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Tauberian Conditions Under Which Convergence Follows from Cesàro Summability of Double Integrals Over $\mathbb{R}_+^2$

Gökşen Fındık[a], İbrahim Çanak[a]

[a]*Ege University, Department of Mathematics, İzmir, Turkey*

Abstract. For a real- or complex-valued continuous function f over $\mathbb{R}_+^2 := [0,\infty) \times [0,\infty)$, we denote its integral over $[0,u] \times [0,v]$ by $s(u,v)$ and its $(C,1,1)$ mean, the average of $s(u,v)$ over $[0,u] \times [0,v]$, by $\sigma(u,v)$. The other means $(C,1,0)$ and $(C,0,1)$ are defined analogously. We introduce the concepts of backward differences and the Kronecker identities in different senses for double integrals over $\mathbb{R}_+^2$. We give one-sided and two-sided Tauberian conditions based on the difference between double integral of $s(u,v)$ and its means in different senses for Cesàro summability methods of double integrals over $[0,u] \times [0,v]$ under which convergence of $s(u,v)$ follows from integrability of $s(u,v)$ in different senses.

1. Introduction

Tauberian theorems for Cesàro (or (C,1)) summability methods for functions of one variable over $\mathbb{R}_+ := [0,\infty)$ have been studied by Móricz and Németh [9] and Laforgia [8]. Namely, Móricz and Németh [9] investigated Tauberian conditions under which convergence of integral follows from summability $(C,1)$ for functions of one variable over $\mathbb{R}_+ := [0,\infty)$ and they obtained a one-sided and two-sided Tauberian conditions of Landau and Hardy type as corollaries of their main results. Laforgia [8] established an alternative proof of a Tauberian theorem for the summability $(C,1)$ for functions of one variable over $\mathbb{R}_+$.

In some recent works [2, 3, 5], Tauberian conditions for Cesàro summability methods have been based on the difference between the integral of a function of one variable and its Cesàro means. Firstly, Çanak and Totur obtained an alternative proof of the generalized Littlewood Tauberian theorem for Cesàro summability of improper integrals in [2]. Secondly, Çanak and Totur generalized some classical type Tauberian theorems given for Cesàro summability in [3]. Finally, Çanak and Totur obtained alternative proofs of some classical Tauberian theorems for the Cesàro summability of integrals for functions of one variable in [5].

Tauberian theorems for Cesàro summability methods for functions of two variables over $\mathbb{R}_+^2$ have been studied by Móricz [10] and Totur and Çanak [6]. Namely, Móricz [10] extended the results in [9] to Cesàro summable double integrals over $\mathbb{R}_+^2$. Following Laforgia [8], Totur and Çanak [6] obtained analogous results for integrals of functions of two variables.

Recently, the concept of generator sequence, which is the difference between a sequence and its Cesàro mean, has been introduced by Çanak and Totur [4]. Belen [1] introduced the double analogue of the concept

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Email addresses: findikgoksen@gmail.com (Gökşen Fındık), ibrahim.canak@ege.edu.tr (İbrahim Çanak)

of generator sequence and obtained Tauberian conditions based on double generator sequences for double sequences of real or complex numbers.

In this paper, we introduce the concepts of backward differences and the Kronecker identities in different senses for double integrals over $\mathbb{R}^2_+$. We give one-sided and two-sided Tauberian conditions based on the difference between the double integral of a function of two variables and its means in different senses for Cesàro summability methods of double integrals over $[0,u]\times[0,v]$ under which convergence of improper double integral follows from integrability of improper double integral in different senses.

2. Tauberian theorems for $(C,1,1)$ summability of improper double integrals over $\mathbb{R}^2_+$

Suppose that f is a real- or complex-valued continuous function defined on $\mathbb{R}^2_+ := [0,\infty)\times[0,\infty)$ and $s(u,v)=\int_0^u\int_0^v f(x,y)dxdy$ for $0<u,v<\infty$. The mean $(C,1,1)$ (or Cesàro mean in sense $(1,1)$) of $s(u,v)$ is defined by

$$\sigma(s(u,v))=\sigma_{11}(s(u,v))=\frac{1}{uv}\int_0^u\int_0^v s(x,y)dxdy$$

for $u,v>0$. The integral

$$\int_0^\infty\int_0^\infty f(x,y)dxdy \tag{1}$$

is said to be $(C,1,1)$ summable (or Cesàro summable in sense $(1,1)$) to a finite number L if

$$\lim_{u,v\to\infty}\sigma(s(u,v))=\lim_{u,v\to\infty}\int_0^u\int_0^v\left(1-\frac{x}{u}\right)\left(1-\frac{y}{v}\right)f(x,y)dxdy=L. \tag{2}$$

The backward difference in sense $(1,1)$ of $s(u,v)$ is defined by

$$\Delta_{11}s(u,v)=\frac{\partial^2 s(u,v)}{\partial u\partial v}=f(u,v)$$

for $u,v>0$.

The Kronecker identity in sense $(1,1)$ has been given independently by Totur [13] and Belen [1] for double sequences of real or complex numbers. The identity defined by Belen is slightly different than the identity defined by Totur. We prefer the identity defined by Belen [1] for our purposes. Next, we obtain an analogous Kronecker identity in sense $(1,1)$ for improper double integrals.

From

$$\begin{aligned}\sigma_{11}(s(u,v)) &= \int_0^u\int_0^v\left(1-\frac{x}{u}\right)\left(1-\frac{y}{v}\right)f(x,y)dxdy\\ &= \int_0^u\int_0^v f(x,y)dxdy-\frac{1}{u}\int_0^u\int_0^v xf(x,y)dxdy\\ &\quad -\frac{1}{v}\int_0^u\int_0^v yf(x,y)dxdy+\frac{1}{uv}\int_0^u\int_0^v xyf(x,y)dxdy,\end{aligned}$$

we have

$$s(u,v)-\sigma_{11}(s(u,v)) =: V_{11}^{(0)}(\Delta_{11}s(u,v)) \tag{3}$$

where $V_{11}^{(0)}(\Delta_{11}s(u,v))=V_{10}^{(0)}(\Delta_{10}s(u,v))+V_{01}^{(0)}(\Delta_{01}s(u,v))-\dfrac{1}{uv}\displaystyle\int_0^u\int_0^v xyf(x,y)dxdy$. This identity is known as the Kronecker identity in sense (1,1). (See Sections 3 and 4 for the definitions of $V_{10}^{(0)}(\Delta_{10}s(u,v))$ and $V_{01}^{(0)}(\Delta_{01}s(u,v))$)

For each nonnegative integer m, the iterative means of $s(u,v)$ and $V^{(0)}_{11}(\Delta_{11}s(u,v))$ in sense $(1,1)$ are defined by

$$\sigma^{(m)}_{11}(s(u,v)) = \begin{cases} \dfrac{1}{uv}\displaystyle\int_0^u\int_0^v \sigma^{(m-1)}_{11}(s(x,y))dxdy, & m \geq 1 \\ s(u,v), & m = 0 \end{cases}$$

and

$$V^{(m)}_{11}(\Delta_{11}s(u,v)) = \begin{cases} \dfrac{1}{uv}\displaystyle\int_0^u\int_0^v V^{(m-1)}_{11}(\Delta_{11}s(x,y))dxdy, & m \geq 1 \\ V^{(0)}_{11}(\Delta_{11}s(u,v)), & m = 0 \end{cases}$$

respectively. Note that $\sigma_{11}(s(u,v)) = \sigma^{(1)}_{11}(s(u,v))$.

Throughout this paper, convergence is always used in Pringsheim's sense for convergence of improper double integral [11]. Namely, both u and v tend to ∞ independently of each other in (2).

A function $s(u,v)$ is bounded if there exists a real number $H > 0$ such that $|s(u,v)| \leq H$ for all $u,v > 0$. In this case, we write $s(u,v) = O(1)$. Moreover, a double integral $s(u,v)$ is said to be one-sided bounded if there exists a real number $H > 0$ such that $s(u,v) \geq -H$ for all $u,v > 0$.

Assume that the function $s(u,v)$ is bounded on $\mathbb{R}^2_+$. If the limit

$$\lim_{u,v\to\infty} s(u,v) = L \tag{4}$$

exists, then the limit (2) also exists. The converse of this statement is not true in general, even if $s(u,v)$ is bounded on $\mathbb{R}^2_+$. Adding some suitable condition, which is called a Tauberian condition, one may get the converse. Any theorem which states that convergence of the improper double integral follows from its Cesàro summability in sense $(1,1)$ and some Tauberian condition is said to be a Tauberian theorem. A similar situation is valid for the Cesàro summability methods in senses $(1,0)$ and $(0,1)$.

For the real-valued functions defined on $\mathbb{R}^2_+$, we need the following definitions:

A real-valued function $s(u,v)$ defined on $\mathbb{R}^2_+$ is said to be slowly decreasing in sense $(1,0)$ [10] if

$$\lim_{\lambda\to 1^+}\liminf_{u,v\to\infty}\min_{u\leq x\leq \lambda u}[s(x,v)-s(u,v)] \geq 0$$

or equivalently

$$\lim_{\lambda\to 1^-}\liminf_{u,v\to\infty}\min_{\lambda u\leq x\leq u}[s(u,v)-s(x,v)] \geq 0.$$

Analogously, a real-valued function $s(u,v)$ defined on $\mathbb{R}^2_+$ is said to be slowly decreasing in sense $(0,1)$ [10] if

$$\lim_{\lambda\to 1^+}\liminf_{u,v\to\infty}\min_{v\leq y\leq \lambda v}[s(u,y)-s(u,v)] \geq 0$$

or equivalently

$$\lim_{\lambda\to 1^-}\liminf_{u,v\to\infty}\min_{\lambda v\leq y\leq v}[s(u,v)-s(u,y)] \geq 0.$$

A real-valued function $s(u,v)$ defined on $\mathbb{R}^2_+$ is said to be strong slowly decreasing in sense $(1,0)$ if

$$\lim_{\lambda\to 1^+}\liminf_{u,v\to\infty}\min_{\substack{u\leq x\leq \lambda u\\ v\leq y\leq \lambda v}}[s(x,y)-s(u,y)] \geq 0$$

or equivalently

$$\lim_{\lambda\to 1^-}\liminf_{u,v\to\infty}\min_{\substack{\lambda u\leq x\leq u\\ \lambda v\leq y\leq v}}[s(u,y)-s(x,y)] \geq 0.$$

Analogously, a real-valued function $s(u,v)$ defined on $\mathbb{R}^2_+$ is said to be strong slowly decreasing in sense $(0,1)$ [10] if

$$\lim_{\lambda\to 1^+}\liminf_{u,v\to\infty}\min_{\substack{u\le x\le\lambda u\\ v\le y\le\lambda v}}[s(x,y)-s(x,v)]\ge 0$$

or equivalently

$$\lim_{\lambda\to 1^-}\liminf_{u,v\to\infty}\min_{\substack{\lambda u\le x\le u\\ \lambda v\le y\le v}}[s(x,v)-s(x,y)]\ge 0.$$

Note that the concept of slow decrease was introduced by Schmidt [12] for the sequences of real numbers.
For the complex-valued functions defined on $\mathbb{R}^2_+$, we need the following definitions:
A complex-valued function $s(u,v)$ defined on $\mathbb{R}^2_+$ is said to be slowly oscillating in sense $(1,0)$ [10] if

$$\lim_{\lambda\to 1^+}\limsup_{u,v\to\infty}\max_{u\le x\le\lambda u}|s(x,v)-s(u,v)|=0$$

or equivalently

$$\lim_{\lambda\to 1^-}\limsup_{u,v\to\infty}\max_{\lambda u\le x\le u}|s(u,v)-s(x,v)|=0.$$

Analogously, a complex-valued function $s(u,v)$ defined on $\mathbb{R}^2_+$ is said to be slowly oscillating in sense $(0,1)$ [10] if

$$\lim_{\lambda\to 1^+}\limsup_{u,v\to\infty}\max_{v\le y\le\lambda v}\left|s(u,y)-s(u,v)\right|=0$$

or equivalently

$$\lim_{\lambda\to 1^+}\limsup_{u,v\to\infty}\max_{\lambda v\le y\le v}\left|s(u,v)-s(u,y)\right|=0.$$

A complex-valued function $s(u,v)$ defined on $\mathbb{R}^2_+$ is said to be strong slowly oscillating in sense $(1,0)$ [10] if

$$\lim_{\lambda\to 1^+}\limsup_{u,v\to\infty}\max_{\substack{u\le x\le\lambda u\\ v\le y\le\lambda v}}|s(x,y)-s(u,y)|=0$$

or equivalently

$$\lim_{\lambda\to 1^-}\limsup_{u,v\to\infty}\max_{\substack{\lambda u\le x\le u\\ \lambda v\le y\le v}}|s(u,y)-s(x,y)|=0.$$

Analogously, a complex-valued function $s(u,v)$ defined on $\mathbb{R}^2_+$ is said to be strong slowly oscillating in sense $(0,1)$ [10] if

$$\lim_{\lambda\to 1^+}\limsup_{u,v\to\infty}\max_{\substack{u\le x\le\lambda u\\ v\le y\le\lambda v}}|s(x,y)-s(x,v)|=0$$

or equivalently

$$\lim_{\lambda\to 1^-}\limsup_{u,v\to\infty}\max_{\substack{\lambda u\le x\le u\\ \lambda v\le y\le v}}|s(x,v)-s(x,y)|=0.$$

Note that the concept of slow oscillation was introduced by Hardy [7] for the sequences of complex numbers.

First, we consider Landau type one-sided Tauberian conditions sufficient for convergence of $s(u,v)$ to follow from its $(C,1,1)$ summability.

Theorem 2.1. *Let the double integral $s(u,v)$ be bounded. If (1) is $(C,1,1)$ summable to a finite number L and there exist constants $H>0$ and $x_0 \geq 0$ such that conditions*

$$u\Delta_{10}V_{11}^{(0)}(\Delta_{11}s(u,v)) \geq -H \tag{5}$$

and

$$v\Delta_{01}V_{11}^{(0)}(\Delta_{11}s(u,v)) \geq -H \tag{6}$$

are satisfied for all $(u,v) \in \mathbb{R}_+^2$ with $u,v > x_0$, then we have (4).

As corollary of Theorem 2.1, we have the following two-sided Hardy type Tauberian theorem.

Corollary 2.2. *Let the double integral $s(u,v)$ be bounded. If (1) is $(C,1,1)$ summable to a finite number L and there exist constants $H>0$ and $x_0 \geq 0$ such that conditions*

$$u\Delta_{10}V_{11}^{(0)}(\Delta_{11}s(u,v)) = O(1) \tag{7}$$

and

$$v\Delta_{01}V_{11}^{(0)}(\Delta_{11}s(u,v)) = O(1) \tag{8}$$

are satisfied for all $(u,v) \in \mathbb{R}_+^2$ with $u,v > x_0$, then we have (4).

Next, we give one-sided and two sided Tauberian conditions sufficient in order that convergence follows from $(C,1,1)$ summability for real and complex-valued functions, respectively.

Theorem 2.3. *Let the double integral $s(u,v)$ be bounded. If (1) is $(C,1,1)$ summable to a finite number L and $V_{11}(\Delta_{11}s(u,v))$ is slowly decreasing in sense $(0,1)$ and strong slowly decreasing in sense $(1,0)$ or slowly decreasing in sense $(1,0)$ and strong slowly decreasing in sense $(0,1)$, then we have (4).*

Theorem 2.4. *Let the double integral $s(u,v)$ be bounded. If (1) is $(C,1,1)$ summable to a finite number L and $V_{11}(\Delta_{11}s(u,v))$ is slowly oscillating in sense $(0,1)$ and strong slowly oscillating in sense $(1,0)$ or slowly oscillating in sense $(1,0)$ and strong slowly oscillating in sense $(0,1)$, then we have (4).*

2.1. Proofs of Main Results

The following lemma gives two representations for the difference $s(u,v) - \sigma(s(u,v))$.

Lemma 2.5. *Let $s(u,v)$ be a double integral over the rectangle $[0,u]\times[0,v]$. For sufficiently large u,v*
(i) *If $\lambda > 1$,*

$$s(u,v) - \sigma(s(u,v)) = \left(\frac{\lambda}{\lambda-1}\right)^2 (\sigma(s(\lambda u,\lambda v)) - \sigma(s(u,v))) + \frac{\lambda}{(\lambda-1)^2}(\sigma(s(u,v)) - \sigma(s(\lambda u,v))) + \frac{\lambda}{(\lambda-1)^2}(\sigma(s(u,v)) - \sigma(s(u,\lambda v))) - \frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}(s(x,y)-s(u,v))\,dxdy.$$

(ii) *If $0<\lambda<1$,*

$$s(u,v) - \sigma(s(u,v)) = \left(\frac{\lambda}{1-\lambda}\right)^2 (\sigma(s(\lambda u,\lambda v)) - \sigma(s(u,v))) + \frac{\lambda}{(1-\lambda)^2}(\sigma(s(u,v)) - \sigma(s(\lambda u,v))) + \frac{\lambda}{(1-\lambda)^2}(\sigma(s(u,v)) - \sigma(s(u,\lambda v))) + \frac{1}{(u-\lambda u)(v-\lambda v)}\int_{\lambda u}^{u}\int_{\lambda v}^{v}(s(u,v)-s(x,y))\,dxdy.$$

Proof. (i) By definition, we have

$$\frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}s(x,y)dxdy=\frac{1}{(\lambda-1)^2uv}\left(\int_0^{\lambda u}-\int_0^{u}\right)\left(\int_0^{\lambda v}-\int_0^{v}\right)s(x,y)dxdy$$
$$=\frac{\lambda^2}{(\lambda-1)^2\lambda u\lambda v}\int_0^{\lambda u}\int_0^{\lambda v}s(x,y)dxdy-\frac{\lambda}{(\lambda-1)^2\lambda uv}\int_0^{\lambda u}\int_0^{v}s(x,y)dxdy$$
$$-\frac{\lambda}{(\lambda-1)^2u\lambda v}\int_0^{u}\int_0^{\lambda v}s(x,y)dxdy+\frac{1}{(\lambda-1)^2uv}\int_0^{u}\int_0^{v}s(x,y)dxdy$$
$$=\left(\frac{\lambda}{\lambda-1}\right)^2\sigma(s(\lambda u,\lambda v))-\frac{\lambda}{(\lambda-1)^2}\sigma(s(\lambda u,v))-\frac{\lambda}{(\lambda-1)^2}\sigma(s(u,\lambda v))$$
$$+\frac{1}{(\lambda-1)^2}\sigma(s(u,v))-\sigma(s(u,v))+\sigma(s(u,v))$$
$$=\left(\frac{\lambda}{\lambda-1}\right)^2\sigma(s(\lambda u,\lambda v))-\frac{\lambda}{(\lambda-1)^2}\sigma(s(\lambda u,v))-\frac{\lambda}{(\lambda-1)^2}\sigma(s(u,\lambda v))$$
$$+\frac{2\lambda}{(\lambda-1)^2}\sigma(s(u,v))-\left(\frac{\lambda}{\lambda-1}\right)^2\sigma(s(u,v))+\sigma(s(u,v))$$
$$=\left(\frac{\lambda}{\lambda-1}\right)^2(\sigma(s(\lambda u,\lambda v))-\sigma(s(u,v)))+\frac{\lambda}{(\lambda-1)^2}(\sigma(s(u,v))-\sigma(s(\lambda u,v)))$$
$$+\frac{\lambda}{(\lambda-1)^2}(\sigma(s(u,v))-\sigma(s(u,\lambda v)))+\sigma(s(u,v)).$$

Adding $s(u,v)$ to both sides of the previous equation and then arranging this equality, we reach to the equality (i) of Lemma 2.5.

(ii) The proof of Lemma 2.5 (ii) can be verified in a similar way. □

Proof of Theorem 2.1 Assume that the double integral $s(u,v)$ is bounded, (1) is $(C,1,1)$ summable to L and conditions (5) and (6) hold. Since the $(C,1,1)$ summability method is regular and $\lim_{u,v\to\infty}\sigma^{(1)}_{11}(s(u,v))=L$, we have $\lim_{u,v\to\infty}\sigma^{(2)}_{11}(s(u,v))=L$. Taking $(C,1,1)$ means of both sides of (3), we obtain $\lim_{u,v\to\infty}V^{(1)}_{11}(\Delta_{11}s(u,v))=0$.

If we replace $s(u,v)$ by $V^{(0)}_{11}(\Delta_{11}s(u,v))$ in Lemma 2.5 (i), we obtain

$$V^{(0)}_{11}(\Delta_{11}s(u,v))-V^{(1)}_{11}(\Delta_{11}s(u,v))=\left(\frac{\lambda}{\lambda-1}\right)^2\left(V^{(1)}_{11}(\Delta_{11}s(\lambda u,\lambda v))-V^{(1)}_{11}(\Delta_{11}s(u,v))\right)$$
$$+\frac{\lambda}{(\lambda-1)^2}\left(V^{(1)}_{11}(\Delta_{11}s(u,v))-V^{(1)}_{11}(\Delta_{11}s(\lambda u,v))\right)$$
$$+\frac{\lambda}{(\lambda-1)^2}\left(V^{(1)}_{11}(\Delta_{11}s(u,v))-V^{(1)}_{11}(\Delta_{11}s(u,\lambda v))\right)$$
$$-\frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}\left(V^{(0)}_{11}(\Delta_{11}s(x,y))-V^{(0)}_{11}(\Delta_{11}s(u,v))\right)dxdy. \quad (9)$$

In addition, we obtain by (5) and (6) that

$$\begin{aligned}V^{(0)}_{11}(\Delta_{11}s(x,y))-V^{(0)}_{11}(\Delta_{11}s(u,v)) &= \int_u^x\frac{\Delta V^{(0)}_{11}(\Delta_{11}s(s,y))}{\Delta s}ds+\int_v^y\frac{\Delta V^{(0)}_{11}(\Delta_{11}s(u,t))}{\Delta t}dt\\ &\geq -H\left(\int_u^x\frac{ds}{s}+\int_v^y\frac{dt}{t}\right)\\ &= -H\left(\ln\left(\frac{x}{u}\right)+\ln\left(\frac{y}{v}\right)\right) \quad (10)\end{aligned}$$

for some $H > 0$. Taking lim sup of both sides of (9) as $u, v \to \infty$ and taking (10) into consideration, we get

$$\limsup_{u,v\to\infty}\left(V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,v))\right) \le \left(\frac{\lambda}{\lambda-1}\right)^2 \limsup_{u,v\to\infty}\left(V_{11}^{(1)}(\Delta_{11}s(\lambda u,\lambda v)) - V_{11}^{(1)}(\Delta_{11}s(u,v))\right)$$
$$+ \frac{\lambda}{(\lambda-1)^2}\limsup_{u,v\to\infty}\left(V_{11}^{(1)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(\lambda u,v))\right)$$
$$+ \frac{\lambda}{(\lambda-1)^2}\limsup_{u,v\to\infty}\left(V_{11}^{(1)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,\lambda v))\right) + H\limsup_{u,v\to\infty}\left(\ln\left(\frac{\lambda u}{u}\right) + \ln\left(\frac{\lambda v}{v}\right)\right).$$

Since $\lim_{u,v\to\infty} V_{11}^{(1)}(\Delta_{11}s(u,v)) = 0$, the first three terms on the right-hand side of the previous inequality are vanished and we get

$$\limsup_{u,v\to\infty}\left(V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,v))\right) \le 2H\ln\lambda.$$

Hence taking the limit of both sides of the last inequality as $\lambda \to 1^+$, we have

$$\limsup_{u,v\to\infty}\left(V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,v))\right) \le 0. \tag{11}$$

If we replace $s(u, v)$ by $V_{11}^{(0)}(\Delta_{11}s(u,v))$ in Lemma 2.5 (ii), we obtain

$$V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,v)) = \left(\frac{\lambda}{1-\lambda}\right)^2\left(V_{11}^{(1)}(\Delta_{11}s(\lambda u,\lambda v)) - V_{11}^{(1)}(\Delta_{11}s(u,v))\right)$$
$$+ \frac{\lambda}{(1-\lambda)^2}\left(V_{11}^{(1)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(\lambda u,v))\right)$$
$$+ \frac{\lambda}{(1-\lambda)^2}\left(V_{11}^{(1)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,\lambda v))\right)$$
$$+ \frac{1}{(u-\lambda u)(v-\lambda v)}\int_{\lambda u}^{u}\int_{\lambda v}^{v}\left(V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(0)}(\Delta_{11}s(x,y))\right)dxdy. \tag{12}$$

In addition, we obtain by (5) and (6) that

$$\begin{aligned} V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(0)}(\Delta_{11}s(x,y)) &= \int_x^u \frac{\Delta V_{11}^{(0)}(\Delta_{11}s(s,y))}{\Delta s}ds + \int_y^v \frac{\Delta V_{11}^{(0)}(\Delta_{11}s(u,t))}{\Delta t}dt \\ &\ge -H\left(\int_x^u \frac{ds}{s} + \int_y^v \frac{dt}{t}\right) \\ &= -H\left(\ln\left(\frac{u}{x}\right) + \ln\left(\frac{v}{y}\right)\right) \end{aligned} \tag{13}$$

for some $H > 0$. Hence taking lim inf of both sides of (12) as $u, v \to \infty$ and (13) into consideration, we get

$$\liminf_{u,v\to\infty}\left(V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,v))\right) \ge \left(\frac{\lambda}{1-\lambda}\right)^2 \liminf_{u,v\to\infty}\left(V_{11}^{(1)}(\Delta_{11}s(\lambda u,\lambda v)) - V_{11}^{(1)}(\Delta_{11}s(u,v))\right)$$
$$+ \frac{\lambda}{(1-\lambda)^2}\liminf_{u,v\to\infty}\left(V_{11}^{(1)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(\lambda u,v))\right)$$
$$+ \frac{\lambda}{(1-\lambda)^2}\liminf_{u,v\to\infty}\left(V_{11}^{(1)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,\lambda v))\right) - H\limsup_{u,v\to\infty}\left(\ln\left(\frac{u}{\lambda u}\right) + \ln\left(\frac{v}{\lambda v}\right)\right).$$

Since $\lim_{u,v\to\infty} V_{11}^{(1)}(\Delta_{11}s(u,v)) = 0$, the first three terms on the right-hand side of the previous inequality are vanished and we get

$$\liminf_{u,v\to\infty}\Big(V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,v))\Big) \quad \geq \quad -2H\ln\left(\frac{1}{\lambda}\right).$$

Hence taking the limit of both sides of the last inequality as $\lambda \to 1^-$, we have

$$\liminf_{u,v\to\infty}\Big(V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,v))\Big) \quad \geq \quad 0. \tag{14}$$

From (11) and (14), we obtain $\lim_{u,v\to\infty} V_{11}^{(0)}(\Delta_{11}s(u,v)) = \lim_{u,v\to\infty} V_{11}^{(1)}(\Delta_{11}s(u,v)) = 0$. Thus we have (4) by the Kronecker identity (3). □

Proof of Corollary 2.2 It is plain that conditions (7) and (8) imply (5) and (6) in Theorem 2.1. □

Proof of Theorem 2.3 Assume that the double integral $s(u,v)$ is bounded and (1) is $(C,1,1)$ summable to a finite number L. Without loss of generality, we assume that $V_{11}(\Delta_{11}s(u,v))$ is slowly decreasing in sense $(0,1)$ and strong slowly decreasing in sense $(1,0)$. If we do the same calculations as in the proof of Theorem 2.1, we obtain that $V_{11}^{(1)}(\Delta_{11}s(u,v))$ convergent to zero.

Taking lim sup of both sides of (9) as $u,v\to\infty$, we have

$$\begin{aligned}
&\limsup_{u,v\to\infty}\Big(V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,v))\Big) \leq \left(\frac{\lambda}{\lambda-1}\right)^2 \limsup_{u,v\to\infty}\Big(V_{11}^{(1)}(\Delta_{11}s(\lambda u,\lambda v)) - V_{11}^{(1)}(\Delta_{11}s(u,v))\Big)\\
&+\frac{\lambda}{(\lambda-1)^2}\limsup_{u,v\to\infty}\Big(V_{11}^{(1)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(\lambda u,v))\Big)\\
&+\frac{\lambda}{(\lambda-1)^2}\limsup_{u,v\to\infty}\Big(V_{11}^{(1)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,\lambda v))\Big)\\
&\qquad -\liminf_{u,v\to\infty}\frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}\Big(V_{11}^{(0)}(\Delta_{11}s(x,y)) - V_{11}^{(0)}(\Delta_{11}s(u,v))\Big)dxdy.
\end{aligned}\tag{15}$$

Since $V_{11}^{(1)}(\Delta_{11}s(u,v))$ is convergent to zero, the first three terms on the right-hand side of (15) are vanished. Moreover, we have

$$\begin{aligned}
&\frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}\Big(V_{11}^{(0)}(\Delta_{11}s(x,y)) - V_{11}^{(0)}(\Delta_{11}s(u,v))\Big)dxdy\\
&\qquad\qquad \geq \min_{\substack{u\leq x\leq\lambda u\\ v\leq y\leq\lambda v}}\Big(V_{11}^{(0)}(\Delta_{11}s(x,y)) - V_{11}^{(0)}(\Delta_{11}s(u,v))\Big).
\end{aligned}$$

Taking lim inf of both sides of the last inequality as $u,v\to\infty$, we have

$$\begin{aligned}
&\liminf_{u,v\to\infty}\frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}\Big(V_{11}^{(0)}(\Delta_{11}s(x,y)) - V_{11}^{(0)}(\Delta_{11}s(u,v))\Big)dxdy\\
&\geq \liminf_{u,v\to\infty}\min_{\substack{u\leq x\leq\lambda u\\ v\leq y\leq\lambda v}}\Big(V_{11}^{(0)}(\Delta_{11}s(x,y)) - V_{11}^{(0)}(\Delta_{11}s(u,y))\Big) + \liminf_{u,v\to\infty}\min_{v\leq y\leq\lambda v}\Big(V_{11}^{(0)}(\Delta_{11}s(u,y)) - V_{11}^{(0)}(\Delta_{11}s(u,v))\Big)
\end{aligned}$$

Since $V_{11}(\Delta_{11}s(u,v))$ is slowly decreasing in sense $(0,1)$ and strong slowly decreasing in sense $(1,0)$, we get

$$\liminf_{u,v\to\infty}\frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}\Big(V_{11}^{(0)}(\Delta_{11}s(x,y)) - V_{11}^{(0)}(\Delta_{11}s(u,v))\Big)dxdy \quad \geq \quad 0 \tag{16}$$

by taking the limit of both sides of the last inequality as $\lambda \to 1^+$. Hence from (15) and (16), we obtain

$$\limsup_{u,v\to\infty}\Big(V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,v))\Big) \leq 0. \tag{17}$$

In addition, taking the lim inf of both sides of (12) as $u, v \to \infty$, we have

$$\liminf_{u,v\to\infty}\Big(V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,v))\Big) \geq \left(\frac{\lambda}{1-\lambda}\right)^2 \liminf_{u,v\to\infty}\Big(V_{11}^{(1)}(\Delta_{11}s(\lambda u,\lambda v)) - V_{11}^{(1)}(\Delta_{11}s(u,v))\Big)$$
$$+ \frac{\lambda}{(1-\lambda)^2}\liminf_{u,v\to\infty}\Big(V_{11}^{(1)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(\lambda u,v))\Big)$$
$$+ \frac{\lambda}{(1-\lambda)^2}\liminf_{u,v\to\infty}\Big(V_{11}^{(1)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,\lambda v))\Big)$$
$$+ \liminf_{u,v\to\infty}\frac{1}{(u-\lambda u)(v-\lambda v)}\int_{\lambda u}^{u}\int_{\lambda v}^{v}\Big(V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(0)}(\Delta_{11}s(x,y))\Big)dxdy. \quad (18)$$

Since $V_{11}^{(1)}(\Delta_{11}s(u,v))$ convergent to zero, the first three terms on the right-hand side of the previous inequality are vanished. Moreover,

$$\frac{1}{(u-\lambda u)(v-\lambda v)}\int_{\lambda u}^{u}\int_{\lambda v}^{v}\Big(V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(0)}(\Delta_{11}s(x,y))\Big)dxdy$$
$$\geq \min_{\substack{\lambda u\leq x\leq u\\ \lambda v\leq y\leq v}}\Big(V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(0)}(\Delta_{11}s(x,y))\Big).$$

Taking lim inf of both sides of the last inequality as $u, v \to \infty$, we have

$$\liminf_{u,v\to\infty}\frac{1}{(u-\lambda u)(v-\lambda v)}\int_{\lambda u}^{u}\int_{\lambda v}^{v}\Big(V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(0)}(\Delta_{11}s(x,y))\Big)dxdy$$
$$\geq \liminf_{u,v\to\infty}\min_{\lambda v\leq y\leq v}\Big(V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(0)}(\Delta_{11}s(u,y))\Big)$$
$$+ \liminf_{u,v\to\infty}\min_{\substack{\lambda u\leq x\leq u\\ \lambda v\leq y\leq v}}\Big(V_{11}^{(0)}(\Delta_{11}s(u,y)) - V_{11}^{(0)}(\Delta_{11}s(x,y))\Big).$$

Since $V_{11}(\Delta_{11}s(u,v))$ is slowly decreasing in sense $(0,1)$ and strong slowly decreasing in sense $(1,0)$, we get

$$\liminf_{u,v\to\infty}\frac{1}{(u-\lambda u)(v-\lambda v)}\int_{\lambda u}^{u}\int_{\lambda v}^{v}\Big(V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(0)}(\Delta_{11}s(x,y))\Big)dxdy \quad \geq \quad 0 \quad (19)$$

by taking limit of both sides of the last inequality as $\lambda \to 1^-$. Hence from (18) and (19) we obtain

$$\liminf_{u,v\to\infty}\Big(V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,v))\Big) \geq 0. \quad (20)$$

Thus we have (4) from (17), (20) and (3) as in the proof of Theorem 2.1. □

Proof of Theorem 2.4 Assume that the double integral $s(u,v)$ is bounded and (1) is $(C,1,1)$ summable to a finite number L. Without loss of generality, we assume that $V_{11}(\Delta_{11}s(u,v))$ is slowly oscillating in sense $(0,1)$ and strong slowly oscillating in sense $(1,0)$. If we do the same calculations as in the proof of Theorem 2.1, we obtain that $V_{11}^{(1)}(\Delta_{11}s(u,v))$ convergent to zero.

From (9),we get

$$|V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,v))| \leq \left(\frac{\lambda}{\lambda-1}\right)^2\Big|V_{11}^{(1)}(\Delta_{11}s(\lambda u,\lambda v)) - V_{11}^{(1)}(\Delta_{11}s(u,v)\Big|$$
$$+\frac{\lambda}{(\lambda-1)^2}\Big|V_{11}^{(1)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(\lambda u,v))\Big|$$
$$+\frac{\lambda}{(\lambda-1)^2}\Big|V_{11}^{(1)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,\lambda v))\Big|$$
$$+\left|\frac{1}{(\lambda u-u)(\lambda v-v)}\int_{u}^{\lambda u}\int_{v}^{\lambda v}\Big(V_{11}^{(0)}(\Delta_{11}s(x,y)) - V_{11}^{(0)}(\Delta_{11}s(u,v))\Big)dxdy\right|. \quad (21)$$

Taking lim sup of both sides of (21) as $u, v \to \infty$, we have

$$\limsup_{u,v\to\infty} |V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,v))| \leq \left(\frac{\lambda}{\lambda-1}\right)^2 \limsup_{u,v\to\infty} \left|V_{11}^{(1)}(\Delta_{11}s(\lambda u, \lambda v)) - V_{11}^{(1)}(\Delta_{11}s(u,v)\right|$$
$$+ \frac{\lambda}{(\lambda-1)^2} \limsup_{u,v\to\infty} \left|V_{11}^{(1)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(\lambda u, v))\right|$$
$$+ \frac{\lambda}{(\lambda-1)^2} \limsup_{u,v\to\infty} \left|V_{11}^{(1)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u, \lambda v))\right|$$
$$+ \limsup_{u,v\to\infty} \left|\frac{1}{(\lambda u - u)(\lambda v - v)} \int_u^{\lambda u} \int_v^{\lambda v} \left(V_{11}^{(0)}(\Delta_{11}s(x,y)) - V_{11}^{(0)}(\Delta_{11}s(u,v))\right) dxdy\right|.$$

Since $V_{11}^{(1)}(\Delta_{11}s(u,v))$ convergent to zero, the first three terms on the right-hand side of the last inequality are vanished. Moreover,

$$\left|\frac{1}{(\lambda u - u)(\lambda v - v)} \int_u^{\lambda u} \int_v^{\lambda v} \left(V_{11}^{(0)}(\Delta_{11}s(x,y)) - V_{11}^{(0)}(\Delta_{11}s(u,v))\right) dxdy\right|$$
$$\leq \max_{\substack{u\leq x\leq \lambda u\\ v\leq y\leq \lambda v}} \left|V_{11}^{(0)}(\Delta_{11}s(x,y)) - V_{11}^{(0)}(\Delta_{11}s(u,v))\right|.$$

Taking lim sup of both sides of the last inequality as $u, v \to \infty$, we have

$$\limsup_{u,v\to\infty} \left|\frac{1}{(\lambda u - u)(\lambda v - v)} \int_u^{\lambda u} \int_v^{\lambda v} \left(V_{11}^{(0)}(\Delta_{11}s(x,y)) - V_{11}^{(0)}(\Delta_{11}s(u,v))\right) dxdy\right|$$
$$\leq \limsup_{u,v\to\infty} \max_{\substack{u\leq x\leq \lambda u\\ v\leq y\leq \lambda v}} \left|V_{11}^{(0)}(\Delta_{11}s(x,y)) - V_{11}^{(0)}(\Delta_{11}s(u,y))\right|$$
$$+ \limsup_{u,v\to\infty} \max_{v\leq y\leq \lambda v} \left|V_{11}^{(0)}(\Delta_{11}s(u,y)) - V_{11}^{(0)}(\Delta_{11}s(u,v))\right|.$$

Since $V_{11}(\Delta_{11}s(u,v))$ is slowly oscillating in sense $(0,1)$ and strong slowly oscillating in sense $(1,0)$, we get

$$\limsup_{u,v\to\infty} \left|\frac{1}{(\lambda u - u)(\lambda v - v)} \int_u^{\lambda u} \int_v^{\lambda v} \left(V_{11}^{(0)}(\Delta_{11}s(x,y)) - V_{11}^{(0)}(\Delta_{11}s(u,v))\right) dxdy\right| \leq 0. \tag{22}$$

by taking the limit of both sides of the last inequality as $\lambda \to 1^+$. From (21) and (22), we obtain

$$\limsup_{u,v\to\infty} |V_{11}^{(0)}(\Delta_{11}s(u,v)) - V_{11}^{(1)}(\Delta_{11}s(u,v))| \leq 0.$$

Hence we have (4) from the last equation and (3) as in the proof of Theorem 2.1. □

3. Tauberian theorems for $(C, 1, 0)$ summability of improper double integrals over $\mathbb{R}_+^2$

Suppose that f is a real- or complex-valued continuous function defined on $\mathbb{R}_+^2$. The mean $(C, 1, 0)$ (or Cesàro mean in sense $(1, 0)$) of $s(u, v)$ is defined by

$$\sigma_{10}(s(u,v)) = \frac{1}{u} \int_0^u s(x,v)dx$$

for $u, v > 0$. The integral (1) is said to be $(C, 1, 0)$ summable (or Cesàro summable in sense $(1, 0)$) to a finite number L if

$$\lim_{u,v\to\infty} \sigma_{10}(s(u,v)) = \lim_{u,v\to\infty} \int_0^u \int_0^v \left(1 - \frac{x}{u}\right) f(x,y)dxdy = L.$$

The backward difference in sense $(1,0)$ of $s(u,v)$ is defined by

$$\Delta_{10}s(u,v) = \frac{\partial s(u,v)}{\partial u} = \int_0^v f(u,y)dy$$

for $u, v > 0$.

From

$$\begin{aligned}\sigma_{10}(s(u,v)) &= \int_0^u \int_0^v \left(1 - \frac{x}{u}\right) f(x,y)dxdy \\ &= \int_0^u \int_0^v f(x,y)dxdy - \frac{1}{u}\int_0^u \int_0^v xf(x,y)dxdy\end{aligned}$$

where $\Delta_{10}s(x,v) = \int_0^v f(x,y)dy$, we have

$$s(u,v) - \sigma_{10}(s(u,v)) = V_{10}^{(0)}(\Delta_{10}s(u,v)), \tag{23}$$

where $V_{10}^{(0)}(\Delta_{10}s(u,v)) = \dfrac{1}{u}\displaystyle\int_0^u x\Delta_{10}s(x,v)dx$. This identity is known as the Kronecker identity in sense $(1,0)$.

For each nonnegative integer m, the iterative means of $s(u,v)$ and $V_{10}^{(0)}(\Delta_{10}s(u,v))$ in sense $(1,0)$ are defined by

$$\sigma_{10}^{(m)}(s(u,v)) = \begin{cases} \dfrac{1}{u}\displaystyle\int_0^u \sigma_{10}^{(m-1)}(s(x,v))dx, & m \geq 1 \\ s(u,v), & m = 0 \end{cases}$$

and

$$V_{10}^{(m)}(\Delta_{10}s(u,v)) = \begin{cases} \dfrac{1}{u}\displaystyle\int_0^u V_{10}^{(m-1)}(\Delta_{10}s(x,v))dx, & m \geq 1 \\ V_{10}^{(0)}(\Delta_{10}s(u,v)), & m = 0 \end{cases}$$

respectively.

First, we consider Landau type a one-sided Tauberian condition sufficient for convergence of $s(u,v)$ to follow from its $(C,1,0)$ summability.

Theorem 3.1. *Let the double integral $s(u,v)$ be bounded. If (1) is $(C,1,0)$ summable to a finite number L and there exist constants $H > 0$ and $x_0 \geq 0$ such that condition*

$$u\Delta_{10}V_{10}^{(0)}(\Delta_{10}s(u,v)) \geq -H \tag{24}$$

is satisfied for all $(u,v) \in \mathbb{R}_+^2$ with $u, v > x_0$, then we have (4).

As corollary of Theorem 3.1, we have the following two-sided Hardy type Tauberian theorem.

Corollary 3.2. *Let the double integral $s(u,v)$ be bounded. If (1) is $(C,1,0)$ summable to a finite number L and there exist constants $H > 0$ and $x_0 \geq 0$ such that condition*

$$u\Delta_{10}V_{10}^{(0)}(\Delta_{10}s(u,v)) = O(1) \tag{25}$$

is satisfied for all $(u,v) \in \mathbb{R}_+^2$ with $u, v > x_0$, then we have (4).

Next, we give one-sided and two sided Tauberian conditions sufficient in order that convergence follows from $(C,1,0)$ summability for real and complex-valued functions, respectively.

Theorem 3.3. *Let the double integral $s(u,v)$ be bounded. If (1) is $(C,1,0)$ summable to a finite number L and $V_{10}(\Delta_{10}s(u,v))$ is slowly decreasing in sense $(1,0)$, then we have (4).*

Theorem 3.4. *Let the double integral $s(u,v)$ be bounded. If (1) is $(C,1,0)$ summable to a finite number L and $V_{10}(\Delta_{10}s(u,v))$ is slowly oscillating in sense $(1,0)$, then we have (4).*

3.1. Proofs of Main Results

The following lemma gives two representations for the difference $s(u,v) - \sigma_{10}(s(u,v))$.

Lemma 3.5. *Let $s(u,v)$ be a double integral over the rectangle $[0,u] \times [0,v]$. For sufficiently large u, v*
(i) *If $\lambda > 1$,*

$$s(u,v) - \sigma_{10}(s(u,v)) = \frac{\lambda}{\lambda - 1}\left(\sigma_{10}(s(\lambda u, v)) - \sigma_{10}(s(u,v))\right) - \frac{1}{\lambda u - u}\int_u^{\lambda u}\left(s(x,v) - s(u,v)\right)dx.$$

(ii) *If $0 < \lambda < 1$,*

$$s(u,v) - \sigma_{10}(s(u,v)) = \frac{\lambda}{1 - \lambda}\left(\sigma_{10}(s(u,v)) - \sigma_{10}(s(\lambda u, v))\right) + \frac{1}{u - \lambda u}\int_{\lambda u}^{u}\left(s(u,v) - s(x,v)\right)dx.$$

Proof. (i) By definition, we have

$$\begin{aligned}
\frac{1}{\lambda u - u}\int_u^{\lambda u} s(x,v)dx &= \frac{1}{(\lambda - 1)u}\left(\int_0^{\lambda u} - \int_0^{u}\right)s(x,v)dx \\
&= \frac{\lambda}{(\lambda - 1)\lambda u}\int_0^{\lambda u} s(x,v)dx - \frac{1}{(\lambda - 1)u}\int_0^{u} s(x,v)dx \\
&= \frac{\lambda}{\lambda - 1}\sigma_{10}(s(\lambda u, v)) - \frac{\lambda}{\lambda - 1}\sigma_{10}(s(u,v)) + \sigma_{10}(s(u,v)) \\
&= \frac{\lambda}{\lambda - 1}\left(\sigma_{10}(s(\lambda u, v)) - \sigma_{10}(s(u,v))\right) + \sigma_{10}(s(u,v)).
\end{aligned}$$

Adding $s(u,v)$ to both sides of the previous equation and then arranging this equality, we reach to the equality (i) of Lemma 3.5.

(ii) The proof of Lemma 3.5 (ii) can be verified in a similar way. □

Proof of Theorem 3.1 Assume that the double integral $s(u,v)$ is bounded, (1) is $(C,1,0)$ summable to L and condition (24) holds. Since the $(C,1,0)$ summability method is regular and $\lim_{u,v\to\infty} \sigma_{10}^{(1)}(s(u,v)) = L$, we have $\lim_{u,v\to\infty} \sigma_{10}^{(2)}(s(u,v)) = L$. Taking $(C,1,0)$ means of both sides of (23), we obtain $\lim_{u,v\to\infty} V_{10}^{(1)}(\Delta_{10}s(u,v)) = 0$.

If we replace $s(u,v)$ by $V_{10}^{(0)}(\Delta_{10}s(u,v))$ in Lemma 3.5 (i), we obtain

$$\begin{aligned}
V_{10}^{(0)}(\Delta_{10}s(u,v)) - V_{10}^{(1)}(\Delta_{10}s(u,v)) &= \frac{\lambda}{\lambda - 1}\left(V_{10}^{(1)}(\Delta_{10}s(\lambda u,v)) - V_{10}^{(1)}(\Delta_{10}s(u,v))\right) \\
&\quad - \frac{1}{\lambda u - u}\int_u^{\lambda u}\left(V_{10}^{(0)}(\Delta_{10}s(x,v)) - V_{10}^{(0)}(\Delta_{10}s(u,v))\right)dx. \qquad (26)
\end{aligned}$$

In addition, we obtain by (24) that

$$\begin{aligned}
V_{10}^{(0)}(\Delta_{10}s(x,v)) - V_{10}^{(0)}(\Delta_{10}s(u,v)) &= \int_u^x \frac{\Delta V_{10}^{(0)}(\Delta_{10}s(t,v))}{\Delta t}dt \\
&\geq -H\int_u^x \frac{dt}{t} \\
&= -H\ln\left(\frac{x}{u}\right) \qquad (27)
\end{aligned}$$

for some $H > 0$. Taking lim sup of both sides of the equality (26) as $u, v \to \infty$ and (27) into consideration, we get

$$\begin{aligned}
\limsup_{u,v\to\infty}\left(V_{10}^{(0)}(\Delta_{10}s(u,v)) - V_{10}^{(1)}(\Delta_{10}s(u,v))\right) &\leq \frac{\lambda}{\lambda - 1}\limsup_{u,v\to\infty}\left(V_{10}^{(0)}(\Delta_{10}s(\lambda u,v)) - V_{10}^{(0)}(\Delta_{10}s(u,v))\right) \\
&\quad + H\limsup_{u,v\to\infty}\ln\left(\frac{\lambda u}{u}\right).
\end{aligned}$$

Since $\lim_{u,v\to\infty} V_{10}^{(1)}(\Delta_{10}s(u,v)) = 0$, the first term on the right-hand side of the previous inequality is vanished and we get

$$\limsup_{u,v\to\infty}\left(V_{10}^{(0)}(\Delta_{10}s(u,v)) - V_{10}^{(1)}(\Delta_{10}s(u,v))\right) \leq H\ln\lambda.$$

Hence, taking the limit of both sides of the last inequality as $\lambda \to 1^+$, we have

$$\limsup_{u,v\to\infty}\left(V_{10}^{(0)}(\Delta_{10}s(u,v)) - V_{10}^{(1)}(\Delta_{10}s(u,v))\right) \leq 0. \tag{28}$$

For $0 < \lambda < 1$, in a similar way from Lemma (3.5) (ii) we have

$$\liminf_{u,v\to\infty}\left(V_{10}^{(0)}(\Delta_{10}s(u,v)) - V_{10}^{(1)}(\Delta_{10}s(u,v))\right) \geq 0. \tag{29}$$

From (28) and (29), we have $\lim_{u,v\to\infty} V_{10}^{(0)}(\Delta_{10}s(u,v)) = \lim_{u,v\to\infty} V_{10}^{(1)}(\Delta_{10}s(u,v)) = 0$. Hence, we have (4) by (23). □

Proof of Corollary 3.2 It is plain that condition (25) implies (24) in Theorem 3.1. □

Proof of Theorem 3.3 Assume that the double integral $s(u,v)$ is bounded, (1) is $(C,1,0)$ summable to a finite number L and $V_{10}(\Delta_{10}s(u,v))$ is slowly decreasing in sense $(1,0)$. If we apply same calculations as in the proof of Theorem 3.1, we obtain that $V_{10}^{(1)}(\Delta_{10}s(u,v))$ convergent to zero.

Taking lim sup of both sides of (26) as $u,v\to\infty$, we have

$$\limsup_{u,v\to\infty}\left(V_{10}^{(0)}(\Delta_{10}s(u,v)) - V_{10}^{(1)}(\Delta_{10}s(u,v))\right) \leq \frac{\lambda}{\lambda-1}\limsup_{u,v\to\infty}\left(V_{10}^{(0)}(\Delta_{10}s(\lambda u,v)) - V_{10}^{(0)}(\Delta_{10}s(u,v))\right)$$
$$- \liminf_{u,v\to\infty}\frac{1}{\lambda u - u}\int_u^{\lambda u}\left(V_{10}^{(0)}(\Delta_{10}s(x,v)) - V_{10}^{(0)}(\Delta_{10}s(u,v))\right)dx. \tag{30}$$

Since $V_{10}^{(1)}(\Delta_{10}s(u,v))$ is convergent to zero, the first term on the right-hand side of the above inequality is vanished. For the second term on the right-hand side of (30), we have

$$\frac{1}{\lambda u - u}\int_u^{\lambda u}\left(V_{10}^{(0)}(\Delta_{10}s(x,v)) - V_{10}^{(0)}(\Delta_{10}s(u,v))\right)dx \geq \min_{u\leq x\leq\lambda u}\left(V_{10}^{(0)}(\Delta_{10}s(x,y)) - V_{10}^{(0)}(\Delta_{10}s(u,v))\right).$$

Taking lim inf of both sides of the last inequality as $u,v\to\infty$, we have

$$\liminf_{u,v\to\infty}\frac{1}{\lambda u - u}\int_u^{\lambda u}\left(V_{10}^{(0)}(\Delta_{10}s(x,v)) - V_{10}^{(0)}(\Delta_{10}s(u,v))\right)dx$$
$$\geq \liminf_{u,v\to\infty}\min_{u\leq x\leq\lambda u}\left(V_{10}^{(0)}(\Delta_{10}s(x,v)) - V_{10}^{(0)}(\Delta_{10}s(u,v))\right).$$

Since $V_{10}(\Delta_{10}s(u,v))$ is slowly decreasing in sense $(1,0)$, we get

$$\liminf_{u,v\to\infty}\frac{1}{\lambda u - u}\int_u^{\lambda u}\left(V_{10}^{(0)}(\Delta_{10}s(x,v)) - V_{10}^{(0)}(\Delta_{10}s(u,v))\right)dx \geq 0 \tag{31}$$

by taking the limit of both sides of the last inequality as $\lambda \to 1^+$. Hence from (30) and (31), we obtain

$$\limsup_{u,v\to\infty}\left(V_{10}^{(0)}(\Delta_{10}s(u,v)) - V_{10}^{(1)}(\Delta_{10}s(u,v))\right) \leq 0. \tag{32}$$

For $0 < \lambda < 1$, in a similar way from Lemma 3.5 (ii) we have

$$\liminf_{u,v\to\infty}\left(V_{10}^{(0)}(\Delta_{10}s(u,v)) - V_{10}^{(1)}(\Delta_{10}s(u,v))\right) \geq 0. \tag{33}$$

Hence, we have (4) from (32), (33) and (23) as in the proof of Theorem 3.1. □

Proof of Theorem 3.4 The proof can be given as in that of Theorem 3.3 by using Lemma 3.5. So we omit the proof of it.

4. Tauberian theorems for $(C, 0, 1)$ summability of improper double integrals over $\mathbb{R}^2_+$

Assume that f is a real- or complex-valued continuous function defined on $\mathbb{R}^2_+$. The mean $(C, 0, 1)$ (or Cesàro mean in sense $(0, 1)$) of $s(u, v)$ is defined by

$$\sigma_{01}(s(u,v)) = \frac{1}{v}\int_0^v s(u,y)dy$$

for $u, v > 0$. The integral (1) is said to be $(C, 0, 1)$ summable (or Cesàro summable in sense $(0, 1)$) to a finite number L if

$$\lim_{u,v\to\infty} \sigma_{01}(s(u,v)) = \lim_{u,v\to\infty} \int_0^u \int_0^v \left(1 - \frac{y}{v}\right) f(x,y)dxdy = L.$$

The backward difference in sense $(0, 1)$ of $s(u, v)$ is defined by

$$\Delta_{01}s(u,v) = \frac{\partial s(u,v)}{\partial v} = \int_0^u f(x,v)dx$$

for $u, v > 0$.

From

$$\begin{aligned}\sigma_{01}(s(u,v)) &= \int_0^u \int_0^v \left(1 - \frac{y}{v}\right) f(x,y)dxdy \\ &= \int_0^u \int_0^v f(x,y)dxdy - \frac{1}{v}\int_0^u \int_0^v yf(x,y)dxdy\end{aligned}$$

where $\Delta_{01}s(u,y) = \int_0^u f(x,y)dx$, we have

$$s(u,v) - \sigma_{01}(s(u,v)) = V_{01}^{(0)}(\Delta_{01}s(u,v))$$

where $V_{01}^{(0)}(\Delta_{01}s(u,v)) = \frac{1}{v}\int_0^v y\Delta_{01}s(u,y)dy$. This identity is known as the Kronecker identity in sense $(0, 1)$.

For each nonnegative integer m, the iterative means of $s(u, v)$ and $V_{01}^{(0)}(\Delta_{01}s(u, v))$ in sense $(0, 1)$ are defined by

$$\sigma_{01}^{(m)}(s(u,v)) = \begin{cases} \frac{1}{v}\int_0^v \sigma_{01}^{(m-1)}(s(u,y))dy, & m \geq 1 \\ s(u,v), & m = 0 \end{cases}$$

and

$$V_{01}^{(m)}(\Delta_{01}s(u,v)) = \begin{cases} \frac{1}{v}\int_0^v V_{01}^{(m-1)}(\Delta_{01}s(u,y))dy, & m \geq 1 \\ V_{01}^{(0)}(\Delta_{01}s(u,v)), & m = 0 \end{cases}$$

respectively.

First, we consider Landau type a one-sided Tauberian condition sufficient for convergence of $s(u, v)$ to follow from its $(C, 0, 1)$ summability.

Theorem 4.1. *Let the double integral $s(u, v)$ be bounded. If (1) is* $(C, 0, 1)$ *summable to a finite number L and there exist constants $H > 0$ and $x_0 \geq 0$ such that condition*

$$v\Delta_{01}V_{01}^{(0)}(\Delta_{01}s(u,v)) \geq -H \tag{34}$$

is satisfied for all $(u, v) \in \mathbb{R}^2_+$ with $u, v > x_0$, then we have (4).

As corollary of Theorem 4.1, we have the following two-sided Hardy type Tauberian theorem.

Corollary 4.2. *Let the double integral $s(u,v)$ be bounded. If (1) is $(C,0,1)$ summable to a finite number L and there exist constants $H>0$ and $x_0 \geq 0$ such that condition*

$$v\Delta_{01}V_{01}^{(0)}(\Delta_{01}s(u,v)) = O(1) \tag{35}$$

is satisfied for all $(u,v)\in\mathbb{R}_+^2$ with $u,v>x_0$, then we have (4).

Next, we give one-sided and two sided Tauberian conditions sufficient in order that convergence follows from $(C,0,1)$ summability for real and complex-valued functions, respectively.

Theorem 4.3. *Let the double integral $s(u,v)$ be bounded. If (1) is $(C,0,1)$ summable to a finite number L and $V_{01}(\Delta_{01}s(u,v))$ is slowly decreasing in sense $(0,1)$, then then we have (4).*

Theorem 4.4. *Let the double integral $s(u,v)$ be bounded. If (1) is $(C,0,1)$ summable to a finite number L and $V_{01}(\Delta_{01}s(u,v))$ is slowly oscillating in sense $(0,1)$, then then we have (4).*

4.1. Proofs of Main Results

The following lemma gives two representations for the difference $s(u,v)-\sigma_{01}(s(u,v))$.

Lemma 4.5. *Let $s(u,v)$ be a double integral over the rectangle $[0,u]\times[0,v]$. For sufficiently large u,v*
(i) *If $\lambda>1$,*

$$s(u,v)-\sigma_{01}(s(u,v)) = \frac{\lambda}{\lambda-1}\left(\sigma_{01}(s(u,\lambda v))-\sigma_{01}(s(u,v))\right)-\frac{1}{\lambda v-v}\int_v^{\lambda v}\left(s(u,y)-s(u,v)\right)dy.$$

(ii) *If $0<\lambda<1$,*

$$s(u,v)-\sigma_{01}(s(u,v)) = \frac{\lambda}{1-\lambda}\left(\sigma_{01}(s(u,v))-\sigma_{01}(s(u,\lambda v))\right)+\frac{1}{v-\lambda v}\int_{\lambda v}^{v}\left(s(u,v)-s(u,y)\right)dy.$$

Proof of Theorem 4.1 The proof can be given as in that of Theorem 3.1 by using Lemma 4.5. □

Proof of Corollary 4.2 It is plain that condition (35) implies (34) in Theorem 4.1. □

Proof of Theorem 4.3 The proof can be given as in that of Theorem 3.3 by using Lemma 4.5. □

Proof of Theorem 4.4 The proof can be given as in that of Theorem 4.3 by using Lemma 4.5. □

Conclusion

In this paper, one-sided and two-sided Tauberian conditions have been obtained in terms of the difference between double integral of $s(u,v)$ and its means in different senses for Cesàro summability methods of double integrals over $[0,u]\times[0,v]$ under which convergence of $s(u,v)$ follows from integrability of $s(u,v)$ in different senses. As a natural continuation of this work, we plan to extend the results obtained to weighted summability of double integrals in different senses over $\mathbb{R}_+^2$ in the forthcoming paper.

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