Filomat 33:11 (2019), 3361–3373
https://doi.org/10.2298/FIL1911361A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Transmission-Type Problems for the Generalized Darcy-Forchheimer-Brinkman and Stokes Systems in Complementary Lipschitz Domains in $\mathbb{R}^3$

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Abstract. The aim of our work is to give a well-posedness result for a boundary value problem of transmission-type for the nonlinear, generalized Darcy-Forchheimer-Brinkman and Stokes systems in complementary Lipschitz domains in $\mathbb{R}^3$. First, we introduce the Sobolev spaces in which we seek our solution, then we define the trace operators and conormal derivative operators that are involved in the boundary conditions of our treated problem. Next, we state a result that concerns the well-posedness of the transmission problem for the generalized Brinkman and Stokes system in complementary Lipschitz domains in $\mathbb{R}^3$. Afterwards, we state and prove an important lemma. Finally, we obtain our desired result by employing the well-posedness of the linearized version of our problem and Banach's fixed point theorem.

1. Introduction

Transmission problems are becoming increasingly important problems not only due to their theoretical value (see, e.g., [4],[22]) but also due to practical applications as well. Also, in the field of elliptic boundary value problems, diverse systems of PDEs arise to model real-world problems (see, e.g., [21]).

In this particular research area, many researchers have obtained a great deal of results. Escauriaza and Mitrea [3] have used a layer potential method in order to tackle transmission problems associated to the Laplace operator on Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$. Girault and Sequeira [5] have obtained well-posedness results for boundary value problems for the Stokes system in $\mathbb{R}^2$ and $\mathbb{R}^3$ and in their work they have used weighted Sobolev spaces as introduced in [8]. Groşan et al. [6] have studied the Dirichlet problem for a generalized version of the Darcy-Forchheimer-Brinkman system in Lipschitz domains in $\mathbb{R}^n$, $n = 2, 3$. Kohr et al. [10] have studied boundary value problems of Robin type for the Brinkman and Darcy-Forchheimer-Brinkman systems in Lipschitz domains in $\mathbb{R}^n$, $n = 2, 3$, with small data in L^2-based Sobolev spaces (see also the work of Kohr et al. in [15]).

Mitrea and Wright [20] have studied transmission-type problems for the Stokes system in Sobolev and Besov spaces. Medkova [18] has used the integral equations method in order to treat a transmission problem for the Stokes system in a Lipschitz domain in $\mathbb{R}^3$. Medkova [17] has studied a transmission problem for

2010 *Mathematics Subject Classification*. Primary 35J25 ; Secondary 35Q35, 46E35
Keywords. Darcy-Forchheimer-Brinkman system, Transmission problems, Banach's fixed point theorem, Sobolev spaces.
Received: 10 December 2018; Accepted: 25 March 2019
Communicated by Dragan S. Djordjević
The author acknowledges the support of the Grant PN-III-P4-ID-PCE-2016-0036 of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI.
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the Brinkman system in Lipschitz domains in $\mathbb{R}^n$, $n > 2$. Kohr et al. in [13] have studied the semilinear Brinkman system with nonlinear Robin boundary condition. Kohr et al. in [11] have studied the nonlinear Neumann-transmission problem for the Stokes and Brinkman operators on Euclidean Lipschitz domains in which they considered nonlinear boundary conditions. Kohr et al. [14] have treated a transmission problem for the Stokes system and the nonlinear Darcy-Forchheimer-Brinkman system in Lipschitz domains in $\mathbb{R}^3$. Kohr et al. in [12] have studied nonlinear Robin-transmission boundary value problems for the nonlinear Darcy-Forchheimer-Brinkman system and Navier-Stokes system, in which the nonlinearity is present in the considered systems and also in the boundary conditions. Kohr et al. [16] have studied transmission problems for the Navier-Stokes and Darcy-Forchheimer-Brinkman systems in Lipschitz domains in the setting of compact Riemannian manifolds of dimension 2 and 3.

The goal of this paper is to give a well-posedness result in L^2-based Sobolev spaces, for a Poission problem of transmission type for the generalized Darcy-Forchheimer-Brinkman and Stokes system in two complementary Lipschitz domains in $\mathbb{R}^3$, namely, a bounded Lipschitz domain Γ (with connected boundary $\partial\Gamma$) and the complementary set $\mathbb{R}^3 \setminus \overline{\Gamma}$. The main idea behind this result is a combination between the well-posedness result in the linear case and Banach's fixed point theorem. In Section 2 we shall introduce the L^2-based (Bessel potential) Sobolev spaces and weighted Sobolev spaces in which we seek our unknown velocity fields. In addition, we give a definition (see Definition 2.1) of what it means for a function to tend to a constant at infinity in the sense of Leray, a very useful corollary (see Corollary 2.2). We conclude this section with lemmas that allow us to introduce the trace and conormal derivative operators that appear in the formulation of the boundary conditions of the problem that is the object of our study (see Lemma 2.3, Lemma 2.4, Lemma 2.5). In Section 3 we give an auxiliary well-posedness result for the transmission problem for the generalized Brinkman and Stokes systems in complementary Lipschitz domains in $\mathbb{R}^3$ (see Theorem 3.2). Using this result and the properties established at Lemma 3.1, we can prove the main result of this paper (see Theorem 3.3) by rewriting our non-linear problem in terms of an non-linear operator that is a contraction on some closed ball in a suitable chosen Hilbert space and then to use Banach's fixed point theorem.

2. Preliminary Results

In this section we recall the tools need for the formulation of the transmission-type problem that we will study further on. We shall start with describing the systems that appear in our paper.

In the latter, $\Gamma_+ := \Gamma \subset \mathbb{R}^3$ denotes a bounded Lipschitz domain (see, e.g., [7, Definition 2.1]) with connected boundary denoted by $\partial\Gamma$ and let $\Gamma_- := \mathbb{R}^3 \setminus \overline{\Gamma}$, the corresponding complementary set.

2.1. Sobolev spaces

We aim to introduce, in this subsection, the Sobolev spaces in which we seek our solutions of our transmission problem for the generalized Darcy-Forchheimer-Brinkman system and Stokes system. To this end, we will use the standard Sobolev spaces and the weighted Sobolev spaces (more suitable when dealing with the Stokes system in the unbounded domain Γ_-).

Let $p = 1, 2$. Let $L^p(\mathbb{R})^3$ be the space of (equivalence classes of) measurable functions, p-th power integrable, defined on $\mathbb{R}^3$. Denote the Fourier transform by $\mathcal{F}$, and its inverse by $\mathcal{F}^{-1}$, which both act on $L^1(\mathbb{R}^3)$ functions and consider their generalizations to the space of the tempered distributions.

Next, we denote by Γ_0 either Γ_+, Γ_- or $\mathbb{R}^3$.

Let $s \in \mathbb{R}$. The scalar L^2-based Sobolev (Bessel potential) spaces are given by (see, e.g., [1, Chapter 1]):

$$H^s(\mathbb{R}^3) := \{\mathcal{F}^{-1}(1 + |\xi|^2)^{-\frac{s}{2}}\mathcal{F}v : v \in L^2(\mathbb{R}^3)\},$$

$$H^s(\Gamma_0) := \{v \in \mathcal{D}'(\Gamma_0) : \exists V \in H^s(\mathbb{R}^3) \text{ such that } V|_{\Gamma_0} = v\},$$

$$\tilde{H}^s(\Gamma_0) := \overline{\mathcal{D}(\Gamma_0)}^{\|\cdot\|_{H^s(\mathbb{R}^3)}} \text{ (i.e., the closure of } \mathcal{D}(\Gamma_0) \text{ in } H^s(\mathbb{R}^3)\text{)},$$

where $\mathcal{D}(\Gamma_0)$ is the space of compactly supported smooth functions $C_0^\infty(\Gamma_0)$, endowed with the inductive limit topology and by $\mathcal{D}'(\Gamma_0)$ we denote its dual, the space of distributions on Γ_0, endowed with the weak-star topology. The vector-valued spaces are introduced component-wise. Note that $'$ refers here and in the sequel to the topological dual.

For $s \in (0,1)$ we introduce the boundary Sobolev spaces $H^s(\partial\Gamma)$ as follows (see also [9, Page 169]):

$$H^s(\partial\Gamma) := \{f \in L^2(\partial\Gamma) : \|f\|_{H^s(\partial\Gamma)} = \|f\|_{L^2(\partial\Gamma)} + \int_{\partial\Gamma}\int_{\partial\Gamma} \frac{|f(x)-f(y)|}{|x-y|^{2+2s}} d\sigma_x d\sigma_y < \infty\}.$$

Introduce also the negative order spaces by the following dualities:

$$H^{-s}(\partial\Gamma) = (H^s(\partial\Gamma))',$$

for $s \in (0,1)$. The vector-valued spaces are introduced component-wise.

Next, we introduce the weighted Sobolev spaces. The motivation behind this introduction is that of the need to compensate the behavior at infinity of the fundamental solution of the Stokes system. We proceed to introduce the weighted spaces as in [8].

We introduce the following weight function $\rho(\mathbf{x}) := (1+|\mathbf{x}|^2)^{\frac{1}{2}}$, for all $x \in \mathbb{R}^3$.

We define the weighted space $L^2(\rho^{-1};\Gamma_-)$ by the following relation

$$v \in L^2(\rho^{-1};\Gamma_-) \Leftrightarrow \rho^{-1}v \in L^2(\Gamma_-).$$

Using the above introduced space, we define the weighted Sobolev spaces as follows:

$$\mathcal{H}^1(\Gamma_-) := \{v \in \mathcal{D}'(\Gamma_-) : \rho^{-1}v \in L^2(\Gamma_-), \nabla v \in L^2(\Gamma_-)^3\},$$
$$\tilde{\mathcal{H}}^1(\Gamma_-) := \overline{\mathcal{D}(\Gamma_-)}^{\|\cdot\|_{\mathcal{H}^1(\Gamma_-)}} (\text{i.e. the closure of } \mathcal{D}(\Gamma_-) \text{ in } \mathcal{H}^1(\Gamma_-)),$$

while their vector-valued counterparts are defined component-wise. We also indicate here, the choice of norm on $\mathcal{H}^1(\Gamma_-)$, as follows:

$$\|v\|_{\mathcal{H}^1(\Gamma_-)} := [\|\rho^{-1}v\|^2_{L^2(\Gamma_-)} + \|\nabla v\|^2_{L^2(\Gamma_-)}]^{\frac{1}{2}}.$$

We also define the negative order weighted Sobolev spaces by the following dualities:

$$\mathcal{H}^{-1}(\Gamma_-) = (\tilde{\mathcal{H}}^1(\Gamma_-))', \quad \tilde{\mathcal{H}}^{-1}(\Gamma_-) = (\mathcal{H}^1(\Gamma_-))'.$$

We conclude the section on Sobolev space by describing the condition satisfied by the velocity $\mathbf{w}_-$ in the set Γ_- (see transmission conditions in (34)), i.e., we shall give the definition of what it means for a function to tend to a constant at infinity in the sense of Leray (see, e.g., [14, Definition 2.3])

Definition 2.1. *A function v tends to a constant v_∞ at ∞ in the sense of Leray if:*

$$\lim_{r\to\infty} \int_{S^2} |v(ry) - v_\infty| d\sigma_y = 0,$$

where S^2 denotes the unit sphere in $\mathbb{R}^3$.

The next result mentions the behavior at infinity of functions that belong to $\mathcal{H}^1(\Gamma_-)$ (see, e.g., [14, Corollary 2.4]).

Corollary 2.2. *If $v \in \mathcal{H}^1(\Gamma_-)$, the v tends to 0 at ∞ in the sense of Leray.*

2.2. The generalized Darcy-Forchheimer-Brinkman system

We introduce the generalized version of the Brinkman system, namely (see also, [13, Section 2.2]):

$$\Delta \mathbf{w} - \mathcal{P}\mathbf{w} - \nabla p = \mathfrak{f} \text{ in } \Gamma_+, \ \operatorname{div} \mathbf{w} = 0 \text{ in } \Gamma_+, \tag{1}$$

where $\mathcal{P} \in L^\infty(\Gamma_+)^{3\times 3}$ satisfies the condition:

$$\langle \mathcal{P}\mathbf{v}, \mathbf{v}\rangle_{\Gamma_+} \geq c_{\mathcal{P}} \|\mathbf{v}\|^2_{L^2(\Gamma_+)^3}, \ \forall \ \mathbf{v} \in L^2(\Gamma_+)^3, \tag{2}$$

where $c_{\mathcal{P}} > 0$ is a constant. The Brinkman system can be used to describe flows through swarms of fixed particles at very low concentration and under precise conditions (we refer the reader to the study provided at [21, Pages 17-18]).

Note that, if $\mathcal{P} \equiv 0$ in (1), we recover the well-known Stokes system.

We introduce now the generalized Darcy-Forchheimer-Brinkman system, which is given by (see, also [6]):

$$\Delta \mathbf{w} - \mathcal{P}\mathbf{w} - k|\mathbf{w}|\mathbf{w} - \beta(\mathbf{w}\cdot\nabla)\mathbf{w} - \nabla p = \mathfrak{f} \text{ in } \Gamma_+, \ \operatorname{div} \mathbf{w} = 0 \text{ in } \Gamma_+, \tag{3}$$

where $\mathcal{P} \in L^\infty(\Gamma_+)^{3\times 3}$ such that condition (2) holds and $k, \beta : \Gamma_+ \to \mathbb{R}_+$ are functions such that $k, \beta \in L^\infty(\Gamma_+)$, i.e., essentially bounded, non-negative functions defined on Γ_+.

If $\mathcal{P} \equiv \alpha\mathbb{I}$, where $\alpha > 0$ is a constant and $k, \beta > 0$ are also constants in (3), one obtains the classical Darcy-Forchheimer-Brinkman system which is used to describe flows through porous media saturated with viscous incompressible fluids, where the inertia of the fluid is not negligible (for additional details, we refer to [21]).

If $\mathcal{P} \equiv 0$, $k = 0$ and $\beta > 0$ is a constant in (3), we recover the Navier-Stokes system.

2.3. Trace and Conormal Derivative Operators

In this section, we will introduce the important operators that appear in the transmission condition in our main result.

First, we will introduce the trace operator by the following lemma (see, e.g., [19, Theorem 2.3]).

Lemma 2.3. *(**Gagliardo Trace Lemma**) Let $\Gamma_+ := \Gamma \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary $\partial\Gamma$ and denote by $\Gamma_- := \mathbb{R}^3 \setminus \overline{\Gamma}$ the complementary Lipschitz set. Then, there exist linear, continuous trace operators $Tr^\pm : H^1(\Gamma_\pm) \to H^{\frac{1}{2}}(\partial\Gamma)$, such that*

$$Tr^\pm v = v|_{\partial\Gamma}, \quad \forall v \in C^\infty(\overline{\Gamma_\pm}). \tag{4}$$

Moreover, these operators are surjective, having (non-unique) linear and continuous right inverse operators $Z_\pm : H^{\frac{1}{2}}(\partial\Gamma) \to H^1(\Gamma_\pm)$.

Note that, one introduces similarly the trace operator $Tr^- : \mathcal{H}^1(\Gamma_-) \to H^{\frac{1}{2}}(\partial\Gamma)$ and this lemma holds due to the continuous embedding $H^1(\Gamma_-) \hookrightarrow \mathcal{H}^1(\Gamma_-)$. We omit the statement of the lemma, but we shall refer the reader to work of Kohr et al. [14, Lemma 2.2].

Note that, whenever we use $\langle\cdot,\cdot\rangle_A$ we indicate the duality pairing of two dual Sobolev spaces defined on A, where A is either an open set or a surface in $\mathbb{R}^3$. Moreover, for a given field $\mathbf{w}$, we denote by $\mathrm{E}(\mathbf{w})$ the symmetric part of $\nabla\mathbf{w}$ which is given by

$$\mathrm{E}(\mathbf{w}) := \frac{1}{2}(\nabla\mathbf{w} + \nabla\mathbf{w}^t),$$

where by the superscript t we refer to the transpose.

Now, we give the lemma that allows us to introduce the conormal derivative operator for the generalized Brinkman system (see, e.g., [13, Lemma 2.3]).

Lemma 2.4. *Let $\Gamma_+ := \Gamma \subset \mathbb{R}^3$, be a bounded Lipschitz domain with connected boundary $\partial\Gamma$. Let $\mathcal{P} \in L^\infty(\Gamma_+)^{3\times 3}$. Consider the following space:*

$$\begin{aligned}\boldsymbol{H}(\Gamma_+, \mathcal{B}_\mathcal{P}) :=& \{(\boldsymbol{w}, p, \tilde{\mathfrak{f}}) \in H^1(\Gamma_+)^3 \times L^2(\Gamma_+) \times \tilde{H}^{-1}(\Gamma_+)^3 : \mathcal{B}_\mathcal{P}(\boldsymbol{w}, p) := \Delta w - \mathcal{P}\boldsymbol{w} - \nabla p = \tilde{\mathfrak{f}}|_{\Gamma_+} \\ & \textit{and } \operatorname{div} \boldsymbol{w} = 0 \textit{ in } \Gamma_+\}.\end{aligned}$$

Define the conormal derivative operator for the generalized Brinkman system,

$$t^+_{\mathcal{P},\nu} : \boldsymbol{H}(\Gamma_+, \mathcal{B}_\mathcal{P}) \to H^{-\frac{1}{2}}(\partial\Gamma)^3, \tag{5}$$

by the following relation:

$$\begin{aligned}\langle t^+_{\mathcal{P},\nu}(\boldsymbol{w}, p, \tilde{\mathfrak{f}}), \boldsymbol{\phi}\rangle_{\partial\Gamma} :=& 2\langle \mathrm{E}(\boldsymbol{w}), \mathrm{E}(Z_+\boldsymbol{\phi})\rangle_{\Gamma_+} + \langle \mathcal{P}\boldsymbol{w}, Z_+\boldsymbol{\phi}\rangle_{\Gamma_+} - \langle p, \operatorname{div}(Z_+\boldsymbol{\phi})\rangle_{\Gamma_+} \\ &+ \langle \tilde{\mathfrak{f}}, Z_+\boldsymbol{\phi}\rangle_{\Gamma_+}, \quad \forall \boldsymbol{\phi} \in H^{-\frac{1}{2}}(\partial\Gamma)^3,\end{aligned} \tag{6}$$

where Z_+ is a right inverse of the trace operator $Tr^+ : H^1(\Gamma_+)^3 \to H^{\frac{1}{2}}(\partial\Gamma)^3$. The operator $t^+_{\mathcal{P},\nu}$ is linear, bounded and does not depend on the choice of the right inverse Z_+ of the trace operator Tr^+.

Moreover, the following Green formula holds:

$$\langle t^+_{\mathcal{P},\nu}(\boldsymbol{w}, p, \tilde{\mathfrak{f}}), Tr^+\boldsymbol{\psi}\rangle_{\partial\Gamma} = 2\langle \mathrm{E}(\boldsymbol{w}), \mathrm{E}(\boldsymbol{\psi})\rangle_{\Gamma_+} + \langle \mathcal{P}\boldsymbol{w}, \boldsymbol{\psi}\rangle_{\Gamma_+} - \langle p, \operatorname{div} \boldsymbol{\psi}\rangle_{\Gamma_+} + \langle \tilde{\mathfrak{f}}, \boldsymbol{\psi}\rangle_{\Gamma_+}, \tag{7}$$

for all $(\boldsymbol{w}, p, \tilde{\mathfrak{f}}) \in \boldsymbol{H}(\Gamma_+, \mathcal{B}_\mathcal{P})$ and for any $\boldsymbol{\psi} \in H^1(\Gamma_+)^3$.

Similarly, we introduce the lemma that allows us to define the conormal derivative operator when we consider the Stokes system in the exterior domain in the setting of weighted Sobolev spaces (see, e.g., [14, Lemma 2.9]).

Lemma 2.5. *Let $\Gamma_+ := \Gamma \subset \mathbb{R}^3$, be a bounded Lipschitz domain with connected boundary $\partial\Gamma$. Consider the following space:*

$$\begin{aligned}\boldsymbol{\mathcal{H}}(\Gamma_-, \mathcal{B}_0) :=& \{(\boldsymbol{w}, p, \tilde{\mathfrak{f}}) \in \mathcal{H}^1(\Gamma_-)^3 \times L^2(\Gamma_-) \times \tilde{\mathcal{H}}^{-1}(\Gamma_-)^3 : \mathcal{B}_0(\boldsymbol{w}, p) := \Delta w - \nabla p = \tilde{\mathfrak{f}}|_{\Gamma_-} \\ & \textit{and } \operatorname{div} \boldsymbol{w} = 0 \textit{ in } \Gamma_-\}.\end{aligned}$$

Define the conormal derivative operator $t^-_{0,\nu}$ associated to the Stokes system in the Lipschitz set Γ_-,

$$t^-_{0,\nu} : \boldsymbol{\mathcal{H}}(\Gamma_-, \mathcal{B}_0) \to H^{-\frac{1}{2}}(\partial\Gamma)^3, \tag{8}$$

by the following relation:

$$\langle t^-_{0,\nu}(\boldsymbol{w}, p, \tilde{\mathfrak{f}}), \boldsymbol{\phi}\rangle_{\partial\Gamma} := -2\langle \mathrm{E}(\boldsymbol{w}), \mathrm{E}(Z_-\boldsymbol{\phi})\rangle_{\Gamma_-} + \langle p, \operatorname{div}(Z_-\boldsymbol{\phi})\rangle_{\Gamma_-} + \langle \tilde{\mathfrak{f}}, Z_-\boldsymbol{\phi}\rangle_{\Gamma_-}, \quad \forall \boldsymbol{\phi} \in H^{-\frac{1}{2}}(\partial\Gamma)^3, \tag{9}$$

where Z_- is a right inverse of the trace operator $Tr^- : \mathcal{H}^1(\Gamma_-)^3 \to H^{\frac{1}{2}}(\partial\Gamma)^3$. The operator $t^-_{0,\nu}$ is linear, bounded and does not depend on the choice of the right inverse Z_- of the trace operator Tr^-.

Moreover, the following Green formula holds:

$$\langle t^-_{0,\nu}(\boldsymbol{w}, p, \tilde{\mathfrak{f}}), Tr^-\boldsymbol{\psi}\rangle_{\partial\Gamma} = -2\langle \mathrm{E}(\boldsymbol{w}), \mathrm{E}(\boldsymbol{\psi})\rangle_{\Gamma_-} + \langle p, \operatorname{div} \boldsymbol{\psi}\rangle_{\Gamma_-} - \langle \tilde{\mathfrak{f}}, \boldsymbol{\psi}\rangle_{\Gamma_-}, \tag{10}$$

for all $(\boldsymbol{w}, p, \tilde{\mathfrak{f}}) \in \boldsymbol{\mathcal{H}}(\Gamma_-, \mathcal{B}_0)$ and for any $\boldsymbol{\psi} \in \mathcal{H}^1(\Gamma_-)^3$.

3. Transmission problems involving the generalized Darcy-Forchheimer-Brinkman system

In this section, we present the main result of the paper, namely the theorem that gives the well-posedness property of the transmission type problem for the generalized Darcy-Forchheimer-Brinkman and Stokes systems in Lipschitz domains in $\mathbb{R}^3$.

To simplify the notations, we will introduce the following spaces:

$$H^1_{\mathrm{div}}(\Gamma_+)^3 := \{\mathbf{w} \in H^1(\Gamma_+)^3 : \ \mathrm{div}\ \mathbf{w} = 0 \text{ in } \Gamma_+\},$$
$$\mathcal{H}^1_{\mathrm{div}}(\Gamma_-)^3 := \{\mathbf{w} \in \mathcal{H}^1(\Gamma_-)^3 : \ \mathrm{div}\ \mathbf{w} = 0 \text{ in } \Gamma_-\},$$
$$\mathfrak{X} := H^1_{\mathrm{div}}(\Gamma_+)^3 \times L^2(\Gamma_+) \times \mathcal{H}^1_{\mathrm{div}}(\Gamma_-)^3 \times L^2(\Gamma_-),$$
$$\mathfrak{Y} := \tilde{H}^{-1}(\Gamma_+)^3 \times \tilde{\mathcal{H}}^{-1}(\Gamma_-)^3 \times H^{\frac{1}{2}}(\partial\Gamma)^3 \times H^{-\frac{1}{2}}(\partial\Gamma)^3,$$
$$\mathfrak{Y}_\infty := \mathfrak{Y} \times \mathbb{R}^3.$$

In the latter, let $\mathring{E}$ denote the extension by zero operator outside Γ_+.

In order to get our desired result, we will state and prove a useful lemma that will intervene in the proof of our main result.

To this end, we have the following lemma (see also, [14, Lemma 5.1]).

Lemma 3.1. *Let $\Gamma_+ := \Gamma \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary. Let $k, \beta : \Gamma_+ \to \mathbb{R}_+$ such that $k, \beta \in L^\infty(\Gamma_+)$ and let*

$$\mathcal{J}_{k,\beta,\Gamma_+}(v) := \mathring{E}(k|v|v + \beta(v \cdot \nabla)v). \tag{11}$$

Then, the nonlinear operator $\mathcal{J}_{k,\beta,\Gamma_+} : H^1_{\mathrm{div}}(\Gamma_+)^3 \to \tilde{H}^{-1}(\Gamma_+)^3$ is continuous, positively homogeneous of order 2, and bounded, in the sense that there is a constant $c_0 = c_0(\Gamma_+, k, \beta) > 0$ such that

$$\|\mathcal{J}_{k,\beta,\Gamma_+}(v)\|_{\tilde{H}^{-1}(\Gamma_+)^3} \leq c_0 \|v\|^2_{H^1(\Gamma_+)^3}. \tag{12}$$

Moreover, the following Lipschitz-like relation holds:

$$\|\mathcal{J}_{k,\beta,\Gamma_+}(v) - \mathcal{J}_{k,\beta,\Gamma_+}(w)\| \leq c_0(\|v\|_{H^1(\Gamma_+)^3} + \|w\|_{H^1(\Gamma_+)^3})\|v - w\|_{H^1(\Gamma_+)^3}, \tag{13}$$

with $c_0 = c_0(\Gamma_+, k, \beta) > 0$, the same constant as in the relation (12).

Proof. We follow similar ideas to those in the proof of [14, Lemma 5.1] and we outline the main ideas.

Now, since Γ_+ is a bounded Lipschitz domain, we have that:

$$H^1(\Gamma_+)^3 \hookrightarrow L^q(\Gamma_+)^3, \tag{14}$$

continuously, for all q such that $2 \leq q \leq 6$. Note that the embedding (14) has dense range and hence

$$L^{q'}(\Gamma_+)^3 \hookrightarrow \tilde{H}^{-1}(\Gamma_+)^3, \tag{15}$$

continuously, in the sense that $\mathring{E}\mathbf{u} \in \tilde{H}^{-1}(\Gamma_+)^3$ for all $\mathbf{u} \in L^{q'}(\Gamma_+)^3$ where $\frac{6}{5} \leq q' \leq 2$ and there is a constant $c_q > 0$ such that

$$\|\mathring{E}\mathbf{u}\|_{\tilde{H}^{-1}(\Gamma_+)^3} \leq c_q \|\mathbf{u}\|_{L^{q'}(\Gamma_+)^3}. \tag{16}$$

Letting $q = 4$ in relation (14) and applying Hölder's inequality, one may show that $|\mathbf{v}|\mathbf{w} \in L^2(\Gamma_+)^3$, for all $\mathbf{v}, \mathbf{w} \in H^1(\Gamma_+)^3$. We introduce now, the operator $b : H^1(\Gamma_+)^3 \times H^1(\Gamma_+)^3 \to \tilde{H}^{-1}(\Gamma_+)^3$ as follows:

$$b(\mathbf{v}, \mathbf{w}) := \mathring{E}(k|\mathbf{v}|\mathbf{w}). \tag{17}$$

This operator is well-defined (see relation (16) when $q = 2$).

We can show, using relation (16) with $q = 2$ and Hölder's inequality, that there is a constant $c_* = c_*(\Gamma_+, k) > 0$ such that

$$\|b(\mathbf{v}, \mathbf{w})\|_{\tilde{H}^{-1}(\Gamma_+)^3} \leq c_* \|\mathbf{v}\|_{H^1(\Gamma_+)^3} \|\mathbf{w}\|_{H^1(\Gamma_+)^3}. \tag{18}$$

Thus, the nonlinear operator $b : H^1(\Gamma_+)^3 \times H^1(\Gamma_+)^3 \to \tilde{H}^{-1}(\Gamma_+)^3$ is bounded.

Consider again relation (14) with $q = 6$ and apply again Hölder's inequality, then one may deduce that $(\mathbf{v} \cdot \nabla)\mathbf{w} \in L^{\frac{3}{2}}(\Gamma_+)^3$, for all $\mathbf{v}, \mathbf{w} \in H^1(\Gamma_+)^3$ and the following inequality holds:

$$\|(\mathbf{v} \cdot \nabla)\mathbf{w}\|_{L^{\frac{3}{2}}(\Gamma_+)^3} \leq c' \|\mathbf{v}\|_{H^1(\Gamma_+)^3} \|\mathbf{w}\|_{H^1(\Gamma_+)^3}, \tag{19}$$

for some constant $c' = c'(\Gamma_+) > 0$.

We can introduce the following operator $t : H^1(\Gamma_+)^3 \times H^1(\Gamma_+)^3 \to \tilde{H}^{-1}(\Gamma_+)^3$ by the following relation:

$$t(\mathbf{v}, \mathbf{w}) := \mathring{E}(\beta(\mathbf{v} \cdot \nabla)\mathbf{w}). \tag{20}$$

This operator is well-defined (see relation (16) with $q = \frac{3}{2}$).

Once again, using relation (16) with $q = \frac{3}{2}$ and relation (19). we may show that there is a constant $c^* = c^*(\Gamma_+, \beta) > 0$ such that

$$\|t(\mathbf{v}, \mathbf{w})\|_{\tilde{H}^{-1}(\Gamma_+)^3} \leq c^* \|\mathbf{v}\|_{H^1(\Gamma_+)^3} \|\mathbf{w}\|_{H^1(\Gamma_+)^3}. \tag{21}$$

Thus, the nonlinear operator $t : H^1(\Gamma_+)^3 \times H^1(\Gamma_+)^3 \to \tilde{H}^{-1}(\Gamma_+)^3$ is bounded.

Let us note that the operator $\mathcal{J}_{k,\beta,\Gamma_+}$ admits the following decomposition:

$$\mathcal{J}_{k,\beta,\Gamma_+}(\mathbf{v}) = b(\mathbf{v}, \mathbf{v}) + t(\mathbf{v}, \mathbf{v}). \tag{22}$$

Then, by using inequalities (18) and (21), we deduce that our operator $\mathcal{J}_{k,\beta,\Gamma_+}$ satisfies relation (12) with the constant $c_0 := c_* + c^*$ and hence it is a bounded operator and positively homogeneous of order 2.

By using again relations (18) and (21) one may show that the Lipschitz-like inequality (13) for the operator $\mathcal{J}_{k,\beta,\Gamma_+}$ holds (using similar arguments as in the proof of [14, Lemma 5.1]).

We have omitted the full argument for the sake of brevity, but we refer the reader to the proof of [14, Lemma 5.1]. Thus, our proof is complete. □

In the latter, we recall a result for the transmission problem related to the generalized Brinkman and Stokes systems. Such a result is very useful to obtain the existence and uniqueness for our nonlinear problem concerning the generalized Darcy-Forchheimer-Brinkman and Stokes system in $\mathbb{R}^3$. Also, let $\boldsymbol{L} \in L^\infty(\partial\Gamma)^{3\times 3}$ be a symmetric matrix-valued function, which satisfies the following positivity condition:

$$\langle \boldsymbol{L}\mathbf{v}, \mathbf{v} \rangle_{\partial\Gamma} \geq 0, \quad \forall \mathbf{v} \in L^2(\partial\Gamma)^3. \tag{23}$$

We have the following theorem (cf. [2, Theorem 4.3], see also [14, Lemma 4.1, Theorem 4.2]).

Theorem 3.2. *Let $\Gamma_+ := \Gamma \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary and let $\Gamma_- := \mathbb{R}^3 \setminus \overline{\Gamma}$ the complementary Lipschitz set. Let $\mathcal{P} \in L^\infty(\Gamma_+)^{3\times 3}$, such that condition (2) holds. Let $\boldsymbol{L} \in L^\infty(\partial\Gamma)^{3\times 3}$ be a symmetric matrix-valued function that satisfied condition (23). Then, for $(\mathfrak{f}_+, \mathfrak{f}_-, \mathfrak{g}_0, \mathfrak{h}_0) \in \mathfrak{Y}$ given, the Poisson problem of transmission-type for the Stokes and the generalized Brinkman systems:*

$$\begin{cases} \Delta \boldsymbol{w}_+ - \mathcal{P}\boldsymbol{w}_+ - \nabla p_+ = \mathfrak{f}|_{\Gamma_+} \ in \ \Gamma_+, \\ \Delta \boldsymbol{w}_- - \nabla p_- = \mathfrak{f}|_{\Gamma_-} \ in \ \Gamma_-, \\ Tr^+\boldsymbol{w}_+ - Tr^-\boldsymbol{w}_- = \mathfrak{g}_0 \ on \ \partial\Gamma, \\ t^+_{\mathcal{P},\nu}(\boldsymbol{w}_+, p_+, \mathfrak{f}_+) - t^-_{0,\nu}(\boldsymbol{w}_-, p_-, \mathfrak{f}_-) + \boldsymbol{L}Tr^+\boldsymbol{w}_+ = \mathfrak{h}_0 \ on \ \partial\Gamma, \end{cases} \tag{24}$$

has a unique solution $(\boldsymbol{w}_+, p_+, \boldsymbol{w}_-, p_-) \in \mathfrak{X}$ *and there is a linear and continuous 'solution' operator,*

$$\mathcal{T} : \mathfrak{Y} \to \mathfrak{X}, \tag{25}$$

that maps the given data $(\mathfrak{f}_+, \mathfrak{f}_-, \mathfrak{g}_0, \mathfrak{h}_0) \in \mathfrak{Y}$ *to the solution* $(\boldsymbol{w}_+, p_+, \boldsymbol{w}_-, p_-) \in \mathfrak{X}$ *of the transmission problem (24).*

Moreover, there is a constant $C \equiv C(\Gamma_+, \Gamma_-, \mathcal{P}, \boldsymbol{L}) > 0$ *such that:*

$$\|(\boldsymbol{w}_+, p_+, \boldsymbol{w}_-, p_-)\|_{\mathfrak{X}} \leq C\|(\mathfrak{f}_+, \mathfrak{f}_-, \mathfrak{g}_0, \mathfrak{h}_0)\|_{\mathfrak{Y}}, \tag{26}$$

and $\boldsymbol{w}_-$ *vanishes at infinity in the sense of Leray.*

Proof. In order to show the statement of our theorem, we use the following arguments (see also, [2, Theorem 4.5]).

First, we note that, the Poisson problem of transmission-type for the Stokes system in the bounded Lipschitz domain Γ_+ and the Stokes system in the complementary Lipschitz set Γ_- in $\mathbb{R}^3$:

$$\begin{cases} \Delta \mathbf{w}_+ - \nabla p_+ = \mathfrak{f}|_{\Gamma_+} \text{ in } \Gamma_+, \\ \Delta \mathbf{w}_- - \nabla p_- = \mathfrak{f}|_{\Gamma_-} \text{ in } \Gamma_-, \\ Tr^+\mathbf{w}_+ - Tr^-\mathbf{w}_- = \mathfrak{g}_0 \text{ on } \partial\Gamma, \\ t^+_{\mathcal{P},\nu}(\mathbf{w}_+, p_+, \mathfrak{f}_+) - t^-_{0,\nu}(\mathbf{w}_-, p_-, \mathfrak{f}_-) + \boldsymbol{L}Tr^+\mathbf{w}_+ = \mathfrak{h}_0 \text{ on } \partial\Gamma, \end{cases} \tag{27}$$

is well-posed (see [18, Proposition 5.1, Theorem 5.1] and [2, Lemma 4.1, Theorem 4.3]). It follows that there exists a ″solution″ operator

$$\mathcal{S} : \mathfrak{Y} \to \mathfrak{X}, \tag{28}$$

that maps the given data given data $(\mathfrak{f}_+, \mathfrak{f}_-, \mathfrak{g}_0, \mathfrak{h}_0) \in \mathfrak{Y}$ to the corresponding solution $(\mathbf{w}_+, p_+, \mathbf{w}_-, p_-) \in \mathfrak{X}$ of the problem (27). The operator $\mathcal{S}$ is well-defined, linear and continuous.

Secondly, for the uniqueness argument, one must consider the homogeneous version of our problem (24). Then one must apply Green's formulas (7), (10), the transmission conditions of the homogeneous version of (24), the positivity conditions satisfied by $\mathcal{P}$ and $\boldsymbol{L}$ and the well-posedness of the exterior Dirichlet problem for the Stokes system with homogeneous Dirichlet boundary condition in order to deduce that, indeed, problem (24) has at most one solution.

Finally, for the existence part, we use the extension by zero operator $\mathring{E}$ in order to rewrite the problem (24) in the form:

$$\begin{cases} \Delta \mathbf{w}_+ - \nabla p_+ = \mathfrak{f}|_{\Gamma_+} + \mathring{E}(\mathcal{P}\mathbf{w}_+) \text{ in } \Gamma_+, \\ \Delta \mathbf{w}_- - \nabla p_- = \mathfrak{f}|_{\Gamma_-} \text{ in } \Gamma_-, \\ Tr^+\mathbf{w}_+ - Tr^-\mathbf{w}_- = \mathfrak{g}_0 \text{ on } \partial\Gamma, \\ t^+_{\mathcal{P},\nu}(\mathbf{w}_+, p_+, \mathfrak{f}_+) - t^-_{0,\nu}(\mathbf{w}_-, p_-, \mathfrak{f}_-) + \boldsymbol{L}Tr^+\mathbf{w}_+ = \mathfrak{h}_0 \text{ on } \partial\Gamma. \end{cases} \tag{29}$$

The problem (29) can be written equivalently in terms of the solution operator $\mathcal{S}$ as:

$$(\mathbf{w}_+, p_+, \mathbf{w}_-, p_-) = \mathcal{S}(\mathfrak{f}_+ + \mathring{E}(\mathcal{P}\mathbf{w}_+), \mathfrak{f}_-, \mathfrak{g}_0, \mathfrak{h}_0). \tag{30}$$

Using the linearity of the operator $\mathcal{S}$, we rewrite (30) in the following form:

$$\mathcal{S}_{\mathcal{B}}(\mathbf{w}_+, p_+, \mathbf{w}_-, p_-) = \mathcal{S}(\mathfrak{f}_+, \mathfrak{f}_-, \mathfrak{g}_0, \mathfrak{h}_0), \tag{31}$$

where $\mathcal{S}_{\mathcal{B}} : \mathfrak{X} \to \mathfrak{X}$ is a Fredholm operator of index 0 given by:

$$\mathcal{S}_{\mathcal{B}}(\mathbf{w}_+, p_+, \mathbf{w}_-, p_-) = \mathbb{I}(\mathbf{w}_+, p_+, \mathbf{w}_-, p_-) - \mathcal{S}_c(\mathbf{w}_+, p_+, \mathbf{w}_-, p_-), \tag{32}$$

where $\mathcal{S}_c : \mathfrak{X} \to \mathfrak{X}$ is given by

$$\mathcal{S}_c(\mathbf{w}_+, p_+, \mathbf{w}_-, p_-) := \mathcal{S}(\mathring{E}(\mathcal{P}\mathbf{w}_+), 0, 0, 0) \tag{33}$$

is a compact operator.

Moreover, the equivalence between equation (31) and our transmission problem (29) implies that the operator $\mathcal{S}_{\mathcal{B}}$ is also injective and hence an isomorphism.

Then, the existence part of our proof is finished and the continuity of the operators $\mathcal{S}$ and $\mathcal{S}_{\mathcal{B}}$ implies that there is a constant $C \equiv C(\Gamma_+, \Gamma_-, \mathcal{P}, \boldsymbol{L}) > 0$ such that the estimate (26) holds and $\mathbf{w}_- \in \mathcal{H}^1(\Gamma_-)^3$ vanishes at infinity in the sense of Leray due to Corollary 2.2.

This concludes the proof. □

The main problem in this paper, as previously announced, is the transmission problem for the generalized Darcy-Forchheimer-Brinkman and Stokes systems in Lipschitz domains in $\mathbb{R}^3$. It requires to determine the unknown fields $(\mathbf{w}_+, p_+, \mathbf{w}_-, p_-) \in \mathfrak{X}$ satisfying:

$$\begin{cases} \Delta\mathbf{w}_+ - \mathcal{P}\mathbf{w}_+ - k|\mathbf{w}_+|\mathbf{w}_+ - \beta(\mathbf{w}_+ \cdot \nabla)\mathbf{w}_+ \\ \quad -\nabla p_+ = \mathfrak{f}|_{\Gamma_+} \text{ in } \Gamma_+, \\ \Delta\mathbf{w}_- - \nabla p_- = \mathfrak{f}|_{\Gamma_-} \text{ in } \Gamma_-, \\ Tr^+\mathbf{w}_+ - Tr^-\mathbf{w}_- = \mathfrak{g}_0 \text{ on } \partial\Gamma, \\ t^+_{\mathcal{P},\nu}(\mathbf{w}_+, p_+, \mathfrak{f}_+ + \mathring{E}(k|\mathbf{w}_+|\mathbf{w}_+ + \beta(\mathbf{w}_+ \cdot \nabla)\mathbf{w}_+)) \\ \quad -t^-_{0,\nu}(\mathbf{w}_-, p_-, \mathfrak{f}_-) + \boldsymbol{L}Tr^+\mathbf{w}_+ = \mathfrak{h}_0 \text{ on } \partial\Gamma, \end{cases} \tag{34}$$

The following result regarding the well-posedness of the transmission problem (34) was obtained (see also [14, Theorem 5.2] in the case $k, \beta > 0$, $\mathcal{P} \equiv \alpha\mathbb{I}$ where $\alpha > 0$ is a constant).

Theorem 3.3. *Let $\Gamma_+ := \Gamma \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary and let $\Gamma_- := \mathbb{R}^3 \setminus \overline{\Gamma}$ the complementary Lipschitz set. Let $\mathcal{P} \in L^\infty(\Gamma_+)^{3\times3}$, such that condition (2) holds. Let $\boldsymbol{L} \in L^\infty(\partial\Gamma)^{3\times3}$ be a symmetric matrix-valued function that satisfied condition (23). Let $\boldsymbol{w}_\infty \in \mathbb{R}^3$ be a constant vector. Then, there exist two constants $\xi = \xi(\Gamma_+, \Gamma_-, \mathcal{P}, k, \beta, \boldsymbol{L}) > 0$ and $\lambda = \lambda(\Gamma_+, \Gamma_-, \mathcal{P}, k, \beta, \boldsymbol{L}) > 0$, such that for all given $(\mathfrak{f}_+, \mathfrak{f}_-, \mathfrak{g}_0, \mathfrak{h}_0, \boldsymbol{w}_\infty) \in \mathfrak{Y}_\infty$ that satisfy*

$$\|(\mathfrak{f}_+, \mathfrak{f}_-, \mathfrak{g}_0, \mathfrak{h}_0, \boldsymbol{w}_\infty)\|_{\mathfrak{Y}_\infty} \leq \xi, \tag{35}$$

the transmission problem (34) has a unique solution $(\boldsymbol{w}_+, p_+, \boldsymbol{w}_-, p_-) \in \mathfrak{X}$ and

$$\|(\boldsymbol{w}_+, p_+, \boldsymbol{w}_- - \boldsymbol{w}_\infty, p_-)\|_{\mathfrak{X}} \leq \lambda. \tag{36}$$

In addition, the solution depends continuously on the given data and satisfies the following estimate:

$$\|(\boldsymbol{w}_+, p_+, \boldsymbol{w}_- - \boldsymbol{w}_\infty, p_-)\|_{\mathfrak{X}} \leq C_0\|(\mathfrak{f}_+, \mathfrak{f}_-, \mathfrak{g}_0, \mathfrak{h}_0, \boldsymbol{w}_\infty)\|_{\mathfrak{Y}_\infty}, \tag{37}$$

where $C_0 = C_0(\Gamma_+, \Gamma_-, \mathcal{P}, \boldsymbol{L}) > 0$ is a constant and $\boldsymbol{w}_- - \boldsymbol{w}_\infty$ vanishes at infinity in the sense of Leray.

Proof. We follow similar ideas as those in the proof of [14, Theorem 5.2].

First, we shall concern ourselves with the existence part.

To this end, consider the new variables

$$\mathbf{u}_+ := \mathbf{w}_+, \quad \mathbf{u}_- := \mathbf{w}_- - \mathbf{w}_\infty, \tag{38}$$

which reduce the problem (34) to the following one:

$$\begin{cases} \Delta\mathbf{u}_+ - \mathcal{P}\mathbf{u}_+ - \nabla p_+ = \mathfrak{f}|_{\Gamma_+} + \mathcal{J}_{k,\beta,\Gamma_+}(\mathbf{u}_+)|_{\Gamma_+} \text{ in } \Gamma_+, \\ \Delta\mathbf{u}_- - \nabla p_- = \mathfrak{f}|_{\Gamma_-} \text{ in } \Gamma_-, \\ Tr^+\mathbf{u}_+ - Tr^-\mathbf{u}_- = \mathfrak{g}_0 + \mathbf{w}_\infty \text{ on } \partial\Gamma, \\ t^+_{\mathcal{P},\nu}(\mathbf{u}_+, p_+, \mathfrak{f}_+ + \mathcal{J}_{k,\beta,\Gamma_+}(\mathbf{u}_+)) - t^-_{0,\nu}(\mathbf{u}_-, p_-, \mathfrak{f}_-) \\ \quad +\boldsymbol{L}Tr^+\mathbf{u}_+ = \mathfrak{h}_0 \text{ on } \partial\Gamma. \end{cases} \tag{39}$$

Further on, we will construct a nonlinear operator $\mathcal{U}_+$ that maps a closed ball of the space $H^1_{\mathrm{div}}(\Gamma_+)^3$ to itself and that is a contraction on that ball. Hence, a solution of the problem (39) will be determined with the unique fixed point of the operator $\mathcal{U}_+$.

In order to construct the operator $\mathcal{U}_+$, we fix $\mathbf{u}_+ \in H^1_{\mathrm{div}}(\Gamma_+)^3$ and we consider the (linear) Poisson problem of transmission type for the generalized Brinkman and Stokes systems with the unknowns $(\mathbf{u}^0_+, p^0_+, \mathbf{u}^0_-, p^0_-)$

$$\begin{cases} \Delta\mathbf{u}^0_+ - \mathcal{P}\mathbf{u}^0_+ - \nabla p^0_+ = \mathfrak{f}|_{\Gamma_+} + \mathcal{J}_{k,\beta,\Gamma_+}(\mathbf{u}_+)|_{\Gamma_+} \text{ in } \Gamma_+, \\ \Delta\mathbf{u}^0_- - \nabla p^0_- = \mathfrak{f}|_{\Gamma_-} \text{ in } \Gamma_-, \\ Tr^+\mathbf{u}^0_+ - Tr^-\mathbf{u}^0_- = \mathfrak{g}_0 + \mathbf{w}_\infty \text{ on } \partial\Gamma, \\ t^+_{\mathcal{P},\nu}(\mathbf{u}^0_+, p^0_+, \tilde{\mathfrak{f}}_+ + \mathcal{J}_{k,\beta,\Gamma_+}(\mathbf{u}_+)) - t^-_{0,\nu}(\mathbf{u}^0_-, p^0_-, \tilde{\mathfrak{f}}_-) \\ \quad +LTr^+\mathbf{u}^0_+ = \mathfrak{h}_0 \text{ on } \partial\Gamma, \end{cases} \tag{40}$$

where $\mathfrak{g}_0 + \mathbf{w}_\infty \in H^{\frac{1}{2}}(\partial\Gamma)^3$ and $\mathcal{J}_{k,\beta,\Gamma_+}(\mathbf{u}_+) \in \tilde{H}^{-1}(\Gamma_+)^3$ due to Lemma 3.1.

Then, by applying Theorem 3.2, we deduce that the linear transmission problem (40) has a unique solution given by

$$\begin{aligned} (\mathbf{u}^0_+, p^0_+, \mathbf{u}^0_-, p^0_-) &= (\mathcal{U}_+(\mathbf{u}_+), \mathcal{R}_+(\mathbf{u}_+), \mathcal{U}_-(\mathbf{u}_+), \mathcal{R}_-(\mathbf{u}_+)) \\ &:= \mathcal{T}(\mathfrak{f}|_{\Gamma_+} + \mathcal{J}_{k,\beta,\Gamma_+}(\mathbf{u}_+)|_{\Gamma_+}, \mathfrak{f}|_{\Gamma_-}, \mathfrak{g}_0 + \mathbf{w}_\infty, \mathfrak{h}_0) \in \mathfrak{X}. \end{aligned} \tag{41}$$

On the other hand, for fixed given data $\tilde{\mathfrak{f}}_\pm, \mathfrak{g}_0, \mathfrak{h}_0, \mathbf{w}_\infty$, the nonlinear operator

$$(\mathcal{U}_+, \mathcal{R}_+, \mathcal{U}_-, \mathcal{R}_-) : H^1_{\mathrm{div}}(\Gamma_+)^3 \to \mathfrak{X}, \tag{42}$$

is continuous and bounded, in the sense that there exists a constant denoted by d_*, $d_* = d_*(\Gamma_+, \Gamma_-, \mathcal{P}, \boldsymbol{L}) > 0$ such that

$$\begin{aligned} &\|(\mathcal{U}_+(\mathbf{u}_+), \mathcal{R}_+(\mathbf{u}_+), \mathcal{U}_-(\mathbf{u}_+), \mathcal{R}_-(\mathbf{u}_+))\|_{\mathfrak{X}} \\ &\quad \le d_*\|(\mathfrak{f}|_{\Gamma_+} + \mathcal{J}_{k,\beta,\Gamma_+}(\mathbf{u}_+)|_{\Gamma_+}, \mathfrak{f}|_{\Gamma_-}, \mathfrak{g}_0, \mathfrak{h}_0, \mathbf{w}_\infty)\|_{\mathfrak{Y}_\infty} \\ &\quad \le d_*\|(\mathfrak{f}|_{\Gamma_+}, \mathfrak{f}|_{\Gamma_-}, \mathfrak{g}_0, \mathfrak{h}_0, \mathbf{w}_\infty)\|_{\mathfrak{Y}_\infty} + d_* c_0\|\mathbf{u}_+\|^2_{H^1(\Gamma_+)^3}, \end{aligned} \tag{43}$$

for all $\mathbf{u}_+ \in H^1_{\mathrm{div}}(\Gamma_+)^3$ where the constant $c_0 = c_0(\Gamma_+, k, \beta) > 0$ is the same constant as in the Lemma 3.1. Moreover, we can rewrite the problem (39) using the operators $(\mathcal{U}_+, \mathcal{R}_+, \mathcal{U}_-, \mathcal{R}_-)$ as follows

$$\begin{cases} \Delta\mathcal{U}_+(\mathbf{u}_+) - \mathcal{P}\mathcal{U}_+(\mathbf{u}_+) - \nabla\mathcal{R}_+(\mathbf{u}_+) = \mathfrak{f}|_{\Gamma_+} + \mathcal{J}_{k,\beta,\Gamma_+}(\mathbf{u}_+)|_{\Gamma_+} \text{ in } \Gamma_+, \\ \Delta\mathcal{U}_-(\mathbf{u}_+) - \nabla\mathcal{R}_-(\mathbf{u}_+) = \mathfrak{f}|_{\Gamma_-} \text{ in } \Gamma_-, \\ Tr^+\mathcal{U}_+(\mathbf{u}_+) - Tr^-\mathcal{U}_-(\mathbf{u}_+) = \mathfrak{g}_0 + \mathbf{w}_\infty \text{ on } \partial\Gamma, \\ t^+_{\mathcal{P},\nu}(\mathcal{U}_+(\mathbf{u}_+), \mathcal{R}_+(\mathbf{u}_+), \tilde{\mathfrak{f}}_+ + \mathcal{J}_{k,\beta,\Gamma_+}(\mathbf{u}_+)) - t^-_{0,\nu}(\mathcal{U}_-(\mathbf{u}_+), \mathcal{R}_-(\mathbf{u}_+), \tilde{\mathfrak{f}}_-) \\ \quad +LTr^+\mathcal{U}_+(\mathbf{u}_+) = \mathfrak{h}_0 \text{ on } \partial\Gamma. \end{cases} \tag{44}$$

The next step in our argument is to show that the operator $\mathcal{U}_+$ has a fixed point $\mathbf{u}_+ \in H^1_{\mathrm{div}(\Gamma_+)^3}$. Indeed, such a fixed point together with $\mathbf{u}_- = \mathcal{U}_-(\mathbf{u}_+), p_\pm = \mathcal{R}_\pm(\mathbf{u}_+)$ will determine a solution of our nonlinear problem (39).

Similar to the proof of Theorem 5.2 in [14], to prove that $\mathcal{U}_+$ has a fixed point, we will show that $\mathcal{U}_+$ maps a closed ball $\mathbb{B}_\lambda$ in $H^1_{\mathrm{div}}(\Gamma_+)^3$ to itself and that $\mathcal{U}_+$ is a contraction on $\mathbb{B}_\lambda$.

We introduce the constants

$$\xi := \frac{3}{16c_0 d_*^2} > 0, \quad \lambda := \frac{1}{4c_0 d_*} > 0 \tag{45}$$

and the closed ball

$$\mathbb{B}_\lambda := \{\mathbf{u}_+ \in H^1_{\mathrm{div}}(\Gamma_+)^3 : \|\mathbf{u}_+\|_{H^1(\Gamma_+)^3} \le \lambda\}, \tag{46}$$

and assume that the given data satisfies the following condition

$$\|(\mathfrak{f}|_{\Gamma_+}, \mathfrak{f}|_{\Gamma_-}, \mathfrak{g}_0, \mathfrak{h}_0, \mathbf{w}_\infty)\|_{\mathfrak{Y}_\infty} \le \xi. \tag{47}$$

Then, by using relations (43), (45), (46), (47), we deduce that

$$\|(\mathcal{U}_+(\mathbf{u}_+), \mathcal{R}_+(\mathbf{u}_+), \mathcal{U}_-(\mathbf{u}_+), \mathcal{R}_-(\mathbf{u}_+))\|_{\mathfrak{X}} \le \frac{1}{4c_0 d_*} = \lambda, \tag{48}$$

for all $\mathbf{u}_+ \in \mathbb{B}_\lambda$.

Relation (48) shows us that $\|\mathcal{U}_+(\mathbf{u}_+)\|_{H^1(\Gamma_+)^3} \le \lambda$ for all $\mathbf{u}_+ in \mathbb{B}_\lambda$, i.e., $\mathcal{U}_+$ maps the ball $\mathbb{B}_\lambda$ to itself.

Next, we shall prove that $\mathcal{U}_+$ is Lipschitz continuous on the ball $\mathbb{B}_\lambda$. To do this, we fix the data $(\mathfrak{f}|_{\Gamma_+}, \mathfrak{f}|_{\Gamma_-}, \mathfrak{g}_0, \mathfrak{h}_0, \mathbf{w}_\infty)$, two arbitrary functions $\mathbf{u}_+, \mathbf{v}_+ \in \mathbb{B}_\lambda$ and use expression (41) in order to deduce that

$$\begin{aligned}
&\|\mathcal{U}_+(\mathbf{u}_+) - \mathcal{U}_+(\mathbf{v}_+)\|_{H^1(\Gamma_+)^3} \\
&\quad \le d_* \|\mathcal{J}_{k,\beta,\Gamma}(\mathbf{u}_+) - \mathcal{J}_{k,\beta,\Gamma}(\mathbf{v}_+)\|_{\tilde{H}^{-1}(\Gamma_+)^3} \\
&\quad \le d_* c_0 (\|\mathbf{u}_+\|_{H^1(\Gamma_+)^3} + \|\mathbf{v}_+\|_{H^1(\Gamma_+)^3})\|\mathbf{u}_+ - \mathbf{v}_+\|_{H^1(\Gamma_+)^3} \\
&\quad \le 2 d_* c_0 \|\mathbf{u}_+ - \mathbf{v}_+\|_{H^1(\Gamma_+)^3} = \frac{1}{2}\|\mathbf{u}_+ - \mathbf{v}_+\|_{H^1(\Gamma_+)^3},
\end{aligned} \tag{49}$$

for all $\mathbf{u}_+, \mathbf{v}_+ \in \mathbb{B}_\lambda$, where the first inequality holds due to the continuity of the "solution" operator $\mathcal{T}$, while the second inequality holds due to inequality (13) from Lemma 3.1 and the constants d_*, c_0 are the same constants as in (43). In view of this argument, we conclude that $\mathcal{U}_+ : \mathbb{B}_\lambda \to \mathbb{B}_\lambda$ is a $\frac{1}{2}$-contraction.

It remains to apply the Banach Fixed Point Theorem to conclude that there exists a unique fixed point $\mathbf{u}_+ \in \mathbb{B}_\lambda$ of the operator $\mathcal{U}_+$. Then, the fixed point $\mathbf{u}_+$ together with the function $\mathbf{u}_- := \mathcal{U}_-(\mathbf{u}_+)$ and $p_\pm := \mathcal{R}_\pm(\mathbf{u}_+)$ given by (41), determine a solution of the nonlinear problem (39) in the space $\mathfrak{X}$. Moreover, $\mathbf{u}_-$ vanishes at infinity in the sense of Leray due to the membership of $\mathbf{u}_- \in \mathcal{H}^1_{\mathrm{div}}(\Gamma_-)^3$.

The original fields $(\mathbf{w}_+, p_+, \mathbf{w}_-, p_-)$. where $\mathbf{w}_+$ and $\mathbf{w}_-$ are determined by (38) represent a solution of our nonlinear Poisson problem of transmission-type (34) satisfying $(\mathbf{w}_+, p_+, \mathbf{w}_- - \mathbf{w}_\infty, p_-) \in \mathfrak{X}$. Taking into account relation (48) and the expressions of $\mathbf{u}_-$ and $p_\pm$ in terms of $\mathbf{u}_+$, we get that our constructed solution satisfies the estimate (36). Also, the quantity $\mathbf{w}_- - \mathbf{w}_\infty$ vanishes at infinity in the sense of Leray.

Moreover, in view of the fact that the field $\mathbf{u}_+ \in \mathbb{B}_\lambda$, we deduce that

$$d_* c_0 \|\mathbf{u}_+\|_{H^1(\Gamma_+)^3} \le \frac{1}{4}$$

and using inequality (43), we get

$$\begin{aligned}
&\|\mathbf{u}_+\|_{H^1(\Gamma_+)^3} + \|p_+\|_{L^2(\Gamma_+)} + \|\mathbf{u}_-\|_{\tilde{H}^1(\Gamma_-)^3} + \|p_-\|_{L^2(\Gamma_-)} \\
&\quad \le d_* \|(\mathfrak{f}|_{\Gamma_+}, \mathfrak{f}|_{\Gamma_-}, \mathfrak{g}_0, \mathfrak{h}_0, \mathbf{w}_\infty)\|_{\mathfrak{Y}_\infty} + \frac{1}{4}\|\mathbf{u}_+\|_{H^1(\Gamma_+)^3},
\end{aligned} \tag{50}$$

which in turn gives (cf. [14, Theorem 5.2]),

$$\|\mathbf{u}_+\|_{H^1(\Gamma_+)^3} \le \frac{4}{3} d_* \|(\mathfrak{f}|_{\Gamma_+}, \mathfrak{f}|_{\Gamma_-}, \mathfrak{g}_0, \mathfrak{h}_0, \mathbf{w}_\infty)\|_{\mathfrak{Y}_\infty}. \tag{51}$$

Now, we substitute relation (51) into relation (50) and using (38), we obtain the desired estimate (37) with $C_0 = \frac{4}{3} d_*$.

With the last arguments, we have established the existence part.

We shall move to the uniqueness part of our proof.

We choose two solutions of the transmission problem (34) and we denote them by $(\mathbf{w}^1_+, p^1_+, \mathbf{w}^1_-, p^1_-)$ and $(\mathbf{w}^2_+, p^2_+, \mathbf{w}^2_-, p^2_-)$. Note that $(\mathbf{w}^1_+, p^1_+, \mathbf{w}^1_- - \mathbf{w}_\infty, p^1_-) \in \mathfrak{X}$ and $(\mathbf{w}^2_+, p^2_+, \mathbf{w}^2_- - \mathbf{w}_\infty, p^2_-) \in \mathfrak{X}$ are both solutions satisfying inequality (36).

Now, for $(\mathbf{u}_+^2, \mathbf{u}_-^2) = (\mathbf{w}_+^2, \mathbf{w}_-^2 - \mathbf{w}_\infty)$, we obtain $\mathbf{u}_+^2 \in \mathbb{B}_\lambda$. Since $\mathbf{u}_+^2 \in \mathbb{B}_\lambda$, we obtain that $\mathcal{U}_+(\mathbf{u}_+^2) \in \mathbb{B}_\lambda$, where $(\mathcal{U}_+(\mathbf{u}_+^2), \mathcal{R}_+(\mathbf{u}_+^2), \mathcal{U}_-(\mathbf{u}_+^2), \mathcal{R}_-(\mathbf{u}_+^2))$ are given by relation (41) and satisfy the problem (44) with $\mathbf{u}_+$ substituted with $\mathbf{u}_+^2$.

Then, we consider the problem (44), as well as the problem (39) written in the variables $(\mathbf{u}_+^2, p_+^2, \mathbf{u}_-^2 - \mathbf{w}_\infty, p_-^2)$ and based on these problems we obtain the following linear problem

$$\begin{cases} \Delta(\mathcal{U}_+(\mathbf{u}_+^2) - \mathbf{u}_+^2) - \mathcal{P}(\mathcal{U}_+(\mathbf{u}_+^2) - \mathbf{u}_+^2) - \nabla(\mathcal{R}_+(\mathbf{u}_+^2) - p_+^2) = \mathbf{0} \text{ in } \Gamma_+, \\ \Delta(\mathcal{U}_-(\mathbf{u}_+^2) - \mathbf{u}_-^2) - \nabla(\mathcal{R}_-(\mathbf{u}_+^2) - p_-^2) = \mathbf{0} \text{ in } \Gamma_-, \\ Tr^+(\mathcal{U}_+(\mathbf{u}_+^2) - \mathbf{u}_+^2) - Tr^-(\mathcal{U}_-(\mathbf{u}_+^2) - \mathbf{u}_-^2) = \mathbf{0} \text{ on } \partial\Gamma, \\ t_{\mathcal{P},\nu}^+((\mathcal{U}_+(\mathbf{u}_+^2) - \mathbf{u}_+^2), (\mathcal{R}_+(\mathbf{u}_+^2) - p_+^2), \mathbf{0}) \\ \quad -t_{0,\nu}^-((\mathcal{U}_-(\mathbf{u}_+^2) - \mathbf{u}_-^2), (\mathcal{R}_-(\mathbf{u}_+^2) - p_-^2), \mathbf{0}) \\ \quad +LTr^+(\mathcal{U}_+(\mathbf{u}_+^2) - \mathbf{u}_+^2) = \mathbf{0} \text{ on } \partial\Gamma. \end{cases} \tag{52}$$

In view of Theorem 3.2, we deduce that the linear problem (52) has only the trivial solution in $\mathfrak{X}$. Hence, $\mathcal{U}_+(\mathbf{u}_+^2) - \mathbf{u}_+^2 = 0$, i.e., $\mathbf{u}_+^2$ is a fixed point of the operator $\mathcal{U}_+$. Recall that $\mathcal{U}_+ : \mathbb{B}_\lambda \to \mathbb{B}_\lambda$ is a contraction, and thus, there is only a unique fixed point $\mathbf{u}_+^1$ in $\mathbb{B}_\lambda$. Therefore $\mathbf{u}_+^2 = \mathbf{u}_+^1$, $\mathbf{u}_-^2 = \mathbf{u}_-^1$ and also $p_\pm^2 = p_\pm^1$. This establishes the uniqueness part.

Finally, due to the continuity of the contraction $\mathcal{U}_+$ with respect to the given data and the continuity of the operator $\mathcal{T}$, we deduce that the solution $(\mathbf{w}_+, p_+, \mathbf{w}_-, p_-) \in \mathfrak{X}$ depends continuously on the given data and hence inequality (50) is satisfied with the constant $C_0 = \frac{4}{3}d_*$.

This concludes the proof. □

Remark 3.4. *In the case $k = 0$ and $\beta : \Gamma_+ \to \mathbb{R}_+$ such that $\beta \in L^\infty(\Gamma_+)$, we obtain a well-posedness result for the nonlinear transmission problem for the generalized Navier-Stokes and Stokes systems.*

Remark 3.5. *In the case $k : \Gamma_+ \to \mathbb{R}_+$ such that $k \in L^\infty(\Gamma_+)$ and $\beta = 0$, we obtain a well-posedness result for a semilinear transmission problem for a generalized semilinear Darcy-Forchheimer-Brinkman system and Stokes system.*

Acknowledgement

The author acknowledges the support of the Grant PN-III-P4-ID-PCE-2016-0036 of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI.

References

[1] M.S. Agranovich, Sobolev Spaces, Their Generalizations and Elliptic Problems in Smooth and Lipschitz Domains, Springer International Publishing, Cham, 2015.

[2] A.F. Albişoru, A Poisson Problem of Transmission-type for the Stokes and Generalized Brinkman Systems in Complementary Lipschitz Domains in $\mathbb{R}^3$, Taiwanese J. Math., advance publication, 14 May 2019, doi:10.11650/tjm/190408, https://projecteuclid.org/euclid.twjm/1557799218.

[3] L. Escauriaza, M. Mitrea, Transmission problems and spectral theory for singular integral operators on Lipschitz domains, J. Funct. Anal. 216 (2004) 141-171.

[4] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol. I, II, Springer, Berlin, 1998.

[5] V. Girault, A. Sequeira, A well-posed problem for the exterior Stokes equations in two and three dimensions, Arch. Ration. Mech. Anal. 114 (1991) 313-333.

[6] T. Groşan, M. Kohr, W.L. Wendland, Dirichlet problem for a nonlinear generalized Darcy-Forchheimer-Brinkman system in Lipschitz domains, Math. Meth. Appl. Sci. 38 (2015) 3615-3628.

[7] R. Gutt, M. Kohr, S.E. Mikhailov, W.L. Wendland, On the mixed problem for the semilinear Darcy-Forchheimer-Brinkman PDE system in Besov spaces on creased Lipschitz domains, Math. Meth. Appl. Sci. 40 (2017) 7780-7829.

[8] B. Hanouzet, Espaces de Sobolev avec poids - application au probleme de Dirichlet dans un demi-space, Rend. Ser. Math. Univ. Padova 46 (1971) 227-272.

[9] G.C. Hsiao, W.L. Wendland, Boundary Integral Equations: Variational Methods, Springer, Heidelberg, 2008.

[10] M. Kohr, M. Lanza de Cristoforis, W.L. Wendland, Boundary Value Problems of Robin Type for the Brinkman and Darcy-Forchheimer-Brinkman Systems in Lipschitz Domains, J. Math. Fluid Mech. 16 (2014) 595-630.

[11] M. Kohr, M. Lanza de Cristoforis, W.L. Wendland, Nonlinear Neumann-Transmission Problems for Stokes and Brinkman Equations on Euclidean Lipschitz Domains, Potential Anal. 38 (2013) 1123-1171.

[12] M. Kohr, M. Lanza de Cristoforis, W.L. Wendland, On the Robin-Transmission Boundary Value Problems for the Darcy-Forchheimer-Brinkman and Navier-Stokes Systems, J. Math. Fluid Mech. 18 (2016), 293-329.

[13] M. Kohr, M. Lanza de Cristoforis, W.L. Wendland, Poisson problems for semilinear Brinkman systems on Lipschitz domains in $\mathbb{R}^n$, Z. Angew. Math. Phys. 66 (2015) 833-864.

[14] M. Kohr, M. Lanza de Cristoforis, S.E. Mikhailov, W.L. Wendland, Integral potential method for a transmission problem with Lipschitz interface in $\mathbb{R}^3$ for the Stokes and Darcy-Forchheimer-Brinkman PDE Systems, Z. Angew. Math. Phys. 67:116 5 (2016) 1-30.

[15] M. Kohr, M. Lanza de Cristoforis, W.L. Wendland, Nonlinear Darcy-Forchheimer-Brinkman system with linear Robin boundary conditions in Lipschitz domains, Complex Analysis and Potential Theory with Applications (T. Aliev Azeroglu, A. Golberg, S. Rogosin eds.), Cambridge Scientific Publishers, 2014, 111-124, ISBN 978-1-908106-40-7.

[16] M. Kohr, S.E. Mikhailov, W.L. Wendland, Transmission Problems for the Navier-Stokes and Darcy-Forchheimer-Brinkman Systems in Lipschitz Domains on Compact Riemannian Manifolds, J. Math. Fluid Mech. 19 (2017) 203-238.

[17] D. Medkova, Transmission problem for the Brinkman system, Complex Var. Elliptic Equ. 59 (2014), 1664-1678.

[18] D. Medkova, Integral equations method and the transmission problem for the Stokes system, Kragujevac J. Math., 39 (2015), 53-71.

[19] S.E. Mikhailov, Traces, extensions and co-normal derivatives for elliptic systems on Lipschitz domains, J. Math. Anal. Appl. 378 (2011) 324-342.

[20] M. Mitrea, M. Wright,Boundary value problems for the Stokes system in arbitrary Lipschitz domains, Asterisque 344 (2012) viii+241.

[21] D.A. Nield, A. Bejan, Convection in Porous Media, 3rd edn., Springer, New York, 2013.

[22] H. Sohr, The Navier-Stokes Equations: An Elementary Functional Analytic Approach, Birkhäuser Verlag, Basel, 2001.