Filomat 33:11 (2019), 3409–3424
https://doi.org/10.2298/FIL1911409E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalized Truncated Distributions with N Intervals Deleted: Mathematical Definition

Mohamed Abd Allah El-Hadidy[a,b]

[a]*Mathematics and Statistics Department, College of Science, Taibah University, Yanbu, Saudi Arabia*
[b]*Mathematics Department, Faculty of Science, Tanta University, Tanta, Egypt*

Abstract. The distributions obtained by N intervals truncations are characterized by its high sensitivity for stochastic volatility data. In stable intervals, we use this method to delete some certain range of data values from a domain of the random variable. A comprehensive treatment of the statistical properties of this distribution is presented. We assume Normal and Log-Lindley distributions to apply the obtained results.

1. Introduction

The truncation method is an important methodology to delete some certain range of values from a domain of the random variable Y. In other words, the truncation happens when we need to delete a dataset which have values that are outside of a usual range. For instance, in some probability distributions you might restrict your y-values between two values a and b. There are three known kinds of truncated distributions: (a) Truncated from above, high values of y are cut off so your range is from negative infinity to some maximum value of y $\{-\infty, y_{max}\}$, (b) Truncated from below, low values of y are cut off so your range is from some minimum value of y to positive infinity $\{y_{min}, \infty\}$; and (c) Double truncation, both the low values and y values are cut off $\{y_{min}, y_{max}\}$. Ali and Nadarajah [14] introduced a truncated version of the Pareto distribution. They derived the explicit expressions for the moments for the truncated version. Nadarajah [19] introduced truncated versions of five of the most commonly known log-tailed distributions which possess finite moments of all orders and could therefore be better models. Zaninetti and Ferraro [10] presented a comparison between the Pareto and truncated Pareto distributions. In [6],[11],[17],[20] and [22] some details about truncated distribution have been discussed.
What should we do to delete some values in some intervals for some probability distributions such as stock price distribution (Wiener process range distribution which has been studied by Feller [21], Withers and Nadarajah [3] and Teamah et al. [1])?. In this paper, we give the answer for this question by presenting a new definition for the N intervals truncated distribution, which generalizes existing double truncation distribution to truncation on multiple intervals.
This paper is organized as follows: in Section 2, we provide the definition of N intervals truncated distribution and discuss some of its statistical properties. The method of order statistics is derived in Section 3

2010 *Mathematics Subject Classification*. Primary 60E05
Keywords. Truncated life time distribution; Characterizations using order statistics; Parameters Estimation
Received: 20 December 2018; Accepted: 18 June 2019
Communicated by Biljana Popović
Email address: melhadidi@science.tanta.edu.eg (Mohamed Abd Allah El-Hadidy)

for some characterization of N intervals truncated distribution. In section 4, we apply the results which obtained in sections 2 and 3 on Normal and Log-Lindley distributions with a real dataset. Section 5 ends the paper with some concluding remarks and future works.

2. N intervals truncated distribution

Let Y be a random variable has a probability density function (PDF) $g_Y(y)$ and cumulative distribution function (CDF) $G_Y(y)$. We need to delete some certain range of values from its domain at some intervals. We are interested in the distribution of N intervals truncated random variable at the intervals (α_j, β_j) where $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < ... < \alpha_N < \beta_N$ and defined by:

Definition 2.1. *Let Y be a random variable with known probability density function $g_Y(y)$, define X as a corresponding N intervals truncated version of the random variable Y with PDF $f_X(x)$. Then, the probability density function of N intervals truncated of Y is given by:*

$$f_X(x) = \begin{cases} \frac{g_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}[G_Y(\alpha_j)-G_Y(\beta_{j-1})]}, & if \quad x \in (-\infty, \alpha_1) \bigcup (\beta_1, \alpha_2) \bigcup (\beta_2, \alpha_3) \bigcup ... \bigcup (\beta_{N-1}, \alpha_N) \bigcup (\beta_N, \infty), \\ 0, & Otherwise, \end{cases} \tag{1}$$

where $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < ... < \alpha_N < \beta_N$ and $\beta_0 = -\infty$.

The idea of the above definition has been drawn from the idea of left, right and double truncation, which studied in [10], [14], [19]. In our definition, we distribute the probability of the truncated parts (i.e., the parts which we need to delete from the domain of X) on the remaining parts (i.e., the undeleted parts from the domain of X) with equal proportions.

2.1. Special cases

For this new definition we get some special cases as follows:

Case 1 If we let $N = 1$ and $(\alpha_1, \beta_1) = (-\infty, \alpha)$, $\beta_{-1} = -\infty$ then we have,

$$f_X(x) = \frac{g_X(x)}{1 - G_Y(\alpha) + [G_Y(-\infty) - G_Y(\beta_{-1})]} = \frac{g_X(x)}{1 - G_Y(\alpha) + [G_Y(-\infty) - G_Y(-\infty)]} = \frac{g_X(x)}{1 - G_Y(\alpha)}, \qquad \alpha < x < \infty, \tag{2}$$

which is a definition of the left truncated distribution which have been studied before by Zaninetti [11].

Case 2 If $(\alpha_1, \beta_1) = (\beta, \infty)$ then we have,

$$f_X(x) = \frac{g_X(x)}{1 - G_Y(\infty) + [G_Y(\beta) - G_Y(\beta_{-1})]} = \frac{g_X(x)}{1 - G_Y(\infty) + [G_Y(\beta) - G_Y(-\infty)]} = \frac{g_X(x)}{G_Y(\beta)}, \qquad -\infty < x < \beta, \tag{3}$$

that is a definition of the right truncated distribution.

Case 3 If we let $N = 2$ and $(\alpha_1, \beta_1) = (-\infty, \alpha)$, $(\alpha_2, \beta_2) = (\beta, \infty)$ then we have,

$$f_X(x) = \begin{cases} \frac{g_X(x)}{1-G_Y(\infty)+[G_Y(-\infty)-G_Y(\beta_{-1})]+[G_Y(\beta)-G_Y(\alpha)]}, & -\infty < x < \alpha, \\ \frac{g_X(x)}{1-G_Y(\infty)+[G_Y(-\infty)-G_Y(\beta_{-1})]+[G_Y(\beta)-G_Y(\alpha)]}, & \beta < x < \infty, \end{cases} = \begin{cases} \frac{g_X(x)}{G_Y(\beta)-G_Y(\alpha)}, & -\infty < x < \alpha, \\ \frac{g_X(x)}{G_Y(\beta)-G_Y(\alpha)}, & \beta < x < \infty, \end{cases} \tag{4}$$

where the complement of $f_X(x)$ gives the PDF of the double-truncated distribution where the random variable will become situated between α and β (i.e., $\alpha < x < \beta$) which have been studied before by Johnson et al. [17].

2.2. Distribution, Survival, Hazard and Reversed Hazard functions

The distribution function of the N intervals truncated random variable is given by:

$$F_X(x)=\begin{cases}\dfrac{G_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\left[G_Y(\alpha_j)-G_Y(\beta_{j-1})\right]}, & -\infty<x<\alpha_1,\\ \dfrac{G_Y(\alpha_1)+G_X(x)-G_Y(\beta_1)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\left[G_Y(\alpha_j)-G_Y(\beta_{j-1})\right]}, & \beta_1<x<\alpha_2,\\ \dfrac{G_Y(\alpha_1)+G_Y(\alpha_2)-G_Y(\beta_1)+G_X(x)-G_Y(\beta_2)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\left[G_Y(\alpha_j)-G_Y(\beta_{j-1})\right]}, & \beta_2<x<\alpha_3,\\ \dots & \\ \dfrac{G_Y(\alpha_1)+G_Y(\alpha_2)-G_Y(\beta_1)+\dots+G_Y(\alpha_{N-1})-G_Y(\beta_{N-2})+G_X(x)-G_Y(\beta_{N-1})}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\left[G_Y(\alpha_j)-G_Y(\beta_{j-1})\right]}, & \beta_{N-1}<x<\alpha_N,\\ \dfrac{G_Y(\alpha_1)+G_Y(\alpha_2)-G_Y(\beta_1)+\dots+G_Y(\alpha_N)-G_Y(\beta_{N-1})+G_X(x)-G_Y(\beta_N)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\left[G_Y(\alpha_j)-G_Y(\beta_{j-1})\right]}, & \beta_N<x<\infty,\end{cases} \tag{5}$$

Also, its survival function is given by:

$$\tilde{F}_X(x)=\begin{cases}1-\left[\dfrac{G_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\left[G_Y(\alpha_j)-G_Y(\beta_{j-1})\right]}\right], & -\infty<x<\alpha_1,\\ 1-\left[\dfrac{2G_Y(\alpha_1)-G_Y(\beta_1)+G_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\left[G_Y(\alpha_j)-G_Y(\beta_{j-1})\right]}\right], & \beta_1<x<\alpha_2,\\ 1-\left[\dfrac{3G_Y(\alpha_1)-2G_Y(\beta_1)+2G_Y(\alpha_2)-G_Y(\beta_2)+G_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\left[G_Y(\alpha_j)-G_Y(\beta_{j-1})\right]}\right], & \beta_2<x<\alpha_3,\\ \dots & \\ 1-\left[\dfrac{NG_Y(\alpha_1)-(N-1)G_Y(\beta_1)+(N-1)G_Y(\alpha_2)-(N-2)G_Y(\beta_2)+\dots+2G_Y(\alpha_{N-1})-G_Y(\beta_{N-1})+G_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\left[G_Y(\alpha_j)-G_Y(\beta_{j-1})\right]}\right], & \beta_{N-1}<x<\alpha_N,\\ 1-\left[\dfrac{(N+1)G_Y(\alpha_1)-NG_Y(\beta_1)+NG_Y(\alpha_2)-(N-1)G_Y(\beta_2)+(N-1)G_Y(\alpha_3)-\dots+2G_Y(\alpha_N)-G_Y(\beta_N)+G_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\left[G_Y(\alpha_j)-G_Y(\beta_{j-1})\right]}\right], & \beta_N<x<\infty,\end{cases} \tag{6}$$

The hazard function is a very useful function in life time analysis, see Marshall and Olkin [16]. For the truncation distribution, the hazard rate function is given by:

$$\Theta_X(x)=\begin{cases}\dfrac{\frac{g_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}[G_Y(\alpha_j)-G_Y(\beta_{j-1})]}}{1-\left[\frac{G_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}[G_Y(\alpha_j)-G_Y(\beta_{j-1})]}\right]}, & -\infty<x<\alpha_1,\\ \dfrac{\frac{g_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}[G_Y(\alpha_j)-G_Y(\beta_{j-1})]}}{1-\left[\frac{2G_Y(\alpha_1)-G_Y(\beta_1)+G_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}[G_Y(\alpha_j)-G_Y(\beta_{j-1})]}\right]}, & \beta_1<x<\alpha_2,\\ \dfrac{\frac{g_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}[G_Y(\alpha_j)-G_Y(\beta_{j-1})]}}{1-\left[\frac{3G_Y(\alpha_1)-2G_Y(\beta_1)+2G_Y(\alpha_2)-G_Y(\beta_2)+G_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}[G_Y(\alpha_j)-G_Y(\beta_{j-1})]}\right]}, & \beta_2<x<\alpha_3,\\ \dots & \\ \dfrac{\frac{g_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}[G_Y(\alpha_j)-G_Y(\beta_{j-1})]}}{1-\left[\frac{NG_Y(\alpha_1)-(N-1)G_Y(\beta_1)+(N-1)G_Y(\alpha_2)-(N-2)G_Y(\beta_2)+\dots+2G_Y(\alpha_{N-1})-G_Y(\beta_{N-1})+G_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}[G_Y(\alpha_j)-G_Y(\beta_{j-1})]}\right]}, & \beta_{N-1}<x<\alpha_N,\\ \dfrac{\frac{g_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}[G_Y(\alpha_j)-G_Y(\beta_{j-1})]}}{1-\left[\frac{(N+1)G_Y(\alpha_1)-NG_Y(\beta_1)+NG_Y(\alpha_2)-(N-1)G_Y(\beta_2)+(N-1)G_Y(\alpha_3)-\dots+2G_Y(\alpha_N)-G_Y(\beta_N)+G_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}[G_Y(\alpha_j)-G_Y(\beta_{j-1})]}\right]}, & \beta_N<x<\infty,\end{cases} \tag{7}$$

Now, we need to obtain the reverse hazard rate function. This function is useful in the analysis of data in the presence of left censored observations and it is used to discuss the lifetime distribution with reversed time scale. The reverse hazard rate is natural if the time scale is reversed. Also, this rate is important in the study of systems because it has an affinity for series systems and it is more appropriate for studying parallel systems. Thus, by using our definition the reverse hazard rate function will become:

$$\tilde{\Theta}_X(x) = \begin{cases} \frac{g_X(x)}{G_X(x)}, & -\infty < x < \alpha_1, \\ \frac{g_X(x)}{G_Y(\alpha_1)-G_Y(\beta_1)+G_X(x)}, & \beta_1 < x < \alpha_2, \\ \frac{g_X(x)}{G_Y(\alpha_1)+G_Y(\alpha_2)-G_Y(\beta_1)-G_Y(\beta_2)+G_X(x)}, & \beta_2 < x < \alpha_3, \\ \dots & \\ \frac{g_X(x)}{G_Y(\alpha_1)+G_Y(\alpha_2)-G_Y(\beta_1)+\dots+G_Y(\alpha_{N-1})-G_Y(\beta_{N-2})-G_Y(\beta_{N-1})+G_X(x)}, & \beta_{N-1} < x < \alpha_N, \\ \frac{g_X(x)}{G_Y(\alpha_1)+G_Y(\alpha_2)-G_Y(\beta_1)+\dots+G_Y(\alpha_N)-G_Y(\beta_{N-1})-G_Y(\beta_N)+G_X(x)}, & \beta_N < x < \infty, \end{cases} \tag{8}$$

2.3. *Kth moment*

Moments are used to study some of the most important features of the distribution. One of these features is the first four moments which can used to describe some characteristics of a distribution such as the skewness and kurtosis. The *Kth* moment of the N intervals truncated random variable is given by:

$$\Omega_t^{(k)} = \frac{\int_{-\infty}^{\alpha_1} x^k f_X(x)dx + \int_{\beta_1}^{\alpha_2} x^k f_X(x)dx + \int_{\beta_2}^{\alpha_3} x^k f_X(x)dx + \dots + \int_{\beta_N}^{\infty} x^k f_X(x)dx}{1 - G_Y(\beta_N) + \sum_{j=1}^{N}\left[G_Y(\alpha_j) - G_Y(\beta_{j-1})\right]}.$$

Using the integration by parts, one can get:

$$\begin{aligned} \Omega_t^{(k)} &= \frac{\alpha_1^k F_X(\alpha_1) - \int_{-\infty}^{\alpha_1} kx^{k-1}F_X(x)dx + \alpha_2^k F_X(\alpha_2) - \beta_1^k F_X(\beta_1) - \int_{\beta_1}^{\alpha_2} kx^{k-1}F_X(x)dx + \dots + 0 - \beta_N^k F_X(\beta_N) + \int_{-\infty}^{\beta_N} kx^{k-1}F_X(x)dx}{1 - G_Y(\beta_N) + \sum_{j=1}^{N}\left[G_Y(\alpha_j) - G_Y(\beta_{j-1})\right]} \\ &= \frac{-\beta_N^k F_X(\beta_N) + \sum_{j=1}^{N}\left[\alpha_j^k F_X(\alpha_j) - \beta_{j-1}^k F_X(\beta_{j-1})\right] - k\left[\sum_{j=1}^{N-1}\int_{\beta_j}^{\alpha_{j+1}} x^{k-1}F_X(x)dx - \int_{-\infty}^{\alpha_1} x^{k-1}F_X(x)dx + \int_{-\infty}^{\beta_N} x^{k-1}F_X(x)dx\right]}{1 - G_Y(\beta_N) + \sum_{j=1}^{N}\left[G_Y(\alpha_j) - G_Y(\beta_{j-1})\right]}. \end{aligned} \tag{9}$$

2.4. *Moment generating and characteristic functions*

The moment generating function of the N intervals truncated random variable is given by:

$$M_X^{(*)}(t) = \frac{\int_{-\infty}^{\alpha_1} e^{tx} f_X(x)dx + \int_{\beta_1}^{\alpha_2} e^{tx} f_X(x)dx + \int_{\beta_2}^{\alpha_3} e^{tx} f_X(x)dx + \dots + \int_{\beta_N}^{\infty} e^{tx} f_X(x)dx}{1 - G_Y(\beta_N) + \sum_{j=1}^{N}\left[G_Y(\alpha_j) - G_Y(\beta_{j-1})\right]},$$

after integration by parts:

$$M_X^{(*)}(t) = \frac{M_X(t) - e^{t\beta_N}F_X(\beta_N) + \sum_{j=1}^{N}\left[e^{t\alpha_j}F_X(\alpha_j) - e^{t\beta_{j-1}}F_X(\beta_{j-1})\right] - t\left[\sum_{j=1}^{N-1}\int_{\beta_j}^{\alpha_{j+1}} e^{tx}F_X(x)dx + \int_{-\infty}^{\alpha_1} e^{tx}F_X(x)dx - \int_{-\infty}^{\beta_N} e^{tx}F_X(x)dx\right]}{1 - G_Y(\beta_N) + \sum_{j=1}^{N}\left[G_Y(\alpha_j) - G_Y(\beta_{j-1})\right]}. \tag{10}$$

Similarly, one can obtain the characteristic function of the N intervals truncated random variable by:

$$\Phi_X^{(*)}(t) = \frac{\Phi_X(t) - e^{it\beta_N}F_X(\beta_N) + \sum_{j=1}^{N}\left[e^{it\alpha_j}F_X(\alpha_j) - e^{it\beta_{j-1}}F_X(\beta_{j-1})\right] - t\left[\sum_{j=1}^{N-1}\int_{\beta_j}^{\alpha_{j+1}} e^{itx}F_X(x)dx + \int_{-\infty}^{\alpha_1} e^{itx}F_X(x)dx - \int_{-\infty}^{\beta_N} e^{itx}F_X(x)dx\right]}{1 - G_Y(\beta_N) + \sum_{j=1}^{N}\left[G_Y(\alpha_j) - G_Y(\beta_{j-1})\right]}. \tag{11}$$

2.5. Likelihood function

The likelihood function is one of the most commonly used method to estimate the parameter θ of the distribution. Let $x_1, x_2, ..., x_h$ be a random sample of size h from N intervals truncated distribution with PDF (1), then the likelihood of this distribution can be derived as:

$$\begin{aligned} L(\theta|x) = & \prod_{i=1}^{h_1} \frac{f_X(x_i;\theta)}{1 - G_Y(\beta_N;\theta) + \sum_{j=1}^{N}\left[G_Y(\alpha_j;\theta) - G_Y(\beta_{j-1};\theta)\right]} \prod_{i=h_1+1}^{h_2} \frac{f_X(x_i;\theta)}{1 - G_Y(\beta_N;\theta) + \sum_{j=1}^{N}\left[G_Y(\alpha_j;\theta) - G_Y(\beta_{j-1};\theta)\right]} \cdots \\ & \prod_{i=h_N+1}^{h} \frac{f_X(x_i;\theta)}{1 - G_Y(\beta_N;\theta) + \sum_{j=1}^{N}\left[G_Y(\alpha_j;\theta) - G_Y(\beta_{j-1};\theta)\right]} \\ = & \frac{\prod_{i=1}^{h} f_X(x_i;\theta)}{\left(1 - G_Y(\beta_N;\theta) + \sum_{j=1}^{N}\left[G_Y(\alpha_j;\theta) - G_Y(\beta_{j-1};\theta)\right]\right)^h}. \end{aligned} \tag{12}$$

Then, the log-likelihood function is given by:

$$\log L(\theta|x) = \sum_{i=1}^{h} \log f_X(x_i;\theta) - h \log\left(1 - G_Y(\beta_N;\theta) + \sum_{j=1}^{N}\left[G_Y(\alpha_j;\theta) - G_Y(\beta_{j-1};\theta)\right]\right).$$

To get the estimated value $\hat{\theta}$ of the parameter θ, we solve the following equation:

$$\frac{\partial \log L(\theta|x)}{\partial\theta} = \frac{\partial}{\partial\theta}\left[\sum_{i=1}^{h} \log f_X(x_i;\theta) - h \log\left(1 - G_Y(\beta_N;\theta) + \sum_{j=1}^{N}\left[G_Y(\alpha_j;\theta) - G_Y(\beta_{j-1};\theta)\right]\right)\right]_{\theta=\hat{\theta}} = 0.$$

3. Characterization using order statistics

Kendall-Stuart [8] and Johnson-Kotz [7] studied the characterizations of statistical distributions by collecting and describing various characterization theorems for each distribution. Kagan et al. [9] used probabilistic and statistical methods and properties which leads to characterizations, including a form of characteristic functions, behaviour of linear statistics, independence, sufficiency, regression properties, etc. In addition, some authors studied the characterizations of statistical distributions from a purely mathematical point of view such as Lukacs [12], [13].
Let $X_1, X_2, ..., X_m$ be an independent identically distributed sequence of random variables with PDF $f_X(x)$ and CDF $F_X(x)$ then $X_{1:m} \leq X_{2:m} \leq ... \leq X_{m:m}$ be the order statistics with PDF:

$$f_{r:m}(x) = \frac{m!}{(r-1)!(m-r)!}[F_X(x)]^{r-1}[1 - F_X(x)]^{m-r} f_X(x), \qquad 1 \leq r \leq m, \tag{13}$$

and the joint PDF of the rth and sth order statistics where $1 \leq r < s \leq m$ is given by:

$$f_{r,s:m}(x,z) = \frac{m!}{(r-1)!(s-r-1)!(m-s)!}[F_X(x)]^{r-1}[F_Z(z) - F_X(x)]^{s-r-1}[1 - F_Z(z)]^{m-s} f_X(x) f_Z(z), \qquad 1 \leq r < s \leq m, \tag{14}$$

(see Arnold et al. [2]). To characterize the N intervals truncated distribution based on this order statistics, we shall prove the following theorem as in Mohie El-Din et al. [15].

Theorem 3.1. *The random variable X has N intervals truncated distribution with PDF (1) if and only if:*

$$(i)\qquad E\Big(X_{r+1:m}^{k}|X_{r:m}=x\Big)=x^{k}+\frac{k}{\Big[1-G_X(x)\Big]^{m-r}}\int_{x}^{\infty}z^{k-1}\left[1-\frac{\sum_{j=1}^{N}\Big[\sum_{\zeta=1}^{j}\Big[G_Y(\alpha_{\zeta-1})-G_Y(\beta_{\zeta-2})\Big]\Big]+\int_{\beta_{j-1}}^{z}f(t)dt}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\Big[G_Y(\alpha_j)-G_Y(\beta_{j-1})\Big]}\right]^{m-r}dz;\tag{15}$$

$$(ii)\qquad Var\Big(X_{r+1:m}^{k}|X_{r:m}=x\Big)=\frac{2\Big[1-G_X(x)\Big]^{m-r}\Phi_X(x)-2x\Big[1-G_X(x)\Big]^{m-r}\Psi_X(x)-\Psi_X^2(x)}{\Big[1-G_X(x)\Big]^{2(m-r)}},\tag{16}$$

where

$$\Phi_X(x)=\int_{x}^{\infty}z\left[1-\frac{\sum_{j=1}^{N}\Big[\sum_{\zeta=1}^{j}\Big[G_Y(\alpha_{\zeta-1})-G_Y(\beta_{\zeta-2})\Big]\Big]+\int_{\beta_{j-1}}^{z}f(t)dt}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\Big[G_Y(\alpha_j)-G_Y(\beta_{j-1})\Big]}\right]^{m-r}dz;$$

$$\Psi_X(x)=\int_{x}^{\infty}\left[1-\frac{\sum_{j=1}^{N}\Big[\sum_{\zeta=1}^{j}[G_Y(\alpha_{\zeta-1})-G_Y(\beta_{\zeta-2})\Big]\Big]+\int_{\beta_{j-1}}^{z}f(t)dt}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\Big[G_Y(\alpha_j)-G_Y(\beta_{j-1})\Big]}\right]^{m-r}dz;$$

Proof. (i) Firstly, if X has N intervals truncated distribution and $X_{r+1:m}^{k}=z$, then we have,

$$E\Big(X_{r+1:m}^{k}|X_{r:m}=x\Big)=\frac{\int_{x}^{\infty}z^{k}f_{r,r+1:m}(x,z)dz}{f_{r:m}(x)}=\frac{m-r}{\Big[1-G_X(x)\Big]^{m-r}}\int_{x}^{\infty}\Big[1-G(z)\Big]^{m-r-1}g(z)dz.$$

As in Mohie El-Din et al. [15]:

$$\begin{aligned}E\Big(X_{r+1:m}^{k}|X_{r:m}=x\Big)&=x^{k}+\frac{k}{\Big[1-G_X(x)\Big]^{m-r}}\int_{x}^{\infty}z^{k-1}\Bigg[1-\frac{1}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\Big[G_Y(\alpha_j)-G_Y(\beta_{j-1})\Big]}\\&\Bigg[\int_{-\infty}^{z}f(t)dt+G_X(\alpha_1)+\int_{\beta_1}^{z}f(t)dt+G_X(\alpha_1)+G_X(\alpha_2)-G_X(\beta_1)+\int_{\beta_2}^{z}f(t)dt\\&+G_X(\alpha_1)+G_X(\alpha_2)-G_X(\beta_1)+...+G_X(\alpha_{N-1})-G_X(\beta_{N-2})+\int_{\beta_{N-1}}^{z}f(t)dt+...\\&+G_X(\alpha_1)+G_X(\alpha_2)-G_X(\beta_1)+...+G_X(\alpha_N)-G_X(\beta_{N-1})+\int_{\beta_N}^{z}f(t)dt\Bigg]\Bigg]^{m-r}dz\\&=x^{k}+\frac{k}{\Big[1-G_X(x)\Big]^{m-r}}\int_{x}^{\infty}z^{k-1}\left[1-\frac{\sum_{j=1}^{N}\Big[\sum_{\zeta=1}^{j}\Big[G_Y(\alpha_{\zeta-1})-G_Y(\beta_{\zeta-2})\Big]\Big]+\int_{\beta_{j-1}}^{z}f(t)dt}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\Big[G_Y(\alpha_j)-G_Y(\beta_{j-1})\Big]}\right]^{m-r}dz;\end{aligned}$$

Now, we need to prove that if (15) holds then X has N intervals truncated distribution (sufficient condition). Let,

$$\frac{(m-r)}{\Big[1-G_X(x)\Big]^{m-r}}\int_{x}^{\infty}z^{k}\Big[1-G_Z(z)\Big]^{m-r-1}g(z)dz=x^{k}+\frac{k}{\Big[1-G_X(x)\Big]^{m-r}}$$
$$\times\int_{x}^{\infty}z^{k-1}\left[1-\frac{\sum_{j=1}^{N}\Big[\sum_{\zeta=1}^{j}\Big[G_Y(\alpha_{\zeta-1})-G_Y(\beta_{\zeta-2})\Big]\Big]+\int_{\beta_{j-1}}^{z}f(t)dt}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\Big[G_Y(\alpha_j)-G_Y(\beta_{j-1})\Big]}\right]^{m-r}dz.$$

By multiplying both sides with $\Big[1-G_X(x)\Big]^{m-r}$, we get

$$(m-r)\int_x^\infty z^k\Big[1-G_Z(z)\Big]^{m-r-1}g_Z(z)dz = x^k\Big[1-G_X(x)\Big]^{m-r} \\ + k\int_x^\infty z^{k-1}\left[1-\frac{\sum_{j=1}^{N}\Big[\sum_{\zeta=1}^{j}\Big[G_Y(\alpha_{\zeta-1})-G_Y(\beta_{\zeta-2})\Big]\Big]+\int_{\beta_{j-1}}^{z}f(t)dt}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\Big[G_Y(\alpha_j)-G_Y(\beta_{j-1})\Big]}\right]dz.$$

Differentiating both sides with respect to x, and using the rule $\frac{\partial}{\partial a}\Big(\int_a^b f_X(x)dx\Big)=-f(a)$ then we get,

$$-(m-r)x^k\Big[1-G_X(x)\Big]^{m-r-1}g_X(x) = kx^{k-1}\Big[1-G_X(x)\Big]^{m-r}-(m-r)x^k\Big[1-G_X(x)\Big]^{m-r-1}g_X(x) \\ - kx^{k-1}\left[1-\frac{\sum_{j=1}^{N}\Big[\sum_{\zeta=1}^{j}\Big[G_Y(\alpha_{\zeta-1})-G_Y(\beta_{\zeta-2})\Big]\Big]+\int_{\beta_{j-1}}^{z}f(t)dt}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\Big[G_Y(\alpha_j)-G_Y(\beta_{j-1})\Big]}\right]^{m-r}.$$

This leads to,

$$\Big[1-G_X(x)\Big]^{m-r} = \left[1-\frac{\sum_{j=1}^{N}\Big[\sum_{\zeta=1}^{j}\Big[G_Y(\alpha_{\zeta-1})-G_Y(\beta_{\zeta-2})\Big]\Big]+\int_{\beta_{j-1}}^{z}f(t)dt}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\Big[G_Y(\alpha_j)-G_Y(\beta_{j-1})\Big]}\right]^{m-r}.$$

Comparing both sides, we get

$$G_X(x) = \frac{\sum_{j=1}^{N}\Big[\sum_{\zeta=1}^{j}\Big[G_Y(\alpha_{\zeta-1})-G_Y(\beta_{\zeta-2})\Big]\Big]+\int_{\beta_{j-1}}^{z}f(t)dt}{1-G_Y(\beta_N)+\sum_{j=1}^{N}\Big[G_Y(\alpha_j)-G_Y(\beta_{j-1})\Big]}.$$

This complete the prove of (i),(ii). Since

$$\begin{aligned}
Var\Big(X_{r+1:m}|X_{r:m}=x\Big) &= E\Big(X_{r+1:m}^2|X_{r:m}=x\Big)-\Big(E\Big(X_{r+1:m}|X_{r:m}=x\Big)\Big)^2 \\
&= x^2+\frac{2\Phi_X(x)}{\Big[1-G_X(x)\Big]^{m-r}}-\left(x+\frac{\Psi_X(x)}{\Big[1-G_X(x)\Big]^{m-r}}\right)^2 \\
&= \frac{2\Phi_X(x)}{\Big[1-G_X(x)\Big]^{m-r}}-\frac{2x\Psi_X(x)}{\Big[1-G_X(x)\Big]^{m-r}}-\frac{\Psi_X^2(x)}{\Big[1-G_X(x)\Big]^{2(m-r)}} \\
&= \frac{2\Big[1-G_X(x)\Big]^{m-r}\Phi_X(x)-2x\Big[1-G_X(x)\Big]^{m-r}\Psi_X(x)-\Psi_X^2(x)}{\Big[1-G_X(x)\Big]^{2(m-r)}}.
\end{aligned}$$

This gives the necessary conditions.
To prove the sufficient condition of (ii), we let (16) satisfied and by the same method which used in Mohie El-Din et al. [15] (Theorem 3.4 p. 26-28). Then we can easily get that X has N intervals truncated distribution. □

4. Applications

In this section, we apply the above Definition 2.1 on some distributions such as Normal and Log-Lindley distributions.

4.1. One and two intervals truncated Normal distribution

For general reliability analysis, we used a normal distribution. It is also commonly used in times-to-failure of simple electronic and mechanical components, equipment or systems. Here, we apply the above results to this distribution in one and two intervals truncated cases. Firstly, we use *Maple* 13 program to show the original curves of the PDF, distribution and survival functions of the normal distribution to compare them with the curves of one and two intervals truncated.
Let Y has a normal distribution with mean μ and variance σ^2. Thus, its PDF is given by:

$$g_Y(y;\mu,\sigma)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left[\frac{-(y-\mu)^2}{2\sigma^2}\right],\qquad -\infty<y,\mu<\infty,\sigma>0. \tag{17}$$

Also, its corresponding CDF is given by:

$$G_Y(y;\mu,\sigma)=\frac{1}{2}\left[1+erf\left(\frac{y-\mu}{\sigma\sqrt{2}}\right)\right],\qquad -\infty<y,\mu<\infty,\sigma>0. \tag{18}$$

In addition, the Survival function is given by:

$$\tilde{G}_Y(y;\mu,\sigma)=1-G_Y(y;\mu,\sigma)=\frac{1}{2}\left[1-erf\left(\frac{y-\mu}{\sigma\sqrt{2}}\right)\right],\qquad -\infty<y,\mu<\infty,\sigma>0. \tag{19}$$

Now, we will apply the N intervals truncated method on (17) as follows.

4.1.1. One interval truncated Normal distribution

From Definition 2.1, the PDF of one interval truncated Normal distribution (i.e., $N=1$) at the interval (α,β) is given by:

$$f^*_X(x)=\begin{cases}\dfrac{\exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]}{\sigma\sqrt{2\pi}\left(1-\frac{1}{2}\left[erf\left(\frac{\beta-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha-\mu}{\sigma\sqrt{2}}\right)\right]\right)}, & -\infty<x<\alpha,\\[2ex] \dfrac{\exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]}{\sigma\sqrt{2\pi}\left(1-\frac{1}{2}\left[erf\left(\frac{\beta-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha-\mu}{\sigma\sqrt{2}}\right)\right]\right)}, & \beta<x<\infty.\end{cases}$$

If we let, $(\alpha,\beta)=(60,80)$, then $f^*_X(x)$ is represented in Figure 1(a). Also, the PDF $g_Y(y;\mu,\sigma)$ (17) in Figure 1(b) is merged with $f^*_X(x)$ as in Figure 1(c) to show the effect of the new Definition 2.1 (one interval truncation) of the area under the PDF original curve. This effectiveness resulted from the redistribute the probability of the truncated area on the non truncated area. And, the one interval truncated CDF is given by:

$$F^*_X(x)=\begin{cases}\dfrac{\frac{1}{2}\left[1+erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]}{\sigma\sqrt{2\pi}\left(1-\frac{1}{2}\left[erf\left(\frac{\beta-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha-\mu}{\sigma\sqrt{2}}\right)\right]\right)}, & -\infty<x<\alpha,\\[2ex] \dfrac{\frac{1}{2}\left[1+erf\left(\frac{\alpha-\mu}{\sigma\sqrt{2}}\right)+erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\beta-\mu}{\sigma\sqrt{2}}\right)\right]}{\sigma\sqrt{2\pi}\left(1-\frac{1}{2}\left[erf\left(\frac{\beta-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha-\mu}{\sigma\sqrt{2}}\right)\right]\right)}, & \beta<x<\infty.\end{cases}$$

Figure 2(b) represent the CDF $F^*_X(x)$ at the interval $(\alpha,\beta)=(60,80)$. The area under the curve is affected after the truncation process. That is clear after the mergers between original CDF (18) (Figure 2(a)) and one interval truncated distribution CDF (Figure 2(b) in Figure 2(c)).
For the one interval truncated distribution, the survival function is given by:

$$\tilde{F}^*_X(x)=\begin{cases}1-\dfrac{\frac{1}{2}\left[1+erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]}{\sigma\sqrt{2\pi}\left(1-\frac{1}{2}\left[erf\left(\frac{\beta-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha-\mu}{\sigma\sqrt{2}}\right)\right]\right)}, & -\infty<x<\alpha,\\[2ex] 1-\dfrac{\frac{1}{2}\left[1+erf\left(\frac{\alpha-\mu}{\sigma\sqrt{2}}\right)+erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\beta-\mu}{\sigma\sqrt{2}}\right)\right]}{\sigma\sqrt{2\pi}\left(1-\frac{1}{2}\left[erf\left(\frac{\beta-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha-\mu}{\sigma\sqrt{2}}\right)\right]\right)}, & \beta<x<\infty.\end{cases}$$

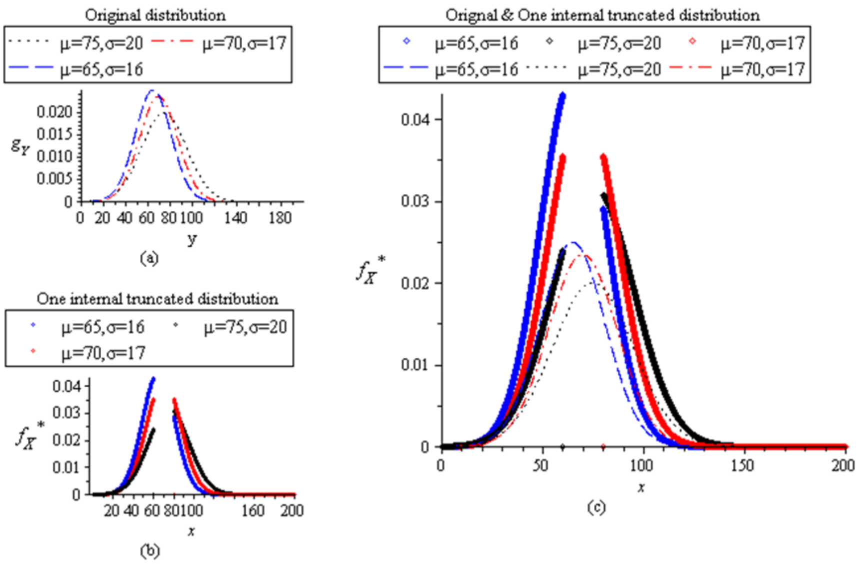

Figure 1: **(a)** The original PDF; **(b)** One interval truncated PDF and **(c)** Mergers between original and one interval truncated PDF of X for different values of μ and σ.

At $(\alpha, \beta) = (60, 80)$, we get $\tilde{F}_X^*(x)$ in Figure 3(b), but for the original distribution Figure 3(a) represents its survival function. The difference between the survival functions of the original and one interval truncated distribution is represented in Figure 3(c).

4.2. *Two intervals truncated Normal distribution*

At $(\alpha_1, \beta_1) = (20, 40)$ and $(\alpha_2, \beta_2) = (90, 100)$, we use Definition 2.1 to get the PDF of two intervals truncated Normal distribution (i.e., $N = 2$) as follows:

$$f_X^{**}(x) = \begin{cases} \dfrac{\exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]}{\sigma\sqrt{2\pi}\left(1-\frac{1}{2}\left[erf\left(\frac{\beta_2-\mu}{\sigma\sqrt{2}}\right)+erf\left(\frac{\beta_1-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha_2-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha_1-\mu}{\sigma\sqrt{2}}\right)\right]\right)}, & -\infty < x < \alpha_1, \\ \dfrac{\exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]}{\sigma\sqrt{2\pi}\left(1-\frac{1}{2}\left[erf\left(\frac{\beta_2-\mu}{\sigma\sqrt{2}}\right)+erf\left(\frac{\beta_1-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha_2-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha_1-\mu}{\sigma\sqrt{2}}\right)\right]\right)}, & \beta_1 < x < \alpha_2, \\ \dfrac{\exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]}{\sigma\sqrt{2\pi}\left(1-\frac{1}{2}\left[erf\left(\frac{\beta_2-\mu}{\sigma\sqrt{2}}\right)+erf\left(\frac{\beta_1-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha_2-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha_1-\mu}{\sigma\sqrt{2}}\right)\right]\right)}, & \beta_2 < x < \infty, \end{cases}$$

and it is represented in Figure 4(a), where the original distribution was represented in Figure 3(a). Here, the effectiveness is differs from the effectiveness in the case of one interval truncated distribution (see Figure 4(b)).

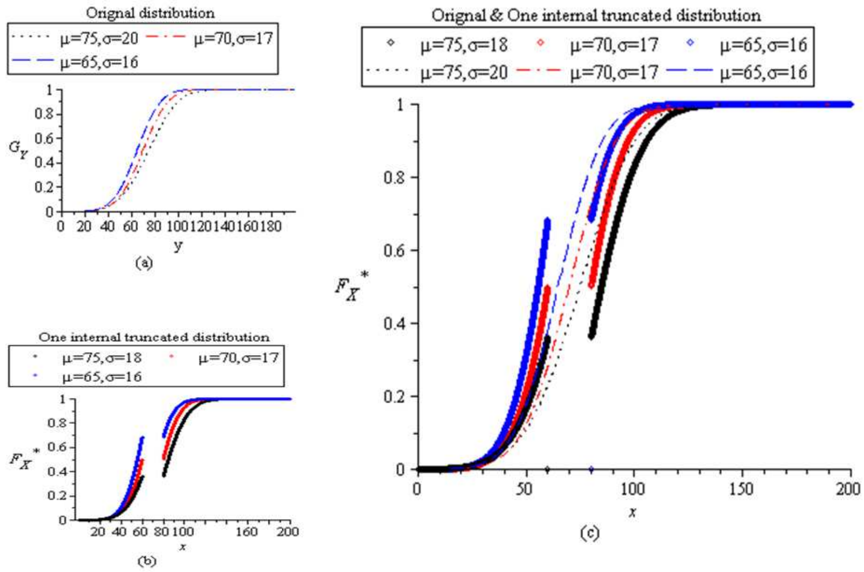

Figure 2: **(a)** The original CDF; **(b)** One interval truncated CDF and **(c)** Mergers between original and one interval truncated CDF of X for different values of μ and σ.

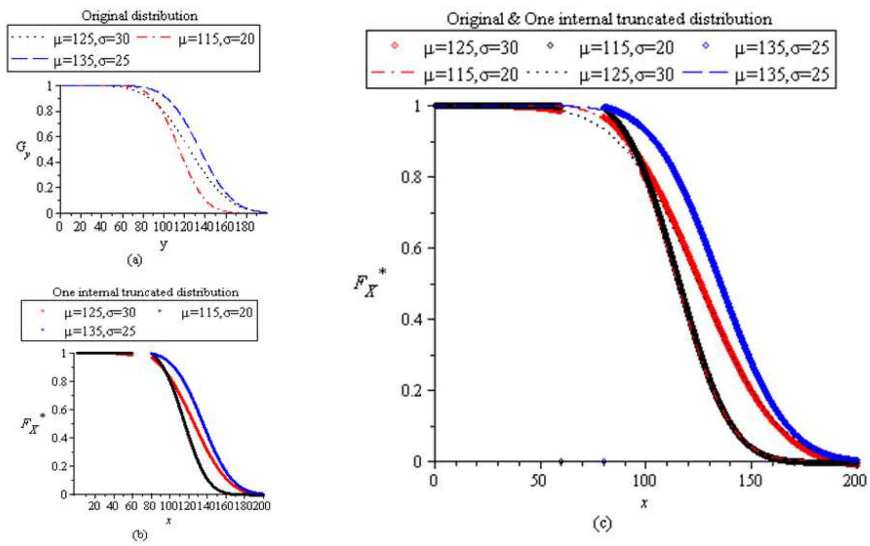

Figure 3: **(a)** The original distribution of the survival function; **(b)** One interval truncated distribution of the survival function and **(c)** Mergers between original and one interval truncated distribution of the survival function of X for different values of μ and σ.

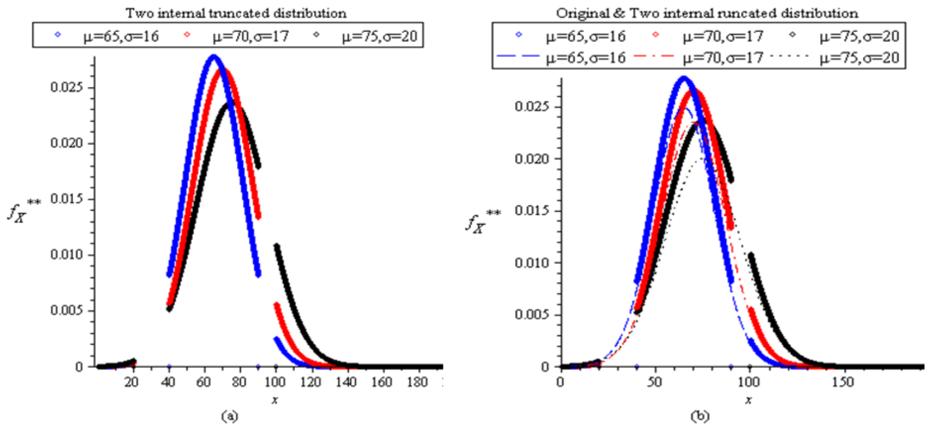

Figure 4: **(a)** The original PDF and **(b)** Mergers between original and two intervals truncated PDF of X for different values of μ and σ.

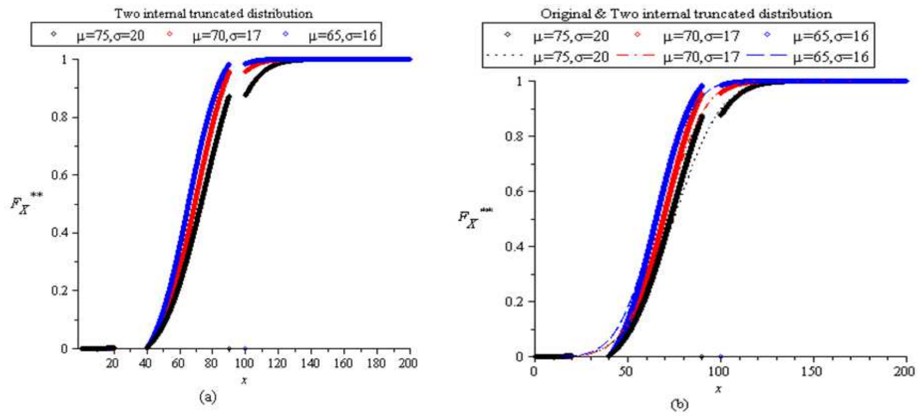

Figure 5: **(a)** The original CDF and **(b)** Mergers between original and two intervals truncated CDF of X for different values of μ and σ.

Also, the two intervals truncated CDF is given by:

$$F_X^{**}(x) = \begin{cases} \dfrac{\frac{1}{2}\left[1+erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]}{\sigma\sqrt{2\pi}\left(1-\frac{1}{2}\left[erf\left(\frac{\beta_2-\mu}{\sigma\sqrt{2}}\right)+erf\left(\frac{\beta_1-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha_2-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha_1-\mu}{\sigma\sqrt{2}}\right)\right]\right)}, & -\infty < x < \alpha_1, \\ \dfrac{\frac{1}{2}\left[1+erf\left(\frac{\alpha_1-\mu}{\sigma\sqrt{2}}\right)+erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\beta_1-\mu}{\sigma\sqrt{2}}\right)\right]}{\sigma\sqrt{2\pi}\left(1-\frac{1}{2}\left[erf\left(\frac{\beta_2-\mu}{\sigma\sqrt{2}}\right)+erf\left(\frac{\beta_1-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha_2-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha_1-\mu}{\sigma\sqrt{2}}\right)\right]\right)}, & \beta_1 < x < \alpha_2, \\ \dfrac{\frac{1}{2}\left[1+erf\left(\frac{\alpha_1-\mu}{\sigma\sqrt{2}}\right)+erf\left(\frac{\alpha_2-\mu}{\sigma\sqrt{2}}\right)+erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\beta_1-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\beta_2-\mu}{\sigma\sqrt{2}}\right)\right]}{\sigma\sqrt{2\pi}\left(1-\frac{1}{2}\left[erf\left(\frac{\beta_2-\mu}{\sigma\sqrt{2}}\right)+erf\left(\frac{\beta_1-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha_2-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha_1-\mu}{\sigma\sqrt{2}}\right)\right]\right)}, & \beta_2 < x < \infty, \end{cases}$$

(see Figure 5(a)). The Mergers of the two CDFs (for original and two intervals truncated distribution) are showed in Figure 5(b).

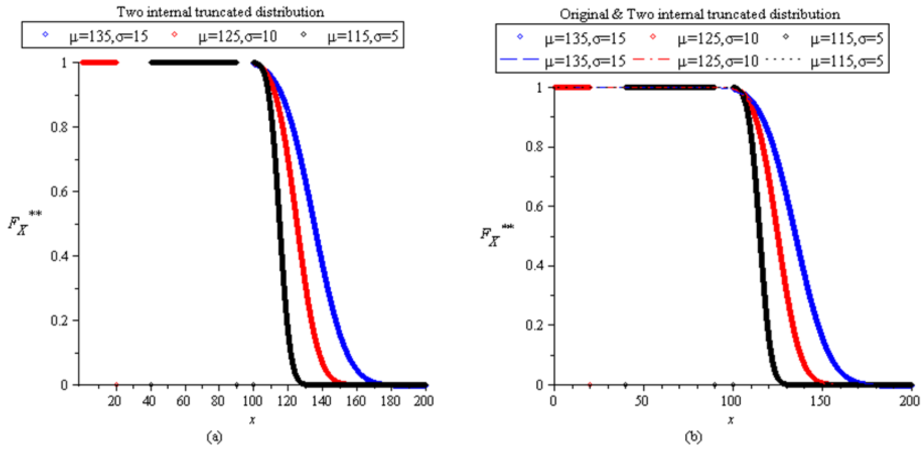

Figure 6: **(a)** The original survival function and **(b)** Mergers between original and two intervals truncated survival of X for different values of μ and σ.

The survival function is given by:

$$\tilde{F}_X^{**}(x) = \begin{cases} 1 - \dfrac{\frac{1}{2}\left[1+erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]}{\sigma\sqrt{2\pi}\left(1-\frac{1}{2}\left[erf\left(\frac{\beta_2-\mu}{\sigma\sqrt{2}}\right)+erf\left(\frac{\beta_1-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha_2-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha_1-\mu}{\sigma\sqrt{2}}\right)\right]\right)}, & -\infty < x < \alpha_1, \\ \dfrac{1+erf\left(\frac{\alpha_1-\mu}{\sigma\sqrt{2}}\right)+\frac{1}{2}\left[erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\beta_1-\mu}{\sigma\sqrt{2}}\right)\right]}{\sigma\sqrt{2\pi}\left(1-\frac{1}{2}\left[erf\left(\frac{\beta_2-\mu}{\sigma\sqrt{2}}\right)+erf\left(\frac{\beta_1-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha_2-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha_1-\mu}{\sigma\sqrt{2}}\right)\right]\right)}, & \beta_1 < x < \alpha_2, \\ \dfrac{\frac{3}{2}\left[1+erf\left(\frac{\alpha_1-\mu}{\sigma\sqrt{2}}\right)\right]+erf\left(\frac{\alpha_2-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\beta_1-\mu}{\sigma\sqrt{2}}\right)+\frac{1}{2}\left[erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\beta_2-\mu}{\sigma\sqrt{2}}\right)\right]}{\sigma\sqrt{2\pi}\left(1-\frac{1}{2}\left[erf\left(\frac{\beta_2-\mu}{\sigma\sqrt{2}}\right)+erf\left(\frac{\beta_1-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha_2-\mu}{\sigma\sqrt{2}}\right)-erf\left(\frac{\alpha_1-\mu}{\sigma\sqrt{2}}\right)\right]\right)}, & \beta_2 < x < \infty, \end{cases}$$

(see Figure 6(a)). The difference between the original and two intervals truncated survival functions showed in Figure 6(b). Definition (2.1) is more interesting to get more effective and applicable life time models after deleting the data from stable time intervals. Of course, because from the above figures (1) to (6), we found that, the redistribution of probabilities from the truncated parts to the remaining parts affects slightly on the shape of the curves with maintaining the original shape of the curves.

4.3. Remarks

To distribute the probability of truncated parts on the remaining parts with unequal proportions one can use the following definition:

Definition 4.1. *Let Y be a random variable with known probability density function $g_Y(y)$, define X as a corresponding N intervals truncated of the random variable Y with PDF $f_X(x)$. And, if $\eta_j, j = 1, 2, ..., N$ are the values of the proportions that used to redistribute the probabilities of the truncated parts on the remaining parts such that $\eta_j \neq \eta_k$, $\forall j \neq k$. Then, the probability density function of N intervals truncated of Y is given by:*

$$\Theta_X(x) = \begin{cases} \dfrac{\eta_1 g_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}[G_Y(\alpha_j)-G_Y(\beta_{j-1})]}, & -\infty < x < \alpha_1, \\ \dfrac{\eta_2 g_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}[G_Y(\alpha_j)-G_Y(\beta_{j-1})]}, & \beta_1 < x < \alpha_2, \\ \dfrac{\eta_3 g_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}[G_Y(\alpha_j)-G_Y(\beta_{j-1})]}, & \beta_2 < x < \alpha_3, \\ \qquad \cdots & \\ \dfrac{\eta_{N-1} g_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}[G_Y(\alpha_j)-G_Y(\beta_{j-1})]}, & \beta_{N-1} < x < \alpha_N, \\ \dfrac{\eta_N g_X(x)}{1-G_Y(\beta_N)+\sum_{j=1}^{N}[G_Y(\alpha_j)-G_Y(\beta_{j-1})]}, & \beta_N < x < \infty, \end{cases}$$

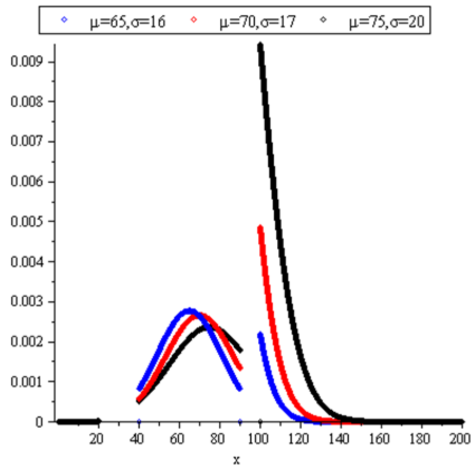

Figure 7: Two intervals truncated probability density function of X for different values of μ and σ and $\eta_1 = 0.03, \eta_2 = 0.1, \eta_3 = 0.87$.

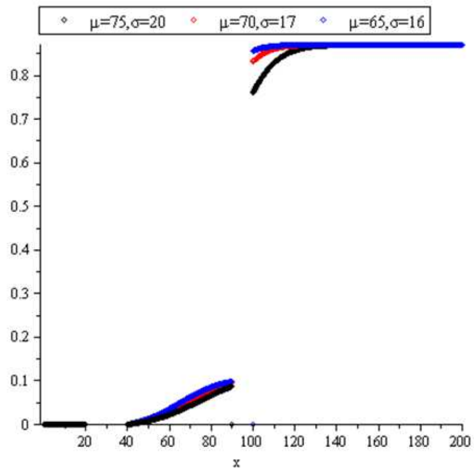

Figure 8: Two intervals truncated distribution function of X for different values of μ and σ and $\eta_1 = 0.03, \eta_2 = 0.1, \eta_3 = 0.87$.

where $\beta_0 = -\infty$ *and* $\sum_{j=1}^{N} \eta_j = 1$.

But this definition will make a complete change in the original shape of the known curve for the density function. For example, if we use the same values of μ and σ which used in Figure 4 and $\eta_1 = 0.03, \eta_2 = 0.1, \eta_3 = 0.87$, we will get the unclear Figure 7. Also, the curve of distribution and survival functions are unclear, see Figures 8 and 9. This show that the Definition 2.1 is more suitable and optimal for N intervals truncated distribution.

4.4. *One interval truncated Log-Lindley distribution*

LogLindley distribution is one of the famous distribution which has a nice application in insurance and inventory management. Gomez-Deniz et al. [4] introduced this distribution as a transformation of the generalized Lindey distribution (see Zakerzadeh and Dolati [5]). The LogLindley distribution depends on two parameters and Shanker et al. [18] studied this distribution for modelling waiting time and survival times data. Sometimes, we need to delete some real data set for cost effectiveness of risk management practice.

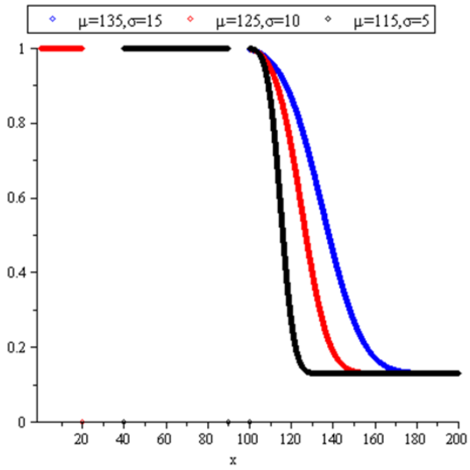

Figure 9: Two intervals truncated survival function of X for different values of μ and σ and $\eta_1 = 0.03, \eta_2 = 0.1, \eta_3 = 0.87$.

The PDF of the LogLindley distribution is given by,

$$\tilde{h}_Y(y;\lambda,\sigma) = \frac{\sigma^2}{1+\lambda\sigma}(\lambda - \log y)y^{\sigma-1}, \tag{20}$$

where $0 < x < 1$, $\lambda \geq 0$ and $\sigma > 0$. Using Definition 2.1 the PDF of one interval truncated LogLindley distribution is given by:

$$f_X(x;\lambda,\sigma) = \begin{cases} \frac{\sigma^2(\lambda-\log x)x^{\sigma-1}}{1+\lambda\sigma-\beta^\sigma[1+\sigma(\lambda-\log\beta)]+\alpha^\sigma[1+\sigma(\lambda-\log\alpha)]}, & 0 < x < \alpha, \\ \frac{\sigma^2(\lambda-\log x)x^{\sigma-1}}{1+\lambda\sigma-\beta^\sigma[1+\sigma(\lambda-\log\beta)]+\alpha^\sigma[1+\sigma(\lambda-\log\alpha)]}, & \beta < x < 1, \end{cases} \tag{21}$$

where the deleting interval $(\alpha, \beta) = (0.3, 0.5)$. Using *Mathematica7*, we can show the difference between the original PDF and the one interval truncated PDF of LogLindley distribution as in Figure [10].
The distribution function of the LogLindley distribution is given by,

$$\tilde{H}_Y(y;\lambda,\sigma) = \frac{y^\sigma[1+\sigma(\lambda-\log y)]}{1+\lambda\sigma} \tag{22}$$

And, the CDF of one interval truncated LogLindley distribution is given by:

$$F_X(x;\lambda,\sigma) = \begin{cases} \frac{x^\sigma[1+\sigma(\lambda-\log x)]}{1+\lambda\sigma-\beta^\sigma[1+\sigma(\lambda-\log\beta)]+\alpha^\sigma[1+\sigma(\lambda-\log\alpha)]}, & 0 < x < \alpha, \\ \frac{\alpha^\sigma[1+\sigma(\lambda-\log\alpha)]-\beta^\sigma[1+\sigma(\lambda-\log\beta)]+x^\sigma[1+\sigma(\lambda-\log x)]}{1+\lambda\sigma-\beta^\sigma[1+\sigma(\lambda-\log\beta)]+\alpha^\sigma[1+\sigma(\lambda-\log\alpha)]}, & \beta < x < 1, \end{cases} \tag{23}$$

Figure 11 presents the difference between the original CDF and the one interval truncated CDF of LogLindley distribution.
Furthermore, the survival function of the LogLindley distribution is given by:

$$\bar{\tilde{H}}_Y(y;\lambda,\sigma) = 1 - \frac{y^\sigma[1+\sigma(\lambda-\log y)]}{1+\lambda\sigma} \tag{24}$$

And, the survival function of the one interval truncated LogLindley distribution is given by:

$$\bar{F}_X(x;\lambda,\sigma) = \begin{cases} 1 - \frac{x^\sigma[1+\sigma(\lambda-\log x)]}{1+\lambda\sigma-\beta^\sigma[1+\sigma(\lambda-\log\beta)]+\alpha^\sigma[1+\sigma(\lambda-\log\alpha)]}, & 0 < x < \alpha, \\ 1 - \frac{2\alpha^\sigma[1+\sigma(\lambda-\log\alpha)]-\beta^\sigma[1+\sigma(\lambda-\log\beta)]+x^\sigma[1+\sigma(\lambda-\log x)]}{1+\lambda\sigma-\beta^\sigma[1+\sigma(\lambda-\log\beta)]+\alpha^\sigma[1+\sigma(\lambda-\log\alpha)]}, & \beta < x < 1. \end{cases} \tag{25}$$

Their exist a difference between the original and one interval truncated LogLindley distribution, see Figure (12).

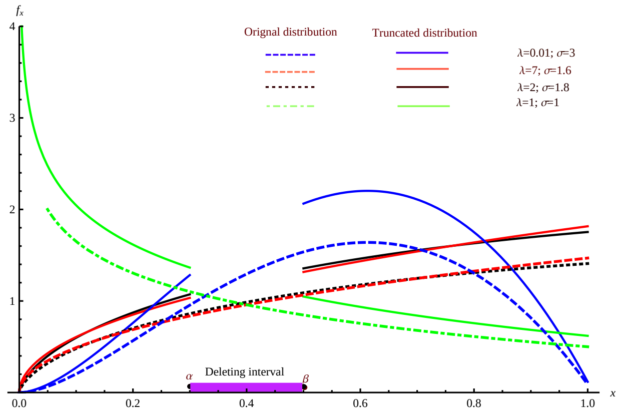

Figure 10: Original and one interval truncated LogLindley PDF distribution for different values of λ and σ.

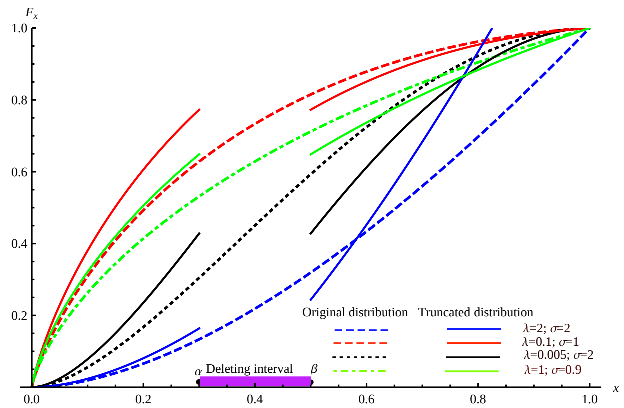

Figure 11: Original and one interval truncated LogLindley distribution CDF for different values of λ and σ.

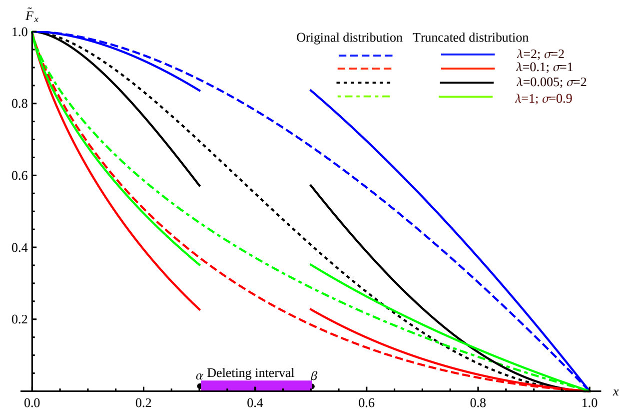

Figure 12: Original and one interval truncated LogLindley distribution survival functions for different values of λ and σ.

5. Concluding Remarks

In this paper, we introduced a new truncation definition for the probability distributions; that is; N intervals truncated. The new distribution has a high sensitivity of stochastic volatility data. It used to delete some certain range of data values from a domain of random variable (case of stable intervals). We provided a distribution, survival, hazard, reversed hazard, moment generating, characteristic and likelihood functions. Beside obtaining the Kth moment of the new distribution, the Characterization using order statistics has been done. When we distribute the probability of truncated parts on the remaining parts with unequal proportions, the natural shape of the known curve for the density function will be changed. As an application for the new definition, we consider two of the famous distributions; that is; a normal and Log-Lindley distributions and applied the obtained results on it. We hope that this new definition may attract a wide application in lifetime modeling.
In the case of two intervals truncated normal distribution, we notice that the shapes of the PDF, CDF and survival curves are construed to the original curve shapes of the normal distribution, see Figures (4),(5) and (6). This will lead us to study the optimal number of truncation intervals in the future. In addition, in future research one can introduce a new type of double-out and N intervals truncated for the lifetime distribution.

References

[1] A. Teamah, M. El-Hadidy and M. El-Ghoul, On bounded range distribution of a Wiener process, Communications in Statistics - Theory and Methods, In press (2019).
[2] B.C. Arnold, N. Balakrish, H.N. Nagarajua. First Course in order statistics. John Wiely and Sons, New York (2008).
[3] C. Withers and S. Nadarajah, The distribution and quantiles of the range of a Wiener process, Appled Mathematices and Computation, 232 (2014) 766–770.
[4] E. Gomez-Deniz, M.A. Sordo and E. Caldern-Ojeda. The LogLindley distribution as an alternative to the beta regression model with applications in insurance. Insurance: Mathematics and Economics, 54 (2014), 49–57.
[5] H. Zakerzadeh and A. Dolati. Generalized Lindley distribution. Journal of Mathematical Extension 3(2)(2009), 13-25.
[6] J. Pender, The truncated normal distribution: Applications to queues with impatient customers, Operations Research Letters 43 (2015) 40-45.
[7] N. L. Johnson, S. Kotz,Distributions in Statistics, 1-4. Wiley, New York (1968) (1970) (1972).
[8] M. G. Kendall, A. Stuart, The Advanced Theory of Statistics, 1-3. C. Griffin, London (1958) (1961) (1966).
[9] A. M. Kagan, Yu. V. Linnik, C. R. Rao, Characterization Problems in Mathematical Statistics, Nauka (in Russian), Moscow (1972). (English translation by B. Ramachandran. J. Wiley (1973)).
[10] L. Zaninetti and M. Ferraro, On the truncated Pareto distribution with applications, Central European Journal of Physics 6(1)(2008) 1–6.
[11] L. Zaninetti, A right and left truncated gamma distribution with application to the stars. Advanced Studies in Theoretical Physics 23 (2014), 1139–1147.
[12] E. Lukacs, Applications of Faa di Bruno's formula in mathematical statistics, American Mathematical Monthly 62(1956) 340-348.
[13] E. Lukacs, Analytic methods in probability theory. Proceedings, Symposium on Prob. Methods in Analysis, Lecture notes No. 31, Springer (1967) 208-238.
[14] M. Ali and S. Nadarajah, A truncated Pareto distribution, Compu. Commun. 30(2006)1–4.
[15] M.M. Mohie El-Din, M.M. Amein, A.M. Abd El-Raheem. Salem, On mid-truncated distributions and its applications, Journal of Advanced Research in Applied Mathematics 5(2)(2013), 20–38.
[16] A. Marshall, I. Olkin, Life Distributions: Structure of Nonparametric, Semiparametric, and Parametric Families. Springer Series in Statistics, Springer New York (2007).
[17] N.L. Johnson, S. Kotz and N. Balakrishnan, Continuous Univariate Distributions, Volume 1, Wiley (1994).
[18] R. Shanker, S. Sharma and R. Shanker, A two-parameter Lindley distribution for modelling waiting time andsurvival times data, Applied Mathematics 4(2)(2013), 363368.
[19] S. Nadarajah, Some truncated distributions, Acta Applicinda Mathematica 106(2009) 105–123.
[20] S. Chattopadhyay, C.A. Murthy and S. K. Pal, Fitting truncated geometric distributions in large scale real world networks, Theoretical Computer Science 551(2014) 22-38.
[21] W. Feller, The asymptotic distribution of the range of sums of independent random variables, Ann. Math. Stat. 22(1951) 427-432.
[22] Y. Dodge, The Oxford Dictionary of Statistical Terms. OUP. (2003).