Rightarrow \rho_B^l(A_1) \leq \rho_B^l(A_2). \end{aligned}$$

(viii) Let $A_1, A_2 \subseteq \mathcal{P}(Y)$ and $0 \leq \lambda \leq 1$. If $\rho_B^l(A_1) = +\infty$ or $\rho_B^l(A_2) = +\infty$, it is nothing to prove. Let $r_1 := \rho_B^l(A_1) < +\infty$ and $r_2 := \rho_B^l(A_2) < +\infty$. By part (iv), we have

$$B \subseteq cl(A_1 + D) + \bar{N}_{r_1}(0) \text{ and } B \subseteq cl(A_2 + D) + \bar{N}_{r_2}(0).$$

Since D is convex, then

$$\begin{aligned} B &\subseteq \lambda B + (1 - \lambda)B \\ &\subseteq \lambda cl(A_1 + D) + (1 - \lambda)cl(A_2 + D) + \lambda \bar{N}_{r_1}(0) + (1 - \lambda)\bar{N}_{r_2}(0) \\ &\subseteq cl(\lambda A_1 + (1 - \lambda)A_2 + \lambda D + (1 - \lambda)D) + \bar{N}_{\lambda r_1 + (1 - \lambda)r_2}(0) \\ &= cl(\lambda A_1 + (1 - \lambda)A_2 + D) + \bar{N}_{\lambda r_1 + (1 - \lambda)r_2}(0). \end{aligned}$$

Then, by part (iv) again,

$$\rho_B^l(\lambda A_1 + (1 - \lambda)A_2) \leq \lambda r_1 + (1 - \lambda)r_2 = \lambda \rho_B^l(A_1) + (1 - \lambda)\rho_B^l(A_2).$$

□

Definition 4.3.

- (i) A set-valued map $F : M \rightrightarrows Y$ is said to be upper D -semicontinuous at $x_0 \in M$ iff for each neighbourhood V of $F(x_0)$ in Y , there exists a neighbourhood U of x_0 such that $F(x) \subseteq V + D$ for all $x \in U \cap M$. F is said to be upper D -semicontinuous on M iff it is upper D -semicontinuous at all $x \in M$.
- (ii) A set-valued map $F : M \rightrightarrows Y$ is said to be lower D -semicontinuous at $x_0 \in M$ iff for any $y \in F(x_0)$, any neighbourhood V of y in Y , there exists a neighbourhood U of x_0 such that $F(x) \cap (V + D) \neq \emptyset$ for all $x \in U \cap M$. F is said to be lower D -semicontinuous on M iff it is lower D -semicontinuous at all $x \in M$.
- (iii) A set-valued map $F : M \rightrightarrows Y$ is said to be D -semicontinuous on M iff it is upper D -semicontinuous and lower D -semicontinuous on M .

If D is a convex cone K , then we get the definition of upper K -continuity and lower K -continuity for set-valued maps, see Definition 7.1 in [4]. Moreover, setting $D = \{0\}$ in above definition, we get the definition of upper continuous and lower continuous for set-valued maps and when $Y = \mathbb{R}$ and $D = \mathbb{R}_+$, the D -semicontinuous ($-D$ -semicontinuous) for single-valued maps collapse to the usual lower semicontinuity (upper semicontinuity) for real-valued functions.

Proposition 4.4. *The following statements hold:*

- (i) If $D + D(0) = D$ and F is lower $-D(0)$ -semicontinuous on M , then $\rho_B^l \circ F$ is upper semicontinuous on M for each $B \in \mathcal{P}(Y)$.
- (ii) If F is upper D -semicontinuous on M , then $\rho_B^l \circ F$ is lower semicontinuous on M for each $B \in \mathcal{P}(Y)$.

Proof. (i) It is enough to show that the set $\{x \in M : \rho_B^l \circ F(x) \geq r\}$ is closed for all $B \in \mathcal{P}(Y)$ and $r \geq 0$. Suppose that there exist $B \in \mathcal{P}(Y)$, $r > 0$, $\bar{x} \in M$ and $\{x_n\} \subseteq M$ such that $x_n \rightarrow \bar{x}$, $\rho_B^l \circ F(x_n) \geq r$ and $\rho_B^l \circ F(\bar{x}) < r$. Then there exists $0 < \varepsilon < r$ such that $\rho_B^l \circ F(x_n) > r - \frac{\varepsilon}{2}$ and $\rho_B^l \circ F(\bar{x}) < r - \varepsilon$. So by Proposition 4.2 (vi), we have $B \not\subseteq F(x_n) + N_{r-\frac{\varepsilon}{2}}(0) + D$ and $B \subseteq F(\bar{x}) + N_{r-\varepsilon}(0) + D$. Since $D + D(0) = D$, it follows that $F(\bar{x}) \not\subseteq F(x_n) + N_{\frac{\varepsilon}{2}}(0) + D(0)$. That is $F(\bar{x}) \cap (F(x_n) + N_{\frac{\varepsilon}{2}}(0) + D(0))^c \neq \emptyset$ and so $F(\bar{x}) \cap (cl(F(x_n) + N_{\frac{\varepsilon}{2}}(0) + D(0)))^c \neq \emptyset$. On the other hand, by $F(x_n) \subseteq cl(F(x_n) + N_{\frac{\varepsilon}{2}}(0) + D(0))$ and Lemma 2.7, we have

$$F(x_n) \cap ((cl(F(x_n) + N_{\frac{\varepsilon}{2}}(0) + D(0)))^c - D(0)) = F(x_n) \cap (cl(F(x_n) + N_{\frac{\varepsilon}{2}}(0) + D(0)))^c = \emptyset,$$

which contradicts with the lower $-D(0)$ -semicontinuity of F at \bar{x} .

(ii) To prove this part, we show that the set $\{x \in M : \rho_B^l \circ F(x) > r\}$ is open for all $B \in \mathcal{P}(Y)$ and $r \geq 0$. Let $B \in \mathcal{P}(Y)$. Suppose that $r \geq 0$ and $\bar{x} \in M$ such that $\rho_B^l \circ F(\bar{x}) > r$. Then by Proposition 4.2 (vi), we have

$$\begin{aligned} \rho_B^l \circ F(\bar{x}) > r &\Rightarrow \exists \varepsilon > 0, \rho_B^l \circ F(\bar{x}) > r + \varepsilon \\ &\Rightarrow B \not\subseteq F(\bar{x}) + N_{r+\varepsilon}(0) + D \\ &\Rightarrow \exists b \in B, b \notin F(\bar{x}) + N_{r+\varepsilon}(0) + D \\ &\Rightarrow \exists b \in B, F(\bar{x}) \cap (b - N_{r+\varepsilon}(0) - D) = \emptyset \\ &\Rightarrow \exists b \in B, F(\bar{x}) \subseteq (b - N_{r+\varepsilon}(0) - D)^c \subseteq (cl(b - N_{r+\frac{\varepsilon}{2}}(0) - D))^c. \end{aligned}$$

Since F is upper D -semicontinuity at \bar{x} , then, there exists a neighbourhood U of \bar{x} such that

$$F(x) \subseteq (cl(b - N_{r+\frac{\varepsilon}{2}}(0) - D))^c + D, \forall x \in U \cap M.$$

It follows from Lemma 2.7 and Proposition 4.2 (vi) that for any $x \in U \cap M$,

$$\begin{aligned} F(x) \subseteq (cl(b - N_{r+\frac{\epsilon}{2}}(0) - D))^c &\Rightarrow F(x) \cap (b - N_{r+\frac{\epsilon}{2}}(0) - D) = \emptyset \\ &\Rightarrow b \notin F(x) + N_{r+\frac{\epsilon}{2}}(0) + D \\ &\Rightarrow \rho_b^l \circ F(x) \geq r + \frac{\epsilon}{2} > r. \end{aligned}$$

So $\rho_B^l \circ F(x) > r$ for any $x \in U \cap M$. □

Theorem 4.5. Let $\bar{x} \in M$ and F be D -closed-valued on M . If $\{x \in M : \rho_{F(\bar{x})}^l \circ F(x) = 0\} \neq \emptyset$, then

- (i) $\bar{x} \in l\text{-Min}(F, M)$ iff $\text{argmin}(\rho_{F(\bar{x})}^l \circ F, M) = \{x \in M : F(x) \sim^l F(\bar{x})\}$.
- (ii) $\bar{x} \in l\text{-SMin}(F, M)$ iff $\text{argmin}(\rho_{F(\bar{x})}^l \circ F, M) = \{\bar{x}\}$.

Proof. By Proposition 4.2 (iv) and $\{x \in M : \rho_{F(\bar{x})}^l \circ F(x) = 0\} \neq \emptyset$, we have

$$\{x \in M : F(x) \leq^l F(\bar{x})\} = \{x \in M : \rho_{F(\bar{x})}^l \circ F(x) = 0\} = \text{argmin}(\rho_{F(\bar{x})}^l \circ F, M). \tag{9}$$

(i) Let $\bar{x} \in l\text{-Min}(F, M)$. Clearly, $\{x \in M : F(x) \sim^l F(\bar{x})\} \subseteq \{x \in M : F(x) \leq^l F(\bar{x})\} = \text{argmin}(\rho_{F(\bar{x})}^l \circ F, M)$. For the reverse inclusion, suppose that $x \in \text{argmin}(\rho_{F(\bar{x})}^l \circ F, M)$, then $F(x) \leq^l F(\bar{x})$ by (9). Since $\bar{x} \in l\text{-Min}(F, M)$, it follows that $F(\bar{x}) \leq^l F(x)$ and $F(x) \sim^l F(\bar{x})$. Therefore $\text{argmin}(\rho_{F(\bar{x})}^l \circ F, M) \subseteq \{x \in M : F(x) \sim^l F(\bar{x})\}$. Conversely, by the assumption and (9), we have $\{x \in M : F(x) \leq^l F(\bar{x})\} = \{x \in M : F(x) \sim^l F(\bar{x})\}$, which means $\bar{x} \in l\text{-Min}(F, M)$.

(ii) The result is a direct consequence of the definition of $l\text{-SMin}(F, M)$ and equality (9). □
 Similar to the proof of Theorem 3.2 in [16], we can get the following theorem.

Theorem 4.6. Let $B \in \mathcal{P}(Y)$. If $\text{argmin}(\rho_B^l \circ F, M) = \{\bar{x}\}$, then $\bar{x} \in l\text{-SMin}(F, M)$.

5. Well-posedness of set optimization using coradiant sets

Consider the following scalar optimization problem:

$$\text{(OP)} \quad \min_{x \in M} f(x)$$

where $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a lower bounded function.

We first recall the notions of well-posedness for (OP).

Definition 5.1. [5] (OP) is called:

- (i) LP well-posed iff $\text{argmin}(f, M)$ is a singleton and every LP-minimizing sequence (i.e. $\{x_n\} \subseteq X, d(x_n, M) \rightarrow 0, f(x_n) \rightarrow \inf_{x \in M} f(x)$) converges to $\text{argmin}(f, M)$.
- (ii) Generalized LP well-posed iff $\text{argmin}(f, M) \neq \emptyset$ and for every LP-minimizing sequence, there exists a subsequence that converges to an element of $\text{argmin}(f, M)$.

Theorem 5.2. [16] If f is lower semicontinuous on a compact set $M \subseteq X$, then (OP) is generalized LP well-posed.

Given $A, B \in \mathcal{P}(Y)$, define

$$e(A, B) = \sup_{a \in A} d(a, B).$$

We say that A is the upper (lower) limit of $\{A_n\}$ in the sense of Hausdorff iff $\lim_{n \rightarrow \infty} e(A_n, A) = 0$ ($\lim_{n \rightarrow \infty} e(A, A_n) = 0$).

Note that $e(B, A + D) = \rho_B^l(A)$.

In the following definitions, we introduce the notions of minimizing sequence and well-posedness for **(SOP)**.

Definition 5.3. A sequence $\{x_n\} \subseteq X$ is said to be a LP-minimizing sequence for **(SOP)** at $\bar{x} \in l\text{-Min}(F, M)$ iff there exists $\{d_n\} \subseteq D(0) \setminus \{0\}$ with $d_n \rightarrow 0$ such that $d(x_n, M) \rightarrow 0$ and $F(x_n) \leq^l F(\bar{x}) + d_n$.

Definition 5.4. **(SOP)** is said to be:

- (i) LP well-posed at $\bar{x} \in l\text{-Min}(F, M)$ iff any LP-minimizing sequence for **(SOP)** at \bar{x} converges to \bar{x} .
- (ii) Strongly DH-well-posed at $\bar{x} \in l\text{-Min}(F, M)$ iff $\inf_{\alpha > 0} \text{diam}_{L_M}(\bar{x}, d, \alpha) = 0$ for all $d \in D(0)$, where $L_M(\bar{x}, d, \alpha) := \{x \in X : d(x, M) \leq \alpha, F(x) \leq^l F(\bar{x}) + \alpha d\}$.
- (iii) Strongly B-well-posed at $\bar{x} \in l\text{-Min}(F, M)$ iff the set-valued map $\Phi_{\bar{x}} : \mathbb{R}_+ \rightrightarrows X$ defined as

$$\Phi_{\bar{x}}(\alpha) = L_M(\bar{x}, e, \alpha) = \{x \in X : d(x, M) \leq \alpha, F(x) \leq^l F(\bar{x}) + \alpha e\},$$

where $e \in \text{int}D(0)$ such that $\|e\| = 1$, is upper semicontinuous at $\alpha = 0$.

Consider a particular case of **(OP)** as follows:

$$\text{(OP}_B) \quad \min_{x \in M} \rho_B^l \circ F(x)$$

where $B \in \mathcal{P}(Y)$.

Now, we give some characterations of LP well-posedness for **(SOP)**.

Theorem 5.5. Let $B \in \mathcal{P}(Y)$. If $M = \text{Dom}(F)$, (OP_B) is LP well-posed and $\text{argmin}(\rho_B^l \circ F, M) = \{\bar{x}\}$, then **(SOP)** is LP well-posed at \bar{x} .

Proof. By Theorem 4.6, $\bar{x} \in l\text{-Min}(F, M)$. Suppose that the sequence $\{x_n\} \subseteq X$ is a LP-minimizing sequence for **(SOP)** at \bar{x} . Then there exists $d_n \in D(0) \setminus \{0\}$ with $d_n \rightarrow 0$ such that $d(x_n, M) \rightarrow 0$ and $F(x_n) \leq^l F(\bar{x}) + d_n$. From $M = \text{Dom}(F)$, it follows that $x_n \in M$. Since \bar{x} is a minimal solution of (OP_B) and ρ_B^l is \leq^l increasing, we have

$$\rho_B^l \circ F(\bar{x}) \leq \rho_B^l \circ F(x_n) \leq \rho_B^l \circ (F(\bar{x}) + d_n) \leq \rho_B^l \circ F(\bar{x}) + \|d_n\|.$$

So, $\rho_B^l \circ F(x_n) \rightarrow \rho_B^l \circ F(\bar{x})$. It follows that $x_n \rightarrow \bar{x}$ because (OP_B) is LP well-posed. □

Theorem 5.6. Let $\bar{x} \in l\text{-Min}(F, M)$. If $M = \text{Dom}(F)$, $D + D(0) = D$ and $\rho_{F(\bar{x})}^l \circ F(\bar{x}) = 0$, then $(\text{OP}_{F(\bar{x})})$ is LP well-posed iff **(SOP)** is LP well-posed at \bar{x} .

Proof. By the definition of $\rho_{F(\bar{x})}^l$, we have $\rho_{F(\bar{x})}^l \circ F(x) \geq 0, \forall x \in M$. Since $\rho_{F(\bar{x})}^l \circ F(\bar{x}) = 0$, then $\bar{x} \in \text{argmin}(\rho_{F(\bar{x})}^l \circ F, M)$ and $\inf_{x \in M} \rho_{F(\bar{x})}^l \circ F(x) = 0$. Suppose that $(\text{OP}_{F(\bar{x})})$ is LP well-posed, then $\text{argmin}(\rho_{F(\bar{x})}^l \circ F, M) = \{\bar{x}\}$. So, according to Theorem 5.5, **(SOP)** is LP well-posed at \bar{x} .

Conversely, suppose that **(SOP)** is LP well-posed at \bar{x} . We first show that $\text{argmin}(\rho_{F(\bar{x})}^l \circ F, M) = \{\bar{x}\}$. Let $e \in \text{int}D$. If there exists a $\hat{x} \in M \setminus \{\bar{x}\}$ such that $\rho_{F(\bar{x})}^l \circ F(\hat{x}) = 0$, then, by Proposition 4.2 (v) and $D + D(0) = D$, we have $F(\bar{x}) \subseteq cl(F(\hat{x}) + D) \subseteq F(\hat{x}) + D - \varepsilon e + D(0) = F(\hat{x}) - \varepsilon e + D$ for all $\varepsilon > 0$. Let $x_n = \hat{x}$ and $\{\varepsilon_n\} \subseteq \mathbb{R}_+ \setminus \{0\}$ such that $\varepsilon_n \rightarrow 0$. It holds that $F(x_n) \leq^l F(\bar{x}) + \varepsilon_n e, \forall n \in \mathbb{N}$, that is to say $\{x_n\}$ is a LP-minimizing sequence for **(SOP)** at \bar{x} . Since **(SOP)** is LP well-posed at \bar{x} , then $x_n \rightarrow \bar{x}$, a contradiction. Now suppose that $\{x_n\} \subseteq X, d(x_n, M) \rightarrow 0$ and $\rho_{F(\bar{x})}^l \circ F(x_n) \rightarrow 0$. It is clear that $\rho_{F(\bar{x})}^l \circ F(x_n) < \varepsilon_n := \rho_{F(\bar{x})}^l \circ F(x_n) + \frac{1}{n}$. Taking into account Proposition 4.2 (vi) and $-\varepsilon_n e + \text{int}D(0)$ is a neighbourhood of zero, then, for some $t_n > 0$, it holds that

$$F(\bar{x}) \subseteq F(x_n) + N_{\varepsilon_n}(0) + D \subseteq F(x_n) + t_n(-\varepsilon_n e + \text{int}D(0)) + D \subseteq F(x_n) - t_n \varepsilon_n e + D.$$

Since $\varepsilon_n \rightarrow 0$, we can choose a bounded sequence $\{t_n\}$ such that $d_n := t_n \varepsilon_n e \in \text{int}D(0)$ and $d_n \rightarrow 0$. Therefore $F(x_n) \leq^l F(\bar{x}) + d_n$. So $\{x_n\}$ is a LP-minimizing sequence for (SOP) at \bar{x} . Thus $x_n \rightarrow \bar{x}$ by the assumption, which means (OP_{F(̄x)}) is LP well-posed. □

Proposition 5.7. Suppose that F is D -closed-valued and upper D -semicontinuous on a compact set M . If $M = \text{Dom}(F)$ and $\rho_{F(\bar{x})}^l \circ F(\bar{x}) = 0$, then (SOP) is LP well-posed at $\bar{x} \in l\text{-SMin}(F, M)$.

Proof. According to Proposition 4.4 (ii), $\rho_{F(\bar{x})}^l \circ F$ is lower semicontinuous on M . Hence, by Theorem 5.2, (OP_{F(̄x)}) is generalized LP well-posed. Moreover, according to Theorem 4.5 (ii), $\text{argmin}(\rho_{F(\bar{x})}^l \circ F, M) = \{\bar{x}\}$. So (OP_{F(̄x)}) is LP well-posed. Taking into account Theorem 5.5, (SOP) is LP well-posed at \bar{x} . □

Proposition 5.8. Let $e \in \text{int}D(0)$. If $D + D(0) = D$, then

$$\inf_{\alpha > 0} \text{diam}L_M(\bar{x}, d, \alpha) = 0, \forall d \in D(0) \Leftrightarrow \inf_{\alpha > 0} \text{diam}L_M(\bar{x}, e, \alpha) = 0.$$

Proof. It is clear that $L_M(\bar{x}, e, \alpha) \subseteq \bigcup_{d \in D(0)} L_M(\bar{x}, d, \alpha)$. Let $d \in D(0)$. Since $-e + \text{int}D(0)$ is a neighbourhood of zero, then there exists $t > 0$ such that $-d \in t(-e + \text{int}D(0))$. By $D + D(0) = D$, we have

$$\begin{aligned} F(x) \leq^l F(\bar{x}) + \alpha d &\Leftrightarrow F(\bar{x}) + \alpha d \subseteq F(x) + D \Leftrightarrow F(\bar{x}) \subseteq F(x) - \alpha d + D \\ &\Rightarrow F(\bar{x}) \subseteq F(x) - t\alpha e + \text{int}D(0) + D \Rightarrow F(\bar{x}) \subseteq F(x) - t\alpha e + D \\ &\Leftrightarrow F(x) \leq^l F(\bar{x}) + t\alpha e. \end{aligned}$$

So, $L_M(\bar{x}, d, \alpha) \subseteq L_M(\bar{x}, t\alpha e, \alpha) = L_M(\bar{x}, e, t\alpha)$. This completes the proof. □

For any $\alpha > 0$, define the LP α -approximating solution set for (OP) as follows:

$$\alpha\text{-argmin}(f, M) := \{x \in X : d(x, M) \leq \alpha, f(x) \leq \inf_{x \in M} f(x) + \alpha\}.$$

Theorem 5.9. [29] Assume that $\text{argmin}(f, M) \neq \emptyset$. If $\inf_{\alpha > 0} \text{diam}(\alpha\text{-argmin}(f, M)) = 0$, then (OP) is LP well-posed.

The following theorem gives a full characterization of strongly DH-well-posedness for (SOP).

Theorem 5.10. Let $\bar{x} \in l\text{-Min}(F, M)$. If $D + D(0) = D$ and $\rho_{F(\bar{x})}^l \circ F(\bar{x}) = 0$, then (SOP) is strongly DH-well-posed at \bar{x} iff $\inf_{\alpha > 0} \text{diam}(\alpha\text{-argmin}(\rho_{F(\bar{x})}^l \circ F, M)) = 0$.

Proof. It clear that $\rho_{F(\bar{x})}^l \circ F(x) \geq 0, \forall x \in M$. Since $\rho_{F(\bar{x})}^l \circ F(\bar{x}) = 0$, then $\inf_{x \in M} \rho_{F(\bar{x})}^l \circ F(x) = 0$. Let $e \in \text{int}D(0)$ such that $\|e\| = 1$ and $\alpha > 0$. By Proposition 4.2 (ii), we have

$$\begin{aligned} L_M(\bar{x}, e, \alpha) &= \{x \in X : d(x, M) \leq \alpha, F(x) \leq^l F(\bar{x}) + \alpha e\} \\ &\subseteq \{x \in X : d(x, M) \leq \alpha, \rho_{F(\bar{x})}^l \circ F(x) \leq \rho_{F(\bar{x})}^l \circ F(\bar{x}) + \alpha \|e\|\} \\ &= \alpha\text{-argmin}(\rho_{F(\bar{x})}^l \circ F, M). \end{aligned}$$

On the other hand, since $-\alpha e + \text{int}D(0)$ is a neighbourhood of zero, then there exists $t > 0$ such that $\bar{N}_\alpha(0) \subseteq t(-\alpha e + \text{int}D(0))$. So, by Proposition 4.2 (v) and Lemma 2.8 (i), we have

$$\begin{aligned} \alpha\text{-argmin}(\rho_{F(\bar{x})}^l \circ F, M) &= \{x \in X : d(x, M) \leq \alpha, \rho_{F(\bar{x})}^l \circ F(x) \leq \alpha\} \\ &\subseteq \{x \in X : d(x, M) \leq \alpha, F(\bar{x}) \subseteq \text{cl}(F(x) + D) + \bar{N}_\alpha(0)\} \\ &\subseteq \{x \in X : d(x, M) \leq \alpha, F(\bar{x}) \subseteq F(x) - \varepsilon \alpha e + D + \bar{N}_\alpha(0)\} \\ &\subseteq \{x \in X : d(x, M) \leq \alpha, F(\bar{x}) \subseteq F(x) - (t + \varepsilon)\alpha e + D + \text{int}D(0)\} \\ &\subseteq \{x \in X : d(x, M) \leq \alpha, F(\bar{x}) \subseteq F(x) - (t + \varepsilon)\alpha e + D\} \\ &= \{x \in X : d(x, M) \leq \alpha, F(x) \leq^l F(\bar{x}) + t_1 \alpha e\} \\ &= L_M(\bar{x}, e, t_1 \alpha), \end{aligned}$$

where $\varepsilon > 0$ and $t_1 := t + \varepsilon$. So $L_M(\bar{x}, e, \alpha) \subseteq \alpha\text{-argmin}(\rho_{F(\bar{x})}^l \circ F, M) \subseteq L_M(\bar{x}, e, t_1\alpha)$. Taking into account Proposition 5.8, this completes the proof. \square

Proposition 5.11. *Let $\bar{x} \in M$. If $\{x \in M : \rho_{F(\bar{x})}^l \circ F(x) = 0\} \neq \emptyset$, then $(\mathbf{OP}_{F(\bar{x})})$ is LP well-posed iff $\inf_{\alpha > 0} \text{diam}(\alpha\text{-argmin}(\rho_{F(\bar{x})}^l \circ F, M)) = 0$.*

Proof. Since $\{x \in M : \rho_{F(\bar{x})}^l \circ F(x) = 0\} \neq \emptyset$, then $\text{argmin}(\rho_{F(\bar{x})}^l \circ F, M) \neq \emptyset$ and $\inf_{x \in M} \rho_{F(\bar{x})}^l \circ F(x) = 0$. Suppose that $\inf_{\alpha > 0} \text{diam}(\alpha\text{-argmin}(\rho_{F(\bar{x})}^l \circ F, M)) = 0$. Then, according to Theorem 5.9, $(\mathbf{OP}_{F(\bar{x})})$ is LP well-posed. Conversely, suppose that $\inf_{\alpha > 0} \text{diam}(\alpha\text{-argmin}(\rho_{F(\bar{x})}^l \circ F, M)) > 0$. Then there exists $\varepsilon > 0$ such that $\inf_{\alpha > 0} \text{diam}(\alpha\text{-argmin}(\rho_{F(\bar{x})}^l \circ F, M)) > \varepsilon$. Let $\alpha > 0$ be arbitrary. It is possible to find some $x_\alpha, y_\alpha \in \alpha\text{-argmin}(\rho_{F(\bar{x})}^l \circ F, M)$ such that $d(x_\alpha, y_\alpha) > \varepsilon$. This implies that

$$d(x_\alpha, M) \leq \alpha, d(y_\alpha, M) \leq \alpha, 0 \leq \rho_{F(\bar{x})}^l \circ F(x_\alpha) \leq \alpha, 0 \leq \rho_{F(\bar{x})}^l \circ F(y_\alpha) \leq \alpha.$$

Setting $\alpha \rightarrow 0$, by LP well-posedness of $(\mathbf{OP}_{F(\bar{x})})$, there exists a $\hat{x} \in M$ such that $\text{argmin}(\rho_{F(\bar{x})}^l \circ F, M) = \{\hat{x}\}$ and $x_\alpha \rightarrow \hat{x}, y_\alpha \rightarrow \hat{x}$, a contradiction. \square

Corollary 5.12. *Let $\bar{x} \in l\text{-Min}(F, M)$. If $D + D(0) = D$ and $\rho_{F(\bar{x})}^l \circ F(\bar{x}) = 0$, then (\mathbf{SOP}) is strongly DH-well-posed at \bar{x} iff $(\mathbf{OP}_{F(\bar{x})})$ is LP well-posed.*

Proof. It follows from Theorem 5.10 and Proposition 5.11. \square

Let $\bar{x} \in M$, define the set-valued maps $H, G : \mathbb{R}_+ \rightrightarrows X$ as follows:

$$H(\alpha) = \alpha\text{-argmin}(f, M), \forall \alpha > 0, H(0) = \text{argmin}(f, M),$$

$$G(\alpha) = \alpha\text{-argmin}(\rho_{F(\bar{x})}^l \circ F, M), \forall \alpha > 0, G(0) = \text{argmin}(\rho_{F(\bar{x})}^l \circ F, M).$$

Next, we give some characterizations of strongly B-well-posedness for (\mathbf{SOP}) .

Theorem 5.13. *Let $\bar{x} \in l\text{-Min}(F, M)$. If $D + D(0) = D$ and $\rho_{F(\bar{x})}^l \circ F(\bar{x}) = 0$, then (\mathbf{SOP}) is strongly B-well-posed at \bar{x} iff $G(\alpha)$ is upper semicontinuous at 0.*

Proof. Let $e \in \text{int}D(0)$ such that $\|e\| = 1$. From the proof of Theorem 5.10, there exists $t_1 > 0$ such that $\Phi_{\bar{x}}(\alpha) \subseteq G(\alpha), G(\alpha) \subseteq \Phi_{\bar{x}}(t_1\alpha)$. This completes the proof. \square

Theorem 5.14. [29] *(\mathbf{OP}) is generalized LP well-posed iff $H(\alpha)$ is upper semicontinuous and compact at 0.*

Corollary 5.15. *Let $D + D(0) = D$ and $\bar{x} \in l\text{-Min}(F, M)$ such that $\rho_{F(\bar{x})}^l \circ F(\bar{x}) = 0$. If $(\mathbf{OP}_{F(\bar{x})})$ is LP well-posed, then (\mathbf{SOP}) is strongly B-well-posed at \bar{x} . The converse is true if F is D-closed-valued on M and $\bar{x} \in l\text{-SMin}(F, M)$.*

Proof. The first part of the result follows from Theorems 5.13 and 5.14. The second part of the result follows from Theorems 4.5, 5.13 and 5.14. \square

Remark 5.16. *According to Remark 2.3, we can see that Theorems 5.5, 5.6, 5.10 and 5.13 are generalizations of Theorems 5.1, 5.2, 5.3 and 5.4 in [16] respectively.*

We conclude this section by giving an example to illustrate the effectiveness of Theorems 5.6, 5.10 and 5.13.

Example 5.17. *Consider (\mathbf{SOP}) with $X = Y = \mathbb{R}, M = [0, 1], D = [1, +\infty)$ and*

$$F(x) = \begin{cases} (-\infty, 0] & x = 0, \\ (0, 1) & 0 < x \leq 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\bar{x} = 0$. It is not difficult to check that $\bar{x} \in l\text{-Min}(F, M)$, $M = \text{Dom}(F)$, $D + D(0) = D$,

$$\rho_{F(\bar{x})}^l \circ F(x) = \begin{cases} 0 & x = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

and for all $\alpha \geq 0$, $\alpha\text{-argmin}(\rho_{F(\bar{x})}^l \circ F, M) = \{0\}$.

(i) According to Definition 5.1, it is easy to see that $(\mathbf{OP}_{F(\bar{x})})$ is LP well-posed, so by Theorem 5.6, (\mathbf{SOP}) is LP well-posed at \bar{x} .

(ii) Since $\inf_{\alpha > 0} \text{diam}(\alpha\text{-argmin}(\rho_{F(\bar{x})}^l \circ F, M)) = 0$, then by Theorem 5.10, (\mathbf{SOP}) is strongly DH -well-posed at \bar{x} .

(iii) As $G(\alpha) = \alpha\text{-argmin}(\rho_{F(\bar{x})}^l \circ F, M) \equiv \{0\}$ is upper semicontinuous at 0, so by Theorem 5.13, (\mathbf{SOP}) is strongly B -well-posed at \bar{x} .

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