Investigation and Corrigendum to Some Results Related to g-Soft Equality and gf-Soft Equality Relations

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Abstract. Since Molodtsov defined the concept of soft sets, many types of soft equality relations between two soft sets were discussed. Among these types are g-soft equality and gf-soft equality relations introduced in [1] and [2], respectively. In this paper, we first aim to show that some results obtained in [1, 2] need not be true, by giving two counterexamples. Second, we investigate under what conditions these results are correct. Finally, we define and study the concepts of gf-soft union and gf-soft intersection for arbitrary family of soft sets.

1. Introduction

During recent past, many researchers endeavored to overcome problems which contain uncertainty by using some mathematical tools other than probability theory. Among these initiated tools are fuzzy set theory and rough set theory. Although these theories are all virtual tools for describing uncertainty, Molodtsov [18] demonstrated that they have their inherent difficulties and suggested a new tool, namely soft sets.

Then efforts were made to constitute a frame work for the soft set theory by defining various types of operators of soft sets. This matter was begun in 2003 by Maji et al. [17]. They formulated the notions of null and absolute soft sets, union and intersection of two soft sets and the complement of a soft set. These notions were improved and investigated by many researchers interested in the soft set theory (see, for example, [3, 13, 21]). In 2010, Qin and Hong [19] made an interesting theoretical study on the soft set theory. They first constructed Algebra structure with respect to the soft operators introduced in [3] and then they studied two new types of soft equality relations. Jun and Yang [15] initiated and probed the notions of generalized soft subset and generalized soft equality relations. Liu et al. [16], in 2012, formulated the definitions of a soft L-equality and proved that the associative laws of soft product operations are satisfied in the sense of a soft L-equality relation. Zhu and Wen [21], in 2013, redefined some notions on the soft set theory such as the codomain of a soft set and a null soft set.

With a view to preserve some classical set-theoretic laws true for soft sets, Abbas et al. [1] gave the notions of g-null soft set, g-absolute soft set, g-soft subset and g-soft equality with illustrative examples. In order to avoid a shortcoming of non existence of a g-soft union of two soft sets, Abbas et al. [2] proposed the concepts of gf-soft subset and gf-soft equality. Then they defined gf-soft union and gf-soft intersection

of two soft sets and discussed main properties. The authors of [11] defined two new soft relations, namely partial belong and total non belong relations. Also, they [9] combined a partial order relation with a soft set to construct a partially ordered soft set. [7] and [8] presented soft separation axioms on supra soft topological spaces and supra soft topological ordered spaces, respectively. Al-shami and Kočinac [10] verified the equivalence between the enriched and extended soft topologies and derived many findings related to soft mappings and soft axioms. Recently, some studies were carried out to correct and restate some results introduced in soft topologies (see, for example, [4–6, 12]).

In [1], the authors concluded that lower and upper soft equality relations are g-soft equality. In fact this result true with respect to lower soft equality, but it need not be true in the case of upper soft equality. To elucidate this matter, we give a counterexample. In [2], the authors obtained two results related to extended soft intersection and gf-soft intersection of two soft sets. However, we observe that these results do not hold true. To support our observations, we provide a counterexample. In the end of this work, we introduced and investigate the concepts of gf-soft union and gf-soft intersection for arbitrary family of soft sets.

2. Preliminaries

We start with some definitions related to the soft sets needed in the sequel. In this study, the notations X and K denote a universe set and a set of parameters, respectively. Also, we consider $A, B, C \subseteq K$.

Definition 2.1. ([18]) A pair (G, K) is said to be a soft set over X provided that G is a map of K into the family of all subsets of X. A soft set is identified as $(G, K) = \{(k, G(k)) : k \in K \text{ and } G(k) \in 2^X\}$. The collection of all soft sets over X under a parameter set K is denoted by $S(X_K)$.

The authors of [14] improved the soft subset definition given in [17] to be as follows.

Definition 2.2. ([14]) A soft set (G, A) is a soft F-subset of a soft set (F, B), denoted by $(G, A) \subseteq_F (F, B)$, if $A \subseteq B$ and $G(a) \subseteq F(a)$ for all $a \in A$. The soft sets (G, A) and (F, B) are soft F-equal, denoted by $(G, A) =_F (F, B)$, if each of them is a soft F-subset of the other.

Definition 2.3. ([17]) A soft set (G, K) over X is said to be:

(i) A null soft set, denoted by $\widetilde{\emptyset}$, if $G(k) = \emptyset$ for each $k \in K$.

(ii) An absolute soft set, denoted by $\widetilde{X}$, if $G(k) = X$ for each $k \in K$.

The definition of null soft set introduced in [21] is among attempts of keeping the property which reads as a null soft set is a soft subset of any soft set. But this definition does not cover certain situations arising in the soft set theory. So Abbas et al. [1] gave the following definition.

Definition 2.4. ([1]) A soft set (G, K) over X is said to be a g-null soft set provided that $K = \emptyset$ or $G(k) = \emptyset$ for each $k \in K$, whenever $K \neq \emptyset$.

However, Definition (2.4) contradicts with the original definition of soft set [18] in the case of $K = \emptyset$. In fact, we note that the notion of a generalized soft subset, see Definition (2.14), is a sufficient guarantee to keep this property.

Definition 2.5. ([17]) The union of two soft sets (G, A) and (F, B) over X, denoted by $(G, A) \widetilde{\cup} (F, B)$, is the soft set (V, D), where $D = A \cup B$ and a map $V : D \to 2^X$ is given as follows:

$$V(d) = \begin{cases} G(d) & : & d \in A \setminus B \\ F(d) & : & d \in B \setminus A \\ G(d) \cup F(d) & : & d \in A \cap B \end{cases}$$

Definition 2.6. ([3]) The restricted union of two soft sets (G, A) and (F, B) over X such that $A \bigcap B \neq \emptyset$, is the soft set (V, D), where $D = A \bigcap B$, and a map $V : D \to 2^X$ is given by $V(d) = G(d) \bigcup F(d)$. It is written as $(G, A) \bigcup_{\Re} (F, B)$.

Definition 2.7. ([17]) The intersection of two soft sets (G, A) and (F, B) over X, denoted by $(G, A) \widetilde{\bigcap} (F, B)$, is the soft set (V, D), where $D = A \bigcap B$, and a map $V : D \to 2^X$ is given by $V(d) = G(d)$ or $F(d)$.

Definition (2.7) suffers from some problems concerning its existence and causes loss many properties of set theory via soft set theory. This leads to emerge other definitions for soft intersection between two soft sets. Hereinafter, we mention some of them.

Definition 2.8. ([13]) The bi-intersection of two soft sets (G, A) and (F, B) over X, denoted by $(G, A) \widetilde{\sqcap} (F, B)$, is the soft set (V, D), where $D = A \bigcap B$, and a map $V : D \to 2^X$ is given by $V(d) = G(d) \bigcap F(d)$.

Definition 2.9. ([3]) The extended intersection of two soft sets (G, A) and (F, B) over X, denoted by $(G, A) \sqcap_{\varepsilon} (F, B)$, is the soft set (V, D), where $D = A \bigcup B$ and a map $V : D \to 2^X$ is given as follows:

$$V(d) = \begin{cases} G(d) & : & d \in A \setminus B \\ F(d) & : & d \in B \setminus A \\ G(d) \bigcap F(d) & : & d \in A \bigcap B \end{cases}$$

Definition 2.10. ([3]) The restricted intersection of two soft sets (G, A) and (F, B) over X such that $A \bigcap B \neq \emptyset$, is the soft set (V, D), where $D = A \bigcap B$, and a map $V : D \to 2^X$ is given by $V(d) = G(d) \bigcap F(d)$. It is written as $(G, A) \Cap (F, B)$.

Remark 2.11. It worthily noting that:

(i) The union of an arbitrary family of soft sets was given in [13].

(ii) The restricted union (extended intersection, restricted intersection) of an arbitrary family of soft sets was given in [20].

Ali et al. [3] redefined the complement of a soft set which given in [15] to satisfy some classical set-theoretic laws such as:

(i) The union (restricted union) of soft set and its complement is an absolute soft set.

(ii) The extended intersection (restricted intersection) of soft set and its complement is a null soft set.

Definition 2.12. ([3]) The relative complement of a soft set (G, K), denoted by (G^c, K), where $G^c : K \to 2^X$ is the map defined by $G^c(k) = X \setminus G(k)$ for each $k \in K$.

Qin and Hong [19] introduced two soft equalities $\approx_s, \approx^s$. We remain their names, lower soft equality $\approx_s$ and upper soft equality $\approx^s$ as they were mentioned in [1].

Definition 2.13. ([19]) Let (G, A) and (F, B) be two soft sets over X. We say that:

(i) (G, A) is a lower soft equal to (F, B), denoted by $(G, A) \approx_s (F, B)$, provided that $G(k) = \emptyset$ for each $k \in A \setminus B$, $F(k) = \emptyset$ for each $k \in B \setminus A$, and $G(k) = F(k)$ for each $k \in A \bigcap B$.

(ii) (G, A) is an upper soft equal to (F, B), denoted by $(G, A) \approx^s (F, B)$, provided that $G(k) = X$ for each $k \in A \setminus B$, $F(k) = X$ for each $k \in B \setminus A$, and $G(k) = F(k)$ for each $k \in A \bigcap B$.

Definition 2.14. ([15]) A soft set (G, A) is a generalized soft subset of a soft set (F, B), denoted by $(G, A) \prec (F, B)$, if for each $a \in A$, there exists $b \in B$ such that $G(a) \subseteq F(b)$. The soft sets (G, A) and (F, B) are generalized soft equal, denoted by $(G, A) \doteq (F, B)$, if each of them is a generalized soft subset of the other.

Definition 2.15. ([16]) A soft set (G, A) is a soft L-subset of a soft set (F, B), denoted by $(G, A) \subseteq_L (F, B)$, if for each $a \in A$, there exists $b \in B$ such that $G(a) = F(b)$. The soft sets (G, A) and (F, B) are soft L-equal, denoted by $(G, A) =_L (F, B)$, if each of them is a soft L-subset of the other.

Generalized soft subset and generalized soft equality relations were studied in [2] under the names of g-soft subset and g-soft equality.

Definition 2.16. ([1]) A soft set (G, A) is a g-soft subset of (F, B), denoted by $(G, A) \sqsubseteq_g (F, B)$, if $A = \emptyset$ or for each $a \in A$, there is $b \in B$ such that $G(a) \subseteq F(b)$. The soft sets (G, A) and (F, B) are g-soft equal, denoted by $(G, A) \approx_g (F, B)$, if each of them is a g-soft subset of the other.

Abbas et al. [2] formulated new soft operators of two soft sets called g-soft union and g-soft intersection by using a g-soft subset relation.

Definition 2.17. ([2]) A g-soft union of two soft sets $(F, A), (G, B) \in S(X_K)$, denoted by $(F, A) \sqcup_g (G, B)$, as the collection of all soft sets $(H, C) \in S(X_K)$ which satisfies the following two conditions:

(i) $(F, A) \sqsubseteq_g (H, C)$ and $(G, B) \sqsubseteq_g (H, C)$.

(ii) If there exists $(J, D) \in S(X_K)$ such that $(F, A) \sqsubseteq_g (J, D)$ and $(G, B) \sqsubseteq_g (J, D)$, then $(H, C) \sqsubseteq_g (J, D)$.

That is, (H, C) is a minimal g-soft superset of $(F, A), (G, B)$ in the sense that if there exists another soft set (J, D) satisfying **(i)**, then (H, C) is g-soft subset of (J, D).

Definition 2.18. ([2]) A g-soft intersection of two soft sets $(F, A), (G, B) \in S(X_K)$, denoted by $(F, A) \sqcap_g (G, B)$, as the collection of all soft sets $(H, C) \in S(X_K)$ which satisfies the following two conditions:

(i) $(H, C) \sqsubseteq_g (F, A)$ and $(H, C) \sqsubseteq_g (G, B)$.

(ii) If there exists $(J, D) \in S(X_K)$ such that $(J, D) \sqsubseteq_g (F, A)$ and $(J, D) \sqsubseteq_g (G, B)$, then $(J, D) \sqsubseteq_g (H, C)$.

That is, (H, C) is a maximal g-soft subset of $(F, A), (G, B)$ in the sense that if there exists another soft set (J, D) satisfying **(i)**, then (J, D) is g-soft subset of H_C.

To address the problem which arise from the non existence of g-soft union of two soft sets (see Example 2.15 of [2]), Abbas et al. [2] formulated the following new definition.

Definition 2.19. ([2]) A soft set (G, A) is a gf-soft subset of (F, B), denoted by $(G, A) \sqsubseteq_{gf} (F, B)$, if for each $a \in A$, there is a finite set $E \subseteq B$ such that $G(a) \subseteq \bigcup_{e \in E} F(e)$. The soft sets (G, A) and (F, B) are gf-soft equal, denoted by $(G, A) \approx_{gf} (F, B)$, if each of them is a gf-soft subset of the other.

By replacing g-soft subset by gf-soft subset in Definition (2.17) and Definition (2.18), the authors of [2] formulated the concepts of gf-soft union and gf-soft intersection of two soft sets.

Definition 2.20. ([14]) A soft set (G, A) over X is said to be a full soft set if $\bigcup_{a \in A} G(a) = X$.

3. Notes on g-Soft Equality and gf-Soft Equality Relations

In Proposition 3.7 of [1], the authors claimed that: For any soft sets (F, A) and (G, B) over X, if $(F, A) \approx^s (G, B)$, then $(F, A) \approx_g (G, B)$. We show that this result need not be true in general by giving the following example.

Example 3.1. Let $A = \{k_1, k_2\}$ and $B = \{k_1, k_2, k_3\}$ be two sets of parameters. Consider the two soft sets (F, A) and (G, B) over $X = \{x_1, x_2, x_3\}$ are given by:

$(F, A) = \{(k_1, \{x_1\}), (k_2, \{x_2\})\}$, and

$(G, B) = \{(k_1, \{x_1\}), (k_2, \{x_2\}), (k_3, X)\}$.

Now, $A \bigcap B = \{k_1, k_2\}$ and $B \setminus A = \{k_3\}$. It is clear that $F(k_1) = G(k_1)$, $F(k_2) = G(k_2)$ and $G(k_3) = X$. So $(F, A) \approx^s (G, B)$. On the other hand, $(G, B) \not\sqsubseteq_g (F, A)$, because $k_3 \in B$ and there does not exist $k_i \in A$ such that $G(k_3) \subseteq F(k_i)$. Hence $(F, A) \not\cong_g (G, B)$.

Remark 3.2. Note that the above example also demonstrates that upper soft equality does not imply gf-soft equality.

Proposition 3.3. *If $(F, A) \approx^s (G, B)$ and $(F, A) \approx_s (G, B)$, then $(F, A) =_F (G, B)$.*

Proof. Let the given conditions be satisfied. Suppose that $A \neq B$. Then there exists a parameter $k \in A \setminus B$ or $k \in B \setminus A$. Say, $k \in A \setminus B$. Since $(F, A) \approx^s (G, B)$, then $F(k) = X$ and since $(F, A) \approx_s (G, B)$, then $F(k) = \emptyset$. But this is a contradiction. So it must $A = B$. From the definitions of $\approx^s$ and $\approx_s$, we find that $F(k) = G(k)$ for each $k \in A \bigcap B = A = B$. Hence we obtain the required result. $\square$

Now, we investigate under what conditions Proposition 3.7 of [1] holds true.

Proposition 3.4. *Let (F, A) and (G, B) be two soft sets over X. Then $(F, A) \approx^s (G, B)$ implies $(F, A) \cong_g (G, B)$ provided that one of the following conditions holds:*

  (i) *$A = B$.*

  (ii) *$A \setminus B$ and $B \setminus A$ are non empty.*

  (iii) *$A \setminus B$ is non empty and there exists $b \in B$ such that $G(b) = X$.*

  (iv) *$B \setminus A$ is non empty and there exists $a \in A$ such that $F(a) = X$.*

Proof. Assume that $(F, A) \approx^s (G, B)$. Then:

(i) If $A = B$, then $F(k) = G(k)$ for each $k \in A = B$. So $(F, A) \cong_g (G, B)$.

(ii) Let k be an arbitrary parameter in A. By hypothesis, there exists $k' \in B \setminus A$ such that $G(k') = X$. So $F(k) \subseteq G(k')$. Thus $(F, A) \sqsubseteq_g (G, B)$. Similarly, it can be proved that $(G, B) \sqsubseteq_g (F, A)$. Hence $(F, A) \cong_g (G, B)$.

(iii) Since $A \setminus B$ is non empty, then there exists $a \in A$ such that $F(a) = X$. This means that for each $b \in B$, we have $G(b) \subseteq F(a) = X$. So $(G, B) \sqsubseteq_g (F, A)$. On the other hand, there exists $b \in B$ such that $G(b) = X$. This implies that for each $a \in A$, we have $F(a) \subseteq G(b) = X$. So $(F, A) \sqsubseteq_g (G, B)$. Hence $(F, A) \cong_g (G, B)$.

(iv) Following similar above arguments, the result is satisfied. $\square$

Corollary 3.5. *Let (F, A) and (G, B) be two soft sets over X. Then:*

  (i) *If there exists $a \in A$ such that $F(a) = X$, then $(G, B) \sqsubseteq_g (F, A)$.*

  (ii) *If there exist $a \in A$ and $b \in B$ such that $F(a) = G(b) = X$, then $(F, A) \cong_g (G, B)$.*

The following result shows under what conditions Proposition (3.4) valid for gf-soft equality.

Proposition 3.6. *Let (F, A) and (G, B) be two soft sets over X. Then $(F, A) \approx^s (G, B)$ implies $(F, A) \cong_{gf} (G, B)$ provided that one of the following conditions holds:*

  (i) *$A = B$.*

  (ii) *$A \setminus B$ and $B \setminus A$ are non empty.*

  (iii) *$A \setminus B$ is non empty and there exists a finite subset $B' \subseteq B$ such that $\bigcup_{b \in B'} G(b) = X$.*

(iv) $B \setminus A$ is non empty and there exists a finite subset $A' \subseteq A$ such that $\bigcup_{a \in A'} F(a) = X$.

Proof. The proof is similar to that of Proposition (3.4). □

Corollary 3.7. *Let* (F, A) *and* (G, B) *be two soft sets over X. Then:*

(i) *If there exists a finite subset* $A' \subseteq A$ *such that* $\bigcup_{a \in A'} F(a) = X$, *then* $(G, B) \sqsubseteq_{gf} (F, A)$.

(ii) *If there exist finite subsets* $A' \subseteq A$ *and* $B' \subseteq B$ *such that* $\bigcup_{a \in A'} F(a) = \bigcup_{b \in B'} G(b) = X$, *then* $(F, A) \cong_{gf} (G, B)$.

Proposition 3.8. *Let* (F, A) *and* (G, B) *be two soft sets over X such that* $A \bigcap B \neq \emptyset$. *Then:*

(i) *If* $(F, A) \approx_s (G, B)$, *then* $(G, A) \cap (F, B) = (G, A) \widetilde{\sqcap} (F, B) = (G, A) \widetilde{\bigcap} (F, B) = (G, A) \bigcup_{\mathfrak{R}} (F, B)$.

(ii) *If* $(F, A) \approx_s (G, B)$, *then* $(G, A) \cap (F, B) =_g (G, A) \sqcap_\varepsilon (F, B) =_g (G, A) \widetilde{\sqcap} (F, B) =_g (G, A) \widetilde{\bigcap} (F, B) =_g (G, A) \bigcup_{\mathfrak{R}} (F, B) =_g (G, A) \widetilde{\bigcup} (F, B)$.

(iii) *If* $(F, A) \approx^s (G, B)$, *then* $(G, A) \sqcap_\varepsilon (F, B) = (G, A) \widetilde{\bigcup} (F, B)$.

Proof. (i) The domain of the four operators are $A \bigcap B$. By noting that $F(k) \bigcap G(k) = F(k) \bigcup G(k)$ for each $k \in A \bigcap B$, the results holds.

(ii) The proof immediately comes by noting that $F(k) = \emptyset$ for each $k \in A \setminus B$, $G(k) = \emptyset$ for each $k \in B \setminus A$ and $F(k) \bigcap G(k) = F(k) \bigcup G(k)$ for each $k \in A \bigcap B$.

(iii) The domain of the four operators are $A \bigcup B$. By noting that $F(k) = X$ for each $k \in A \setminus B$, $G(k) = X$ for each $k \in B \setminus A$ and $F(k) \bigcap G(k) = F(k) \bigcup G(k)$ for each $k \in A \bigcap B$, the result holds. □

In [2], the authors mentioned the following two incorrect results which were numbered as Proposition 4.3 and Theorem 4.4, respectively.

Proposition 3.9. *For any two soft sets* (F, A) *and* (G, B) *over X such that* $A \bigcap B = \emptyset$, *we have* $(F, A) \sqcap_\varepsilon (G, B) \in (F, A) \sqcap_{gf} (G, B)$.

Theorem 3.10. *For any two soft sets* (F, A) *and* (G, B) *over X such that* $A \bigcap B = \emptyset$, *we have* $(F, A) \sqcap_{gf} (G, B) = \{(H, C) \in S(X_K) : (H, C) \cong_{gf} (F, A) \sqcap_\varepsilon (G, B)\}$.

We give the following example to illustrate that the above two results need not be true in general.

Example 3.11. Let $A = \{k_1, k_2\}$ and $B = \{k_3\}$ be two sets of parameters. Consider the two soft sets (F, A) and (G, B) over $X = \{x_1, x_2, x_3, x_4\}$ are defined as follows:
$(F, A) = \{(k_1, \{x_1, x_2\}), (k_2, \{x_2, x_3\})\}$, and
$(G, B) = \{(k_3, \{x_1, x_4\})\}$.
Now, we have $A \bigcap B = \emptyset$ and $(F, A) \sqcap_\varepsilon (G, B) = \{(k_1, \{x_1, x_2\}), (k_2, \{x_2, x_3\}), (k_3, \{x_1, x_4\})\}$. Obviously, $(F, A) \sqcap_\varepsilon (G, B) \not\sqsubseteq_{gf} (F, A)$ and $(F, A) \sqcap_\varepsilon (G, B) \not\sqsubseteq_{gf} (G, B)$. So $(F, A) \sqcap_\varepsilon (G, B) \in (F, A) \sqcap_{gf} (G, B)$ does not hold. Hence $(F, A) \sqcap_{gf} (G, B) = \{(H, C) \in S(X_K) : (H, C) \cong_{gf} (F, A) \sqcap_\varepsilon (G, B)\}$ does not hold as well.

Depending on a gf-soft subset relation, we define the concepts of soft union and intersection for arbitrary family of soft sets.

Definition 3.12. For $(F_i, A_i) \in S(X_K)$, we define the generalized finite soft union (gf-soft union, in short) of (F_i, A_i), denoted by $\sqcup_{gf}(F_i, A_i)$, as the collection of all soft sets $(H, C) \in S(X_K)$ which satisfies the following two conditions:

(i) $(F_i, A_i) \sqsubseteq_{gf} (H, C)$ for each $i \in I$.

(ii) If there exists $(J, D) \in S(X_K)$ such that $(F_i, A_i) \sqsubseteq_{gf} (J, D)$ for each $i \in I$, then $(H, C) \sqsubseteq_{gf} (J, D)$.

That is, (H, C) is a minimal gf-soft superset of (F_i, A_i) in the sense that if there exists another soft set (J, D) satisfying **(i)**, then (H, C) is gf-soft subset of (J, D).

Proposition 3.13. *Let (F_i, A_i) be soft sets over X such that $\bigcap_{j \in J} A_j = \emptyset$ for every infinite subset J of I. Then $\widetilde{\bigcup}(F_i, A_i) \in \sqcup_{gf}(F_i, A_i)$.*

Proof. Obviously, $(F_i, A_i) \sqsubseteq_{gf} \widetilde{\bigcup}(F_i, A_i)$ for each $i \in I$. Then $\widetilde{\bigcup}(F_i, A_i)$ satisfies the condition **(i)** of the above definition. Let $(L, D) \in S(X_K)$ such that $(F_i, A_i) \sqsubseteq_{gf} (L, D)$ for each $i \in I$. Taking $\widetilde{\bigcup}_{i \in C}(F_i, A_i) = (H, C)$, where $C = \bigcup_{i \in I} A_i$. So for each $k \in C$, we have $k \in \bigcup_{s \in S} A_s \setminus \bigcup_{j \in J} A_j$ for some sets $S \subset I$ and $J \subset I$, then $H(k) = \bigcup_{s \in S} F_s(k)$. Since $\bigcap_{j \in J} A_j = \emptyset$ for every infinite subset J of I, then S must be that finite. It follows from the fact $(F_i, A_i) \sqsubseteq_{gf} \widetilde{\bigcup}_{i \in I}(F_i, A_i)$, that $F_s(k) \subseteq \bigcup_{m \in M_i} L(m)$ for some finite sets $M_s \subseteq D$. So $H(k) \subseteq \bigcup_{m \in \bigcup_{i \in S} M_i} L(m)$. Obviously, $\bigcup_{i \in S} M_i$ is finite. So $(H, C) = \widetilde{\bigcup}(F_i, A_i) \sqsubseteq_{gf} (L, D)$. This means that the condition **(ii)** of the above definition holds. Hence $\widetilde{\bigcup}(F_i, A_i) \in \sqcup_{gf}(F_i, A_i)$. $\square$

Proposition 3.14. *Let (F_i, A_i) be soft sets over X such that $\bigcap_{j \in J} A_j = \emptyset$ for every infinite subset J of I. Then $\sqcup_{gf}(F_i, A_i) = \{(H, C) \in S(X_K) : (H, C) \approxeq_{gf} \widetilde{\bigcup}(F_i, A_i)\}$.*

Proof. Consider that $(H, C) \in \sqcup_{gf}(F_i, A_i)$, then (H, C) satisfies the second condition of Definition (3.12). Proposition (3.13) gives that $(F_i, A_i) \sqsubseteq_{gf} \widetilde{\bigcup}(F_i, A_i)$ for each $i \in I$. So we derive that $(H, C) \sqsubseteq_{gf} \widetilde{\bigcup}(F_i, A_i)$. Also, Proposition (3.13) gives that $\widetilde{\bigcup}(F_i, A_i)$ satisfies second condition of Definition (3.12). So we derive that $\widetilde{\bigcup}(F_i, A_i) \sqsubseteq_{gf} (H, C)$. Hence we deduce that $(H, C) \approxeq_{gf} \widetilde{\bigcup}(F_i, A_i)$.

On the other hand, let $(H, C) \approxeq_{gf} \widetilde{\bigcup}(F_i, A_i)$. Then $(F_i, A_i) \sqsubseteq_{gf} (H, C)$ for each $i \in I$. This implies that (H, C) satisfies the first condition of Definition (3.12). Suppose that $(J, D) \in S(X_K)$ such that $(F_i, A_i) \sqsubseteq_{gf} (J, D)$ for each $i \in I$. So $\widetilde{\bigcup}(F_i, A_i) \sqsubseteq_{gf} (J, D)$. Thus $(H, C) \sqsubseteq_{gf} (J, D)$. Hence (H, C) satisfies the second condition of Definition (3.12) which ultimately implies that $(H, C) \in \sqcup_{gf}(F_i, A_i)$. $\square$

Definition 3.15. For $(F_i, A_i) \in S(X_K)$, we define the generalized finite soft intersection (gf-soft intersection, in short) of (F_i, A_i), denoted by $\sqcap_{gf}(F_i, A_i)$, as the collection of all soft sets $(H, C) \in S(X_K)$ which satisfies the following two conditions:

(i) $(H, C) \sqsubseteq_{gf} (F_i, A_i)$ for each $i \in I$.

(ii) If there exists $(J, D) \in S(X_K)$ such that $(J, D) \sqsubseteq_{gf} (F_i, A_i)$ for each $i \in I$, then $(J, D) \sqsubseteq_{gf} (H, C)$.

That is, (H, C) is a maximal gf-soft subset of (F_i, A_i) in the sense that if there exists another soft set (J, D) satisfying **(i)**, then (J, D) is gf-soft subset of (H, C).

Before proceeding forward consider the following

Example 3.16. Let $A = \{k_1, k_2, k_3\}$ be a set of parameters and consider the soft sets (F_n, A) and (F_m, A) over the natural numbers set $\mathcal{N}$ are defined as follows:
$(F_n, A) = \{(k_1, \{1, 3, n\}), (k_2, \{2\}), (k_3, \{n\}) : n \text{ is odd number }\}$.
$(F_m, A) = \{(k_1, \{1, m\}), (k_2, \{2, 3\}), (k_3, \{m\}) : m \text{ is even number }\}$.
Then for each $i \in \mathcal{N}$, $\sqcap_{\varepsilon}(F_i, A) = \{(k_1, \{1\}), (k_2, \{2\}), (k_3, \emptyset)\}$. Let a soft set (J, A) over $\mathcal{N}$ be defined as follows: $(J, A) = \{(k_1, \{1\}), (k_2, \{2\}), (k_3, \{3\})\}$. Obviously, $(J, A) \sqsubseteq_{gf} (F_i, A)$ for each $i \in \mathcal{N}$, but $(J, A) \not\sqsubseteq_{gf} \sqcap_{\varepsilon}(F_i, A)$.

Definition 3.17. A soft set (G, A) over X is said to be:

(i) a huge soft set if there exists $a \in A$ such that $G(a) = X$.

(ii) a large soft set if there exists a finite set $A' \subseteq A \bigcup_{a \in A'} G(a) = X$.

It can be observed that every large soft set is a huge soft set and every huge soft set is a full soft set.

Proposition 3.18. (*i*) *Any two huge soft sets are g-soft equal.*

(*ii*) *The large and huge soft sets are gf-soft equal.*

Proof. Straightforward. □

Proposition 3.19. *Let* (F_i, A_i) *be soft sets over X such that* $\sqcap_\varepsilon (F_i, A_i)$ *is a large soft set. Then if* $(J, D) \sqsubseteq_{gf} (F_i, A_i)$ *for each* $i \in I$, *we have* $(J, D) \sqsubseteq_{gf} \sqcap_\varepsilon (F_i, A_i)$.

Proof. Consider $(J, D) \sqsubseteq_{gf} (F_i, A_i)$ for each $i \in I$ and let $(H, C) = \sqcap_\varepsilon (F_i, A_i)$, where $C = \bigcup_{i \in I} A_i$. Since $\sqcap_\varepsilon (F_i, A_i)$ is a large soft set, then there exists a finite subset B of C such that $\bigcup_{b \in B} G(b) = X$. So $J(d) \subseteq \bigcup_{b \in B} G(b)$ for each $d \in D$. Hence the desired result is proved. □

4. Conclusion

Through this work, we review the historical development of some operators on the soft set theory and discuss the motivations that led to initiate them. Then, we provide some examples to demonstrate some errors which were done via some studies. Also, we probe the interrelations among some types of soft equality relations. Finally, we introduce and study the concepts of gf-soft union and gf-soft intersection for arbitrary family of soft sets. In an upcoming paper, we explore new type of soft equality relations which may be useful to keep some classical set-theoretic laws true for the soft set theory.

Acknowledgments

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