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Infinite system of Integral Equations in Two Variables of Hammerstein Type in c_0 and ℓ_1 spaces

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Abstract. The principal aim of this paper is to study the solvability of infinite system of integral equations in two variables of Hammerstein type in the Banach spaces c_0 and ℓ_1 using Meir-Keeler condensing operators and measure of noncompactness. In this study we give some examples.

1. Introduction

The theory of integral equations is an important branch of nonlinear functional analysis and has attracted the interest of many researchers. In 1895 Le Roux [16] introduced integral equations as a powerful tool in investigating partial differential equations. This theory has many applicabilities like in population dynamics, Economic theory, feedback systems, stability of nuclear reactors [9, 10, 20]. In this paper, our aim is to study the infinite system of Hammerstein type integral equations in two variables of the form

$$v_n(s,t) = \mathsf{r}_n(s,t) + \int_a^b \int_a^b K_n(s,t,\tau_1,\tau_2) f_n(\tau_1,\tau_2,v(\tau_1,\tau_2)) d\tau_1 d\tau_2 \tag{1}$$

where $(s,t) \in [a,b] \times [a,b]$ in the Banach spaces c_0 and ℓ_1. The solvability of (1) is studied using the idea of measure of noncompactness (MNC).

Introduced in 1930 by Kuratowski [15] the concept of measure of noncompactness was further extendend to general Banach space by Banaś and Goebel [6]. The classical Schauder fixed point theorem and Banach contraction principle were generalized by Darbo [11] for condensing operators using the idea of MNC. The method of fixed point arguments has been widely used to study the existence of solutions of functional equations, like Banach contraction principle in [1, 18] and Schauder's fixed point theorem in [14, 17]. But if compactness and Lipschitz condition are not satisfied these results can not be used.

The idea of MNC has been studied by many researcher and applied in various problems. Many properties of MNC in different sequence spaces can be found in [1, 8]. Different types of infinite systems of integral equations in two variables had been studied in [4, 5, 12, 13] by making use of MNC.

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2. Preliminaries

Notation $\mathbb{N}$ is used for set natural numbers, $\mathbb{R}$ is used for set of real numbers and $\mathbb{R}_+$ is used for set of positive real numbers, interval $[a,b]$ is denoted by I. By $C(I^2,\mathbb{R})$ we denote the space of continuously differentiable functions on $I^2 = [a,b]\times[a,b]$. The Hausdorff measure of noncompactness is used frequently in finding the existence of solutions of various functional equations and is defined as:

Definition 2.1. *[6] Let (Ω,d) be a metric space and A be a bounded subset of Ω. Then the* Hausdorff measure of noncompactness *(the ball-measure of noncompactness) of the set A, denoted by $\chi(A)$ is defined to be the infimium of the set of all real $\epsilon > 0$ such that A can be covered by a finite number of balls of radii $< \epsilon$, that is*

$$\chi(\varepsilon) = \inf\left\{\epsilon > 0 : A \subset \bigcup_{i=1}^{n} \bar{B}(x_i,R_i), x_i \in \Omega, R_i < \epsilon\ (i = 1,\ldots,n)\ , n \in \mathbb{N}\right\},$$

where $\bar{B}(x_i,R_i)$ denotes ball of radius R_i centered at $x_i \in A$.

Let $(\mathsf{X},\|\cdot\|)$ be a Banach space, for any $E \subset \mathsf{X}$, $\bar{E}$ denotes closure of E and $conv(E)$ denotes the closed convex hull of E. We denote the family of non-empty bounded subsets of X by M_X and family of non-empty and relatively compact subsets of X by N_X. The axiomatic definition of measures of noncompactness is

Definition 2.2. *[8] A mapping $\mu : \mathsf{M}_\mathsf{X} \to \mathbb{R}_+$ is said to be the measure of noncompactness if the following conditions hold:*

- *(i) The family Ker $\mu = \{E \in \mathsf{M}_\mathsf{X} : \mu(E) = 0\}$ is non-empty and Ker $\mu \subset \mathsf{N}_\mathsf{X}$;*
- *(ii) $E_1 \subset E_2 \Rightarrow \mu(E_1) \le \mu(E_2)$;*
- *(iii) $\mu(\bar{E}) = \mu(E)$;*
- *(iv) $\mu(convE) = \mu(E)$;*
- *(v) $\mu\left[\lambda E_1 + (1-\lambda)E_2\right] \le \lambda\mu(E_1) + (1-\lambda)\mu(E_2)$ for $0 \le \lambda \le 1$;*
- *(vi) If (E_n) is a sequence of closed sets from M_X such that $E_{n+1} \subset E_n$ and $\lim_{n\to\infty}\mu(E_n) = 0$ then the intersection set $E_\infty = \bigcap_{n=1}^{\infty} E_n$ is non-empty.*

Definition 2.3. *[3] Let X_1 and X_2 be two Banach spaces and μ_1 and μ_2 be the measures of noncompactness on X_1 and X_2 respectively. An operator T from X_1 to X_2 is called a $(\mu_1-\mu_2)$ condensing operator if it is continuous and $\mu_2(T(E)) < \mu_1(E)$ for every bounded noncompact set $E \subset \mathsf{X}_1$.*
If $\mathsf{X}_1 = \mathsf{X}_2$ and $\mu_1 = \mu_2 = \mu$ then T is called μ-condensing operator.

Lemma 2.4. *[11, Darbo's fixed point theorem] Let E be a non-empty, bounded, closed, and convex subset of Banach space X and let $T : E \to E$ be a continuous mapping. Assume that there exists a constant $k \in [0,1)$ such that $\mu(T(E)) \le k\mu(E)$. Then T has a fixed point in the set E.*

We know state the fixed point theorem for Meir-Keeler condensing operator which we use in this paper to obtain the main results

Definition 2.5. *[2] Let E be a non-empty subset of a Banach space X and μ be an arbitrary measure of noncompactness on E. An operator $T : E \to E$ is a Meir-Keeler condensing operator if for any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\epsilon \le \mu(E) < \epsilon + \delta \Longrightarrow \mu\left[T(E)\right] < \epsilon$$

for any bounded subset E of X.

Theorem 2.6. *[2] Let E be a non-empty, bounded, closed and convex subset of a Banach space X and let μ be an arbitrary measure of noncompactness on X. If $T : E \to E$ is a continuous and Meir-Keeler condensing operator, then T has at least one fixed point and the set of all fixed points of T in E is compact.*

In order to apply Lemma 2.4 in a given Banach space X, we need a formula expressing the measure of noncompactness by a simple formula. Such formulas are known only in a few spaces [6, 8].

The c_0 sequence space is the set of all sequences converging to 0. Norm $\|\cdot\|_{c_0}$, on c_0 is defined as

$$\|(a_k)\|_{c_0} = \sup_{k\geq 1}\{|a_k|\} \ , \ (a_k) \in c_0.$$

Under the norm $\|\cdot\|_{c_0}$, c_0 is a Banach space, and the Hausdorff measure of noncompactness in c_0 is given by

$$\chi(E) = \lim_{n\to\infty}\left\{\sup_{u\in E}\left(\max_{k\geq n}|u_k|\right)\right\} \tag{2}$$

where $u = \left(u_j\right)_{j=1}^{\infty} \in c_0$ and $E \in \mathsf{M}_{c_0}$.

The ℓ_1 sequence space is the set of all sequences whose series is absolutely convergent. Norm $\|\cdot\|_{\ell_1}$ on ℓ_1 is defined as

$$\|(a_k)\|_{\ell_1} = \sum_{k=1}^{\infty}|a_k| \ , \ (a_k) \in \ell_1.$$

Under the norm $\|\cdot\|_{\ell_1}$, ℓ_1 is a Banach space, and the Hausdorff measure of noncompactness in ℓ_1 is given by

$$\chi(E) = \lim_{n\to\infty}\left\{\sup_{u\in E}\left(\sum_{k\geq n}|u_k|\right)\right\} \tag{3}$$

where $u = \left(u_j\right)_{j=1}^{\infty} \in \ell_1$ and $E \in \mathsf{M}_{\ell_1}$.
The above formulas will be used in the sequel of the paper.

3. Solution in c_0 space

In order to find the condition under which the system (1) has a solution in c_0 we need the following assumptions:

(A_1) Functions $(f_j)_{j=1}^{\infty}$ are real valued and continuous defined on the set $I^2 \times \mathbb{R}^{\infty}$. The operator Q defined on the space $I^2 \times c_0$ as

$$(s,t,v) \mapsto (\mathsf{Q}v)\,(s,t) = (f_1(s,t,v), f_2(s,t,v), f_3(s,t,v), \ldots)$$

maps $I^2 \times c_0$ into c_0. The set of all such functions $\{(\mathsf{Q}v)\,(s,t)\}_{(s,t)\in I^2}$ is equicontinuous at every point of the space c_0, that is given $\epsilon, \delta > 0$

$$\|u - v\|_{c_0} \leq \delta \Longrightarrow \|(\mathsf{Q}u)(s,t) - (\mathsf{Q}v)(s,t)\|_{c_0} \leq \epsilon.$$

(A_2) For each fixed $(s,t) \in I^2$, $v(s,t) = \left(v_j(s,t)\right) \in C(I^2, c_0)$, the following inequality holds

$$\left|f_n(s,t,v(s,t))\right| \leq p_n(s,t) + q_n(s,t)\sup_{j\geq n}\left\{|v_j|\right\} \qquad n \in \mathbb{N},$$

where $p_j(s,t)$ and $q_j(s,t)$ are real valued continuous functions on I^2. The function sequence $\left(q_j(s,t)\right)_{j\in\mathbb{N}}$ is equibounded on I^2 and the function sequence $\left(p_j(s,t)\right)_{j\in\mathbb{N}}$ converges uniformly on I^2 to a function vanishing identically on I^2.

(A_3) The functions $K_n : I^4 \to \mathbb{R}$ are continuous on $I^4, (n = 1, 2, \ldots)$. Also these functions $K_n(s, t, x, y)$ are equicontinuous with respect to (s, t) that is, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|K_n(s_2, t_2, x, y) - K_n(s_1, t_1, x, y)| \le \epsilon \text{ whenever } |s_2 - s_1| \le \delta, |t_2 - t_1| \le \delta,$$

for all $(x, y) \in I^2$. Also the function sequence $(K_n(s, t, x, y))$ is equibounded on the set I^4 and the constant K defined as

$$\mathsf{K} = \sup\left\{|K_n(s, t, x, y)| : (s, t), (x, y) \in I^2, n = 1, 2, \ldots\right\},$$

is finite.

(A_4) Functions $\mathsf{r}_n : I^2 \to \mathbb{R}$ are continuous and the function sequence (r_n) is uniformly convergent to zero on I^2. The constant R defined as

$$\mathsf{R} = \sup\left\{|\mathsf{r}_n(s, t)| : (s, t) \in I^2 : n = 1, 2, \ldots\right\},$$

is finite.

Keeping assumption (A_2) under consideration we define the following finite constants as

$$\mathcal{Q} = \sup\left\{q_n(s, t) : (s, t) \in I^2, n \in \mathbb{N}\right\},$$

$$\mathcal{P} = \sup\left\{p_n(s, t) : (s, t) \in I^2, n \in \mathbb{N}\right\}.$$

Theorem 3.1. *Under assumptions (A_1) – (A_4), the infinite system of integral equations* (1) *has at least one solution* $v(s, t) = \left(v_j(s, t)\right)_{j\in\mathbb{N}}$ *in* c_0 *for fixed* $(s, t) \in I^2$, *whenever* $(b - a)^2\mathsf{K}\mathcal{Q} < 1$.

Proof. We define the operator F on the space $\Gamma = C(I^2, c_0)$ by

$$\begin{aligned}(\mathsf{F}v)(s, t) = ((\mathsf{F}v)_n(s, t)) &= \mathsf{r}_n(s, t) + \int_a^b \int_a^b K_n(s, t, \tau_1, \tau_2) f_n(\tau_1, \tau_2, v(\tau_1, \tau_2)) d\tau_1 d\tau_2 \\ &= \Big(\mathsf{r}_1(s, t) + \int_a^b \int_a^b K_1(s, t, \tau_1, \tau_2) f_1(\tau_1, \tau_2, v_1(\tau_1, \tau_2), v_2(\tau_1, \tau_2), \ldots) d\tau_1 d\tau_2\,, \\ &\quad \mathsf{r}_2(s, t) + \int_a^b \int_a^b K_2(s, t, \tau_1, \tau_2) f_2(\tau_1, \tau_2, v_1(\tau_1, \tau_2), v_2(\tau_1, \tau_2), \ldots) d\tau_1 d\tau_2\,, \ldots\Big),\end{aligned} \tag{4}$$

for all $(s, t) \in I^2$ and $v = \left(v_j\right)_{j\in\mathbb{N}} \in c_0$. We first show that F maps the space Γ into itself. Let $n \in \mathbb{N}$ and $(s, t) \in I^2$ then using assumptions (A_1) and (A_3) we have

$$\begin{aligned}|(\mathsf{F}v)_n(s, t)| &\le |\mathsf{r}_n(s, t)| + \left|\int_a^b \int_a^b K_n(s, t, \tau_1, \tau_2) f_n(\tau_1, \tau_2, v(\tau_1, \tau_2)) d\tau_1 d\tau_2\right| \\ &\le |\mathsf{r}_n(s, t)| + \mathsf{K} \int_a^b \int_a^b |f_n(\tau_1, \tau_2, v(\tau_1, \tau_2))| d\tau_1 d\tau_2.\end{aligned}$$

Thus, by assumption (A_4) and the fact that $(f_n(s, t, v(s, t)))$ is in c_0 space we have

$$\lim_{n\to\infty} (|(\mathsf{F}v)_n(s, t)|) = 0.$$

Hence, $(\mathsf{F}v)(s,t) \in c_0$ for any arbitrarily fixed $(s,t) \in I^2$.
Then we have,

$$\begin{aligned}
\|v(s,t)\|_{c_0} &= \max_{n\geq 1}\left| \mathsf{r}_n(s,t) + \int_a^b\int_a^b K_n(s,t,\tau_1,\tau_2) f_n(\tau_1,\tau_2,v(\tau_1,\tau_2))d\tau_1 d\tau_2\right| \\
&\leq \max_{n\geq 1}|\mathsf{r}_n(s,t)| + \left|\int_a^b\int_a^b K_n(s,t,\tau_1,\tau_2) f_n(\tau_1,\tau_2,v(\tau_1,\tau_2))d\tau_1 d\tau_2\right| \\
&\leq \mathsf{R} + \max_{n\geq 1}\int_a^b\int_a^b |K_n(s,t,\tau_1,\tau_2)|\left| f_n(\tau_1,\tau_2,v(\tau_1,\tau_2))\right| d\tau_1 d\tau_2 \\
&\leq \mathsf{R} + \mathsf{K}\int_a^b\int_a^b \max_{n\geq 1}\left(p_n(\tau_1,\tau_2) + q_n(\tau_1,\tau_2)\sup_{j\geq n}\left\{|v_j(\tau_1,\tau_2)|\right\}\right) d\tau_1 d\tau_2 \\
&\leq \mathsf{R} + \mathsf{K}(b-a)^2[\mathcal{P} + \mathcal{Q}\|v(t,s)\|_{c_0}].
\end{aligned}$$

So we have

$$\begin{aligned}
[1-(b-a)^2\mathsf{K}\mathcal{Q}]\|v(t,s)\|_{c_0} &\leq \mathsf{R} + (b-a)^2\mathsf{K}\mathcal{P} \\
\|v(t,s)\|_{c_0} &\leq \frac{\mathsf{R} + (b-a)^2\mathsf{K}\mathcal{P}}{[1-(b-a)^2\mathsf{K}\mathcal{Q}]}\ (= R_0).
\end{aligned} \tag{5}$$

Therefore, using (4) we conclude that F is a self mapping on Γ.
Also $\|(\mathsf{F}v)(s,t) - 0\| \leq R_0$, so the operator F maps B_{R_0} (ball of radius R_0 centered at origin) in Γ into itself.
We now show that F is continuous on B_{R_0}. To do so fix $\epsilon > 0$ and $v \in \mathsf{B}_{R_0}$. Then for arbitrary $u \in \mathsf{B}_{R_0}$ with $\|u - v\| \leq \epsilon$, arbitrary fixed $(s,t) \in I^2$ and $n \in \mathbb{N}$. We have

$$\begin{aligned}
&|(\mathsf{F}u)(s,t) - (\mathsf{F}v)(s,t)| \\
&= \left|\int_a^b\int_a^b K_n(s,t,\tau_1,\tau_2)\left[f_n(\tau_1,\tau_2,u(\tau_1,\tau_2))d\tau_1 d\tau_2 - f_n(\tau_1,\tau_2,v(\tau_1,\tau_2))\right]d\tau_1 d\tau_2\right| \\
&\leq \int_a^b\int_a^b |K_n(s,t,\tau_1,\tau_2)|\left|f_n(\tau_1,\tau_2,u(\tau_1,\tau_2))d\tau_1 d\tau_2 - f_n(\tau_1,\tau_2,v(\tau_1,\tau_2))\right|d\tau_1 d\tau_2
\end{aligned} \tag{6}$$

Now, using the assumptions (A_1), define the set $\delta(\epsilon)$ as

$$\delta(\epsilon) = \sup\left\{|f_n(s,t,u) - f_n(s,t,v)| : u,v \in c_0, \|u-v\|_{c_0} < \epsilon, (s,t) \in I^2, n = 1,2,\ldots\right\}$$

Then, $\delta(\epsilon) \to 0$ as $\epsilon \to 0$.
So, by (6) and assumption (A_3) we have

$$|(\mathsf{F}u) - (\mathsf{F}v)| \leq (b-a)^2\mathsf{K}\delta(\epsilon).$$

Hence, F is a continuous operator on B_{R_0}. We now show that F is a Meir-Keeler condensing operator, that is given $\epsilon > 0$ and $\delta > 0$

$$\epsilon \leq \chi(\mathsf{B}_{R_0}) < \epsilon + \delta \implies \chi(\mathsf{F}(\mathsf{B}_{R_0})) < \epsilon.$$

Using the definition of measure of noncompactness in c_0 (2) and the assumptions (A_2), (A_3) and (A_3) we have

$$\begin{aligned}
\chi\left(\mathsf{F}\left(\mathsf{B}_{R_0}\right)\right) &= \lim_{n\to\infty}\left[\sup_{v(s,t)\in\mathsf{B}_{R_0}}\left\{\max_{k\geq n}|v_k(t,s)|\right\}\right]\\
&\leq \lim_{n\to\infty}\left[\sup_{v(s,t)\in\mathsf{B}_{R_0}}\left\{\max_{k\geq n}\left(|\mathsf{r}_k(s,t)|+\left|\int_a^b\int_a^b K_k(s,t,\tau_1,\tau_2)f_k(\tau_1,\tau_2,v(\tau_1,\tau_2))d\tau_1 d\tau_2\right|\right)\right\}\right]\\
&\leq \lim_{n\to\infty}\left[\sup_{v(s,t)\in\mathsf{B}_{R_0}}\left\{\max_{k\geq n}\left(\mathsf{K}\int_a^b\int_a^b\left|f_k(\tau_1,\tau_2,v(\tau_1,\tau_2))\right|d\tau_1 d\tau_2\right)\right\}\right]\\
&\leq \lim_{n\to\infty}\left[\sup_{v(s,t)\in\mathsf{B}_{R_0}}\left\{\max_{k\geq n}\left(\mathsf{K}\int_a^b\int_a^b\left(p_k(\tau_1,\tau_2)+q_k(\tau_1,\tau_2)\sup_{j\geq k}\left\{|v_j(\tau_1,\tau_2)|\right\}\right)d\tau_1 d\tau_2\right)\right\}\right]\\
&\leq (b-a)^2\mathsf{K}Q\,\chi\left(\mathsf{B}_{R_0}\right).
\end{aligned}$$

Thus,

$$\chi\left(\mathsf{F}\left(\mathsf{B}_{R_0}\right)\right)\leq (b-a)^2\mathsf{K}Q\,\chi\left(\mathsf{B}_{R_0}\right)<\epsilon,\Longrightarrow \chi\left(\mathsf{B}_{R_0}\right)<\frac{\epsilon}{(b-a)^2\mathsf{K}Q}.$$

Taking, $\delta=\dfrac{\epsilon(1-(b-a)^2\mathsf{K}Q)}{(b-a)^2\mathsf{K}Q}$ we obtain $\epsilon\leq\chi\left(\mathsf{B}_{R_0}\right)<\epsilon+\delta$. Therefore F is a Meir-Keeler condensing operator on $\mathsf{B}_{R_0}\subset c_0$. Since F satisfies Theorem 2.6, F has a fixed point in B_{R_0}. Therefore the system (1) has a solution in c_0. □

Example 3.2. *Consider the following infinite system of Hammerstein type integral equations in two variables*

$$\begin{aligned}
v_n(s,t) = \frac{1}{n}\arctan(s+t)^n + \int_1^2\int_1^2 &\sin\left(\frac{s+t+\tau_1+\tau_2}{n}\right)\\
&\ln\left(\frac{1+4n^2+(\tau_1+\tau_2)^2[4+\sup_{k\geq n}\{|v_k(\tau_1,\tau_2)|\}]}{4[(\tau_1+\tau_2)^2+n^2]}\right)d\tau_1 d\tau_2
\end{aligned}\tag{7}$$

for $(s,t)\in[1,2]\times[1,2]$ *and* $n=1,2,\cdots$.
Comparing (7) *with* (1) *we have*

$$\begin{aligned}
\mathsf{r}_n(s,t) = \frac{1}{n}\arctan(s+t)^n\ ,\ &K_n(s,t,x,y)=\sin\left(\frac{s+t+x+y}{n}\right),\\
f_n(\tau_1,\tau_2,v(\tau_1,\tau_2)) &= \ln\left(\frac{1+4n^2+(\tau_1+\tau_2)^2[4+\sup_{k\geq n}\{|v_k(\tau_1,\tau_2)|\}]}{4[(\tau_1+\tau_2)^2+n^2]}\right)\\
&= \ln\left(1+\frac{1+(\tau_1+\tau_2)^2\sup_{k\geq n}\{|v_k(\tau_1,\tau_2)|\}}{4[(\tau_1+\tau_2)^2+n^2]}\right).
\end{aligned}$$

Denoting, by I_2 *the interval* $[1,2]$, *we show that the assumptions of the Theorem 3.1 are satisfied. It is obvious that the operator* F_1 *defined by*

$$\left(\mathsf{F}_1 v\right)(s,t)=\left(f_n(s,t,v(s,t))\right),$$

transforms the space $I_2^2\times c_0$ *into* c_0. *Next we show that the family of functions* $\{(\mathsf{F}_1 v)(s,t)\}_{(s,t)\in I_2^2}$ *is equicontinuous at* *an arbitrary point $v \in c_0$. Fix $\epsilon > 0, n \in \mathbb{N}, v \in c_0$ and $(s,t) \in I_2^2$, let $u \in c_0$ such that $\|u - v\|_{c_0} \leq \epsilon$. Then,*

$$\begin{aligned}
&\left|f_n(s,t,v) - f_n(s,t,u)\right| \\
&= \left|\ln\left(1 + \frac{1 + (\tau_1+\tau_2)^2 \sup\limits_{k\geq n}\{|v_k(\tau_1,\tau_2)|\}}{4[(\tau_1+\tau_2)^2 + n^2]}\right) - \ln\left(1 + \frac{1 + (\tau_1+\tau_2)^2 \sup\limits_{k\geq n}\{|u_k(\tau_1,\tau_2)|\}}{4[(\tau_1+\tau_2)^2 + n^2]}\right)\right| \\
&\leq \left|\frac{(\tau_1+\tau_2)^2}{4[(\tau_1+\tau_2)^2+n^2]}\left[\sup_{k\geq n}\{|v_k(\tau_1,\tau_2)|\} - \sup_{k\geq n}\{|u_k(\tau_1,\tau_2)|\}\right]\right| \\
&\leq \frac{1}{16}\sup_{k\geq n}\{|v_k - u_k|\}.
\end{aligned}$$

Hence, $\left\|f_n(s,t,v) - f_n(s,t,u)\right\| \leq \frac{1}{16}\|v-u\|_{c_0} \leq \frac{\epsilon}{16}$, so the family of functions $\{(\mathsf{F}_1 v)(s,t)\}_{(s,t)\in I_2^2}$ is equicontinuous.
Now, fix $(s,t) \in I_2^2, v \in c_0$ and $n \in \mathbb{N}$, then

$$\begin{aligned}
\left|f_n(s,t,v)\right| &= \left|\ln\left(1 + \frac{1 + (\tau_1+\tau_2)^2 \sup\limits_{k\geq n}\{|v_k(\tau_1,\tau_2)|\}}{4[(\tau_1+\tau_2)^2 + n^2]}\right)\right| \\
&\leq \frac{1 + (\tau_1+\tau_2)^2 \sup\limits_{k\geq n}\{|v_k(\tau_1,\tau_2)|\}}{4[(\tau_1+\tau_2)^2 + n^2]} \\
&= \frac{1}{4[(\tau_1+\tau_2)^2+n^2]} + \frac{(\tau_1+\tau_2)^2}{4[(\tau_1+\tau_2)^2+n^2]}\sup_{k\geq n}\{|v_k(\tau_1,\tau_2)|\}
\end{aligned}$$

Letting, $p_n(s,t) = \dfrac{1}{4[(s+t)^2+n^2]}$ and $q_n(s,t) = \dfrac{(s+t)^2}{4[(s+t)^2+n^2]}$ it is clear that $p_n(s,t)$ and $q_n(s,t)$ are real valued functions and $p_n(s,t)$ converges uniformly to zero. Also $|q_n(s,t)| \leq \dfrac{1}{4}$ for all $n = 1,2,\cdots$.
Hence, $\mathcal{P} = \dfrac{1}{4}$ and $\mathcal{Q} = \sup\limits_{s,t)\in I^2}\{q_n(s,t)\} = \dfrac{1}{4}$.
The functions $K_n(s,t,x,y)$ are continuous on $I_2^4 = [1,2]\times[1,2]\times[1,2]\times[1,2]$ and the function sequence $(K_n(s,t,x,y))$ is equibounded on I_2^4. Also

$$\mathsf{K} = \sup\left\{|K_n(s,t,x,y)| : (s,t),(x,y) \in I_2^2, n \in \mathbb{N}\right\} = 1.$$

Now, fix $\epsilon > 0, (x,y) \in I_2^2$ and $n \in \mathbb{N}$ then for arbitrary $(s_1,t_1),(s_2,t_2) \in I^2$ with

$$|s_2 - s_1| \leq \frac{\epsilon}{2}, |t_2 - t_1| \leq \frac{\epsilon}{2}.$$

We have

$$\begin{aligned}
\left|K_n(s_2,t_2,x,y) - K_n(s_1,t_1,x,y)\right| &\leq \left|\frac{s_2+t_1+x+y}{n} - \frac{s_1+t_1+x+y}{n}\right| \\
&= \frac{1}{n}\left|(s_2-s_1) + (t_2-t_1)\right| \\
&\leq \frac{1}{n}(|s_2-s_1| + |t_2-t_1|) \\
&\leq \epsilon.
\end{aligned}$$

Therefore, $(K_n(s,t,x,y))$ is equicontinuous.
Also, $\mathsf{r}_n(s,t)$, is continuous for all $(s,t) \in I_2^2$ and for all n and $\mathsf{r}_n(s,t)$ converges uniformly to zero.

The value of the factor $(b-a)^2\mathsf{K}Q = \dfrac{1}{4} < 1$. *Since the conditions in Theorem 3.1 are satisfied, the infinite system in* (7) *has a solution in* c_0. *This solution belongs to the ball* $\mathsf{B}_{R_0} \subset c_0$ *where*

$$R_0 = \frac{\mathsf{R} + (b-a)^2\mathsf{KQ}}{1-(b-a)^2\mathsf{K}Q} = \frac{\arctan 4 + \frac{1}{4}}{1-\frac{1}{4}} = \frac{4}{3}\arctan(4).$$

4. Solution in ℓ_1 space

In this section we consider the system of equations (1). The existence of solution for the system (1) is found in ℓ_1 space keeping the following assumptions under consideration:

(C_1) Functions $(f_j)_{j=1}^{\infty}$ are real valued and continuous defined on the set $I^2 \times \mathbb{R}^\infty$. The operator Q defined on the space $I^2 \times \ell_1$ as

$$(s,t,v) \mapsto (\mathsf{Q}v)(s,t) = (f_1(s,t,v), f_2(s,t,v), f_3(s,t,v), \ldots),$$

maps $I^2 \times \ell_1$ into ℓ_1. The set of all such functions $\{(\mathsf{Q}v)(s,t)\}_{(s,t)\in I^2}$ is equicontinuous at every point of the space ℓ_1, that is given $\epsilon, \delta > 0$

$$\|u-v\|_{\ell_1} \le \delta \Longrightarrow \|(\mathsf{Q}u)(s,t) - (\mathsf{Q}v)(s,t)\|_{\ell_1} \le \epsilon.$$

(C_2) For fixed $(s,t) \in I^2,\ v(s,t) = \left(v_j(s,t)\right) \in C(I^2, \ell_1)$, the following inequality holds

$$\left|f_n(s,t,v(s,t))\right| \le a_n(s,t) + d_n(s,t)|v_n|, \qquad n = 1,2,3,\ldots$$

where $a_j(s,t)$ and $d_j(s,t)$ are real valued continuous functions on I^2. The function series $\sum_{n=1}^{\infty} a_n(s,t)$ is uniformly convergent on I^2 and the function sequence $\left(d_j(s,t)\right)_{j\in\mathbb{N}}$ is equibounded on I^2. The function $a(s,t)$ given by $a(s,t) = \sum_{n=1}^{\infty} a_n(s,t)$ is continuous on I^2 and the constants D , A defined as

$$\mathsf{D} = \sup\left\{d_n(s,t) : (s,t) \in I^2\ ,\ n \in \mathbb{N}\right\},$$
$$\mathsf{A} = \max\left\{a(s,t) : (s,t) \in I^2\right\},$$

are finite.

(C_3) The functions $K_n : I^4 \to \mathbb{R}$ are continuous on $I^4, (n = 1,2,\ldots)$. Also these functions $K_n(s,t,x,y)$ are equicontinuous with respect to (s,t) that is for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|K_n(s_2,t_2,x,y) - K_n(s_1,t_1,x,y)| \le \epsilon \text{ whenever } |s_2-s_1| \le \delta, |t_2-t_1| \le \delta,$$

for all $(x,y) \in I^2$. Also the function sequence $(K_n(s,t,x,y))$ is equibounded on the set I^4 and the constant K defined as

$$\mathsf{K} = \sup\left\{|K_n(s,t,x,y)| : (s,t),(x,y) \in I^2, n = 1,2,\ldots\right\},$$

is finite.

(C_4) Functions $\mathsf{r}_n : I^2 \to \mathbb{R}$ are continuous and the function sequence $(\mathsf{r}_n) \in C(I^2, \ell_1)$.

Remark 4.1. *Since $I^2 = [a,b] \times [a,b]$ is a compact subset of $\mathbb{R}^2$, so the assumption of continuity in (C_4) implies that $\mathsf{r}_n : I^2 \to \mathbb{R}$ is uniformly continuous, which implies that the function sequence $(\mathsf{r}_n(s,t))$ is equicontinuous on I^2, as for every $\epsilon > 0$ there is a $\delta > 0$, such that for all $(s_1,t_1),(s_2,t_2) \in I^2$,*

$$\begin{aligned} \|(\mathsf{r}_n(s_1,t_1)) - (\mathsf{r}_n(s_2,t_2))\|_{\ell_1} &\le \sum_{n=1}^{\infty} |\mathsf{r}_n(s_2,t_2) - \mathsf{r}_n(s_2,t_2)| \\ &\le \epsilon, \end{aligned} \tag{8}$$

whenever, $|(s_1,t_1) - (s_2,t_2)| < \delta$. Also from inequality (8) it is clear that the function series $\sum_{n=1}^{\infty} \mathsf{r}_n(s,t)$ is convergent on I^2 and the function $\mathsf{r}(s,t)$ defined as

$$\mathsf{r}(s,t) = \sum_{n=1}^{\infty} \mathsf{r}_n(s,t),$$

is continuous on I^2. Further the constant given by

$$R = \max\{\mathsf{r}(s,t) : (s,t) \in I^2\},$$

is finite.

Theorem 4.2. *Under assumptions $(C_1) - (C_4)$, the infinite system of integral equations (1) has at least one solution $v(s,t) = \big(v_j(s,t)\big)_{j\in\mathbb{N}}$ in ℓ_1 for fixed $(s,t) \in I^2$, whenever $(b-a)^2\mathsf{KD} < 1$.*

Proof. We define the operator G on the space $\Gamma_1 = C(I^2, \ell_1)$ by

$$\begin{aligned} (\mathsf{G}v)(s,t) = ((\mathsf{G}v)_n(s,t)) &= \mathsf{r}_n(s,t) + \int_a^b \int_a^b K_n(s,t,\tau_1,\tau_2) f_n(\tau_1,\tau_2,v(\tau_1,\tau_2)) d\tau_1 d\tau_2 \\ &= \Big(\mathsf{r}_1(s,t) + \int_a^b \int_a^b K_1(s,t,\tau_1,\tau_2) f_1(\tau_1,\tau_2,v_1(\tau_1,\tau_2),v_2(\tau_1,\tau_2),\ldots) d\tau_1 d\tau_2\,, \\ &\quad \mathsf{r}_2(s,t) + \int_a^b \int_a^b K_2(s,t,\tau_1,\tau_2) f_2(\tau_1,\tau_2,v_1(\tau_1,\tau_2),v_2(\tau_1,\tau_2),\ldots) d\tau_1 d\tau_2\,, \ldots\Big), \end{aligned} \tag{9}$$

for all $(s,t) \in I^2$ and $v = \big(v_j\big)_{j\in\mathbb{N}} \in \ell_1$. We first show that G maps the space Γ_1 into itself. Let $n \in \mathbb{N}$ and $(s,t) \in I^2$ then assumptions (C_1) and (C_3) give

$$\begin{aligned} &\sum_{n=1}^{\infty} |(\mathsf{G}v)_n(s,t)| \\ &\quad \le \sum_{n=1}^{\infty} |\mathsf{r}_n(s,t)| + \sum_{n=1}^{\infty} \left| \int_a^b \int_a^b K_n(s,t,\tau_1,\tau_2) f_n(\tau_1,\tau_2,v(\tau_1,\tau_2)) d\tau_1 d\tau_2 \right| \\ &\quad \le R + \sum_{n=1}^{\infty} \int_a^b \int_a^b |K_n(s,t,\tau_1,\tau_2)| \left| f_n(\tau_1,\tau_2,v(\tau_1,\tau_2)) \right| d\tau_1 d\tau_2 \\ &\quad \le R + \mathsf{K} \sum_{n=1}^{\infty} \int_a^b \int_a^b [a_n(\tau_1,\tau_2) + d_n(\tau_1,\tau_2)|v_j(\tau_1,\tau_2)|] d\tau_1 d\tau_2 \\ &\quad \le R + \mathsf{K} \sum_{n=1}^{\infty} \left(\int_a^b \int_a^b a_n(\tau_1,\tau_2) d\tau_1 d\tau_2 \right) + \mathsf{KD} \sum_{n=1}^{\infty} \left(\int_a^b \int_a^b |v_j(\tau_1,\tau_2)| \right) d\tau_1 d\tau_2. \end{aligned}$$

Using, Lebesgue monotone convergence theorem [19] we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}|(\mathsf{G}v)_n(s,t)| &\leq R+\mathsf{K}\int_a^b\int_a^b a(\tau_1,\tau_2)d\tau_1 d\tau_2+\mathsf{KD}\left(\int_a^b\int_a^b\sum_{n=1}^{\infty}|v_j(\tau_1,\tau_2)|\right)d\tau_1 d\tau_2\\ &\leq R+\mathsf{KA}(b-a)^2+\mathsf{KD}(b-a)^2\sup\{\|v(s,t)\|_{\ell_1}:(s,t)\in I^2\}\\ &\leq R+\mathsf{KA}(b-a)^2+\mathsf{KD}(b-a)^2\|v\|_{\Gamma_1}\\ &<\infty.\end{aligned} \tag{10}$$

Thus, $(\mathsf{G}v)(s,t)$ belongs to ℓ_1 space for arbitrarily fixed $(s,t)\in I^2$.
Further,

$$\begin{aligned}\|v(s,t)\|_{\ell_1}&=\sum_{n=1}^{\infty}\left|\mathsf{r}_n(s,t)+\int_a^b\int_a^b K_n(s,t,\tau_1,\tau_2)f_n(\tau_1,\tau_2,v(\tau_1,\tau_2))d\tau_1 d\tau_2\right|\\ &\leq R+\mathsf{KA}(b-a)^2+\mathsf{KD}(b-a)^2\|v\|_{\ell_1}. \qquad \text{using (10)}\end{aligned}$$

As, $\mathsf{KD}(b-a)^2<1$ so we get

$$\begin{aligned}[1-(b-a)^2\mathsf{KD}]\|v(t,s)\|_{\ell_1}&\leq R+(b-a)^2\mathsf{KA}\\ \|v(t,s)\|_{\ell_1}&\leq\frac{R+(b-a)^2\mathsf{KA}}{[1-(b-a)^2\mathsf{KD}]}\ (=R_1)\end{aligned} \tag{11}$$

Therefore, using (9) we conclude that G is a self mapping on Γ_1.
Also, $\|(\mathsf{G}v)(s,t)-0\|\leq R_1$, so the operator G maps B_{R_1} (ball of radius R_1 centered at the origin) into itself.
We now show that G is continuous on B_{R_1}. To do so fix $\epsilon>0$ and $v\in\mathsf{B}_{R_1}$. Then for arbitrary $u\in\mathsf{B}_{R_1}$ with $\|u-v\|_{\ell_1}\leq\epsilon$, arbitrary fixed $(s,t)\in I^2$ and $n\in\mathbb{N}$.

$$\begin{aligned}&\|(\mathsf{G}u)(s,t)-(\mathsf{G}v)(s,t)\|_{\ell_1}\\ &=\sum_{n=1}^{\infty}\left|\int_a^b\int_a^b K_n(s,t,\tau_1,\tau_2)\left[f_n(\tau_1,\tau_2,u(\tau_1,\tau_2))-f_n(\tau_1,\tau_2,v(\tau_1,\tau_2))\right]d\tau_1 d\tau_2\right|\\ &\leq\sum_{n=1}^{\infty}\int_a^b\int_a^b|K_n(s,t,\tau_1,\tau_2)|\left|f_n(\tau_1,\tau_2,u(\tau_1,\tau_2))-f_n(\tau_1,\tau_2,v(\tau_1,\tau_2))\right|d\tau_1 d\tau_2\\ &\leq\mathsf{K}\sum_{n=1}^{\infty}\int_a^b\int_a^b\left|f_n(\tau_1,\tau_2,u(\tau_1,\tau_2))-f_n(\tau_1,\tau_2,v(\tau_1,\tau_2))\right|d\tau_1 d\tau_2.\end{aligned} \tag{12}$$

Now, using the assumptions (C_1), define the set $\delta(\epsilon)$ as

$$\delta(\epsilon)=\sup\left\{|f_n(s,t,u)-f_n(s,t,v)|:u,v\in\mathsf{B}_{R_1},\|u-v\|_{\ell_1}<\epsilon,(s,t)\in I^2,n=1,2,\ldots\right\}.$$

Then, $\delta(\epsilon)\to 0$ as $\epsilon\to 0$.
So, by assumption (C_3) using Lebesgue monotone convergence theorem [19], we obtained from inequality (12)

$$\begin{aligned}\|(\mathsf{G}u)(s,t)-(\mathsf{G}v)(s,t)\|_{\ell_1}&\leq\mathsf{K}\int_a^b\int_a^b\sum_{n=1}^{\infty}\left|f_n(\tau_1,\tau_2,u(\tau_1,\tau_2))-f_n(\tau_1,\tau_2,v(\tau_1,\tau_2))\right|d\tau_1 d\tau_2\\ &\leq\mathsf{K}\int_a^b\int_a^b\|f_n(\tau_1,\tau_2,u)-f_n(\tau_1,\tau_2,v)\|_{\ell_1}d\tau_1 d\tau_2\\ &\leq(b-a)^2\mathsf{K}\delta(\epsilon).\end{aligned} \tag{13}$$

As, (13) holds for arbitrary fixed $(s,t) \in I^2$, so

$$\begin{aligned}\|\mathsf{G}u - \mathsf{G}v\|_{\ell_1} &\le \sup_{(s,t)\in I^2}\left\{\|(\mathsf{G}u)(s,t) - (\mathsf{G}v)(s,t)\|_{\ell_1}\right\}\\ &\le (b-a)^2\mathsf{K}\delta(\epsilon).\end{aligned}$$

Hence, G is a continuous operator on B_{R_1}.
We now show that G is a Meir-Keeler condensing operator, that is given $\epsilon > 0$ and $\delta > 0$

$$\epsilon \le \chi\left(\mathsf{B}_{R_1}\right) < \epsilon + \delta \implies \chi\left(\mathsf{G}\left(\mathsf{B}_{R_1}\right)\right) < \epsilon.$$

Using, the definition of measure of noncompactness in ℓ_1 (3) and the assumptions (C_2), (C_3) and (C_4) we have

$$\begin{aligned}\chi\left(\mathsf{G}\left(\mathsf{B}_{R_1}\right)\right) &= \lim_{n\to\infty}\left[\sup_{v(s,t)\in\mathsf{B}_{R_1}}\left\{\sum_{k\ge n}|v_k(t,s)|\right\}\right]\\ &\le \lim_{n\to\infty}\left[\sup_{v(s,t)\in\mathsf{B}_{R_1}}\left\{\sum_{k\ge n}\left(|\mathsf{r}_k(s,t)| + \left|\int_a^b\int_a^b K_k(s,t,\tau_1,\tau_2)f_k(\tau_1,\tau_2,v(\tau_1,\tau_2))d\tau_1 d\tau_2\right|\right)\right\}\right]\\ &\le \lim_{n\to\infty}\left[\sup_{v(s,t)\in\mathsf{B}_{R_1}}\left\{\sum_{k\ge n}\left(|\mathsf{r}_k(s,t)| + \mathsf{K}\int_a^b\int_a^b \left|f_k(\tau_1,\tau_2,v(\tau_1,\tau_2))\right| d\tau_1 d\tau_2\right)\right\}\right]\\ &\le \lim_{n\to\infty}\left[\sum_{k\ge n}|\mathsf{r}_k(s,t)| + \mathsf{K}\sup_{v(s,t)\in\mathsf{B}_{R_1}}\left\{\sum_{k\ge n}\left(\int_a^b\int_a^b a_k(\tau_1,\tau_2) + d_k(\tau_1,\tau_2)|v_k(\tau_1,\tau_2)|d\tau_1 d\tau_2\right)\right\}\right].\end{aligned}$$

Using, Lebesgue dominated convergence theorem gives

$$\begin{aligned}\chi\left(\mathsf{G}\left(\mathsf{B}_{R_1}\right)\right) &\le \lim_{n\to\infty}\Bigg[\sum_{k\ge n}|\mathsf{r}_k(s,t)| + \mathsf{K}\Bigg\{\int_a^b\int_a^b\sum_{k\ge n}a_k(\tau_1,\tau_2)d\tau_1 d\tau_2 +\\ &\qquad \mathsf{D}\int_a^b\int_a^b \sup_{v(s,t)\in\mathsf{B}_{R_1}}\left(\sum_{k\ge n}|v_k(\tau_1,\tau_2)|\right)d\tau_1 d\tau_2\Bigg\}\Bigg]\\ &\le \mathsf{KD}\int_a^b\int_a^b \lim_{n\to\infty}\left\{\sup_{v(s,t)\in\mathsf{B}_{R_1}}\left(\sum_{k\ge n}|v_k(\tau_1,\tau_2)|\right)d\tau_1 d\tau_2\right\}\\ &\le (b-a)^2\mathsf{KD}\,\chi\left(\mathsf{B}_{R_1}\right).\end{aligned}$$

Therefore,

$$\chi\left(\mathsf{G}\left(\mathsf{B}_{R_1}\right)\right) \le (b-a)^2\mathsf{KD}\,\chi\left(\mathsf{B}_{R_1}\right) < \epsilon, \Longrightarrow \chi\left(\mathsf{B}_{R_1}\right) < \frac{\epsilon}{(b-a)^2\mathsf{KD}}.$$

Taking, $\delta = \dfrac{\epsilon(1-(b-a)^2\mathsf{KD})}{(b-a)^2\mathsf{KD}}$ we obtained $\epsilon \le \chi\left(\mathsf{B}_{R_1}\right) < \epsilon + \delta$. Therefore G is a Meir-Keeler condensing operator on $\mathsf{B}_{R_1} \subset \ell_1$. As G satisfies all the conditions of Theorem 2.6, so G has a fixed point in B_{R_1}. Therefore the system (1) has a solution in ℓ_1. □

We now give an example to support the result

Example 4.3. *Consider the following infinite system of Hammerstein type integral equations in two variables*

$$\begin{aligned}v_n(s,t) = \frac{\alpha}{n^2}\ln[(s+t)+n] + \int_1^2\int_1^2 \arctan(s+t+\tau_1+\tau_2+n)\Big((\tau_1+\tau_2)^2 e^{-n(\tau_1+\tau_2)}\\ + \frac{\sin n(\tau_1+\tau_2)}{(\tau_1+\tau_2)^2+n^3}\cdot\frac{v_n^2(\tau_1,\tau_2)}{1+v_1^2(\tau_1,\tau_2)+\cdots+v_n^2(\tau_1,\tau_2)}\Big)d\tau_1 d\tau_2\end{aligned} \tag{14}$$

for $(s,t) \in [1,2] \times [1,2]$, $\alpha > 0$ *a constant.*
Comparing the system with (1) *we have*

$$\begin{aligned} \mathsf{r}_n(s,t) &= \frac{\alpha}{n^2} \ln[(s+t)+n], \\ K_n(s,t,x,y) &= \arctan(s+t+x+y+n), \\ f_n(s,t,v_1,v_2,\ldots) &= (s+t)^2 e^{-n(s+t)} + \frac{\sin n(s+t)}{(s+t)^2+n^3} \cdot \frac{v_n^2(s,t)}{1+v_1^2(s,t)+\cdots+v_n^2(s,t)}. \end{aligned}$$

for $(s,t),(\tau_1,\tau_2) \in [1,2]\times[1,2]$ *and* $n = 1,2,\cdots$.
Clearly, $\mathsf{r}_n(s,t)$ *is continuous on* $I_1^2 = [1,2]\times[1,2]$.
Moreover, for fixed $(s_1,t_1),(s_2,t_2) \in I_1^2$, *we see that*

$$\begin{aligned} \left\| (\mathsf{r}_n)(s_1,t_1) - (\mathsf{r}_n)(s_2,t_2) \right\| &= \sum_{n=1}^{\infty} |\mathsf{r}_n(s_1,t_1) - \mathsf{r}_n(s_2,t_2)| \\ &= \alpha \sum_{n=1}^{\infty} \frac{1}{n^2} |\ln[(s_1+t_1)+n] - \ln[(s_2+t_2)+n]| \\ &= \alpha \sum_{n=1}^{\infty} \frac{1}{n^2} \left| \ln\left(1 + \frac{s_1+t_1-s_2-t_2}{s_2+t_2+n}\right) \right| \\ &\le \alpha \sum_{n=1}^{\infty} \frac{1}{n^3} |s_1+t_1-s_2-t_2| \\ &\le \alpha[|s_1-s_2| + |t_1-t_2|]\zeta(3). \end{aligned}$$

where, $\zeta(s)$ *denotes Riemann zeta function.*
Choosing $\delta = \dfrac{\epsilon}{\alpha\zeta(3)}$, *so that* $|s_1 - s_2| < \frac{\delta}{2}, |t_1 - t_2| < \frac{\delta}{2}$, *we obtain*

$$\| (\mathsf{r}_n)(s_1,t_1) - (\mathsf{r}_n)(s_2,t_2) \| < \epsilon.$$

Also, for every $(s,t) \in I_1^2$ *we have*

$$\mathsf{r}_n(s,t) \le \frac{\alpha}{n^2} \ln(4+n) \le \frac{\alpha}{n^2} \sqrt{4+n} \le \alpha\left(\frac{2}{n^2} + \frac{1}{n^{3/2}}\right).$$

Hence,

$$\begin{aligned} \mathsf{R} &= \max\{\sum_{n=1}^{\infty} \mathsf{r}_n(s,t) : (s,t) \in I_1^2\} \\ &= \alpha(2\zeta(2) + \zeta(1.5)). \end{aligned} \tag{15}$$

which is finite. Thus, assumption (C_4) *and Remark 4.1 are satisfied.*
Then, the function $K_n(s,t,x,y)$ *is continuous in* I_1^4 *and*

$$\begin{aligned} K_n(s,t,x,y) &= |\arctan(s+t+x+y+n)| \\ &\le \frac{\pi}{2}. \end{aligned}$$

Thus, the function sequence (K_n) *is equibounded on* I_1^4. *Also, for fixed* $(s_1,t_1),(s_2,t_2) \in I_1^2$ *and* $n \in \mathbb{N}$ *then for* $(x,y) \in I_1^2$ *we have*

$$\begin{aligned} &|K_n(s_1,t_1,x,y) - K_n(s_2,t_2,x,y)| \\ &\qquad = |\arctan(s_1+t_1+x+y+n) - \arctan(s_2+t_2+x+y+n)| \\ &\qquad \le |s_1-s_2| + |t_1-t_2|. \end{aligned}$$

Therefore, the function sequence $K_n(s,t,x,y)$ is equicontinuous with respect to $(s,t) \in I_1^2$ uniformly with respect to $(x,y) \in I_1^2$, the value of the constant K *given as*

$$\begin{aligned}\mathsf{K} &= \sup\{K_n(s,t,x,y) : (s,t),(x,y) \in I_1^2, n \in \mathbb{N}\}\\ &= \frac{\pi}{2}.\end{aligned} \tag{16}$$

Hence, all assumptions of (C_3) *are satisfied.*
Again,

$$\begin{aligned}|f_n(s,t,v)| &\leq (s+t)^2 e^{-n(s+t)} + \left|\frac{\sin n(s+t)}{(s+t)^2+n^3} \cdot \frac{v_n^2}{1+v_1^2+\cdots+v_n^2}\right|\\ &\leq (s+t)^2 e^{-n(s+t)} + \frac{1}{(s+t)^2+n^3} \cdot \left|\frac{v_n^2}{1+v_1^2+\cdots+v_n^2}\right|\\ &\leq (s+t)^2 e^{-n(s+t)} + \frac{1}{(s+t)^2+n^3} \cdot \frac{|v_n|}{1+v_n^2}(|v_n|)\\ &\leq (s+t)^2 e^{-n(s+t)} + \frac{1}{2[(s+t)^2+n^3]}|v_n|.\end{aligned}$$

Taking, $a_n(s,t) = (s+t)^2 e^{-n(s+t)}$ *and* $d_n(s,t) = \dfrac{1}{2[(s+t)^2+n^3]}$ *gives*

$$|f_n(s,t,v)| \leq a_n(s,t) + d_n(s,t)|v_n|.$$

Obviously, the functions $a_n(s,t)$ are continuous on I_1^2, for any $(s,t) \in I_1^2$ we have $|a_n(s,t)| \leq \dfrac{4}{n^3}e^{-2}$, and the function series $a(s,t) = \displaystyle\sum_{n=1}^{\infty} a_n(s,t) = \frac{(s+t)^2}{e^{s+t}-1}$, is uniformly convergent on the interval I_1^2.
Also,

$$|d_n(s,t)| = \frac{1}{2[(s+t)^2+n^3]} \leq \frac{1}{2n^3} \leq \frac{1}{2},$$

for all $n \in \mathbb{N}$. Hence, the function sequence $(h_n(s,t))$is equibounded on I_1^2. The value of the constants A, D *are*

$$\mathsf{A} = \max\left\{a(s,t) : (s,t) \in I_1^2\right\} = \frac{16}{e^2-1}; \mathsf{D} = \frac{1}{2}, \tag{17}$$

and $(b-a)^2\mathsf{K}\mathsf{D} = \dfrac{\pi}{8}$. *Using* (11), (15), (16), (17) *we obtain*

$$\begin{aligned}R_1 &= \frac{\alpha\,(2\zeta(2)+\zeta(3)) + (2-1)^2 \times \frac{1}{2} \times \frac{16}{e^2-1}}{1-\frac{\pi}{8}}\\ &\approx 1.84 \qquad\qquad \textit{for } \alpha = 0.10\end{aligned} \tag{18}$$

Finally, we check whether the assumption (C_1) *is satisfied. Fix* $v = (v_n) \in \mathsf{B}_{R_1} \subset \ell_1$ *and* $\epsilon > 0$, *then for any*

$u = (u_n) \in \mathsf{B}_{R_1}$ with $\|u - v\|_{\ell_1} \leq \epsilon$, then for fixed $(s,t) \in I_1^2$, we have

$$\left\| (\mathsf{Q}u)(s,t) - (\mathsf{Q}v)(s,t) \right\|_{\ell_1} = \sum_{n=1}^{\infty} \left| f_n(s,t,u) - f_n(s,t,v) \right|$$

$$\leq \sum_{n=1}^{\infty} \left| \frac{\sin n(s+t)}{(s+t)^2 + n^3} \right| \left| \frac{u_n^2}{1 + u_1^2 + \cdots + u_n^2} - \frac{v_n^2}{1 + v_1^2 + \cdots + v_n^2} \right|$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n^3} |u_n^2(1 + v_1^2 + \cdots + v_n^2) - v_n^2(1 + u_1^2 + \cdots + u_n^2)|$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n^3} \Big[|u_n^2 - v_n^2| + |u_n^2(v_1^2 + \cdots + v_n^2) - u_n^2(u_1^2 + \cdots + u_n^2)| + |u_n^2(u_1^2 + \cdots + u_n^2) - v_n^2(u_1^2 + \cdots + u_n^2)| \Big]$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n^3} \Big[|u_n^2 - v_n^2| + u_n^2(|v_1^2 - u_1^2| + \cdots + |v_n^2 - u_n^2|) + |u_n^2 - v_n^2|(u_1^2 + \cdots + u_n^2) \Big].$$

Since, $v_n, u_n \in \mathsf{B}_{R_1}, n \in \mathbb{N}$ so $|v_n| \leq R_1, |u_n| < R_1$ so

$$\begin{aligned}
\left\| (\mathsf{Q}u)(s,t) - (\mathsf{Q}v)(s,t) \right\|_{\ell_1} &\leq \sum_{n=1}^{\infty} \frac{1}{n^3} \Big(|u_n - v_n|(|u_n| + |v_n|)(1 + u_1^2 + \cdots + u_n^2) + \\
&\qquad u_n^2(|v_1 - u_1|(|v_1| + |u_1|) + \cdots + |v_n - u_n|(|v_n| + |u_n|) \Big) \\
&< 2R_1 \sum_{n=1}^{\infty} \left[\frac{1}{n^3} |u_n - v_n|(1 + nR_1^2) + R_1^2 \left(\sum_{i=1}^{n} |v_i - u_i| \right) \right] \\
&= 2R_1 \|u - v\|_{\ell_1} \sum_{n=1}^{\infty} \frac{1}{n^3} \left[(1 + nR_1^2) + R_1^2 \right] \\
&= 2R_1 \|u - v\|_{\ell_1} \left([1 + R_1^2]\zeta(3) + R_1^2 \zeta(2) \right).
\end{aligned}$$

Thus, choose $\delta = \dfrac{\epsilon}{2R_1 \left([1 + R_1^2]\zeta(3) + R_1^2\zeta(2) \right)}$, then for $\|u - v\|_{\ell_1} < \delta$ we have

$$\left\| (\mathsf{Q}u)(s,t) - (\mathsf{Q}v)(s,t) \right\|_{\ell_1} < \epsilon.$$

Hence the assumption (C_1) is also satisfied, therefore by Theorem 4.2 we conclude that the system in (14) *has a solution in $\mathsf{B}_{R_1} \subset \ell_1$ where R_1 is given by* (18).

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