Filomat 33:11 (2019), 3291–3305
https://doi.org/10.2298/FIL1911291C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Sufficient Oscillation Conditions for Deviating Difference Equations

George E. Chatzarakis[a]

[a] *Department of Electrical and Electronic Engineering Educators,
School of Pedagogical and Technological Education (ASPETE), 14121, N. Heraklio, Athens, Greece*

Abstract. In this paper sufficient oscillation conditions are established, for deviating difference equations with non-monotone arguments and nonnegative coefficients. An example, numerically solved in MATLAB, is also given to demonstrate the applicability and strength of the obtained conditions over known ones.

1. INTRODUCTION

In this paper we consider the difference equation with a variable deviating argument of either retarded (RDE)

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n \in \mathbb{N}_0 \tag{E}$$

or advanced type (ADE)

$$\nabla x(n) - q(n)x(\sigma(n)) = 0, \quad n \in \mathbb{N}. \tag{E$'$}$$

Equations (E) and (E′) are studied under the following assumptions: everywhere $(p(n))_{n\geq 0}$ and $(q(n))_{n\geq 1}$ are sequences of nonnegative real numbers, and $(\tau(n))_{n\geq 0}$, $(\sigma(n))_{n\geq 1}$ are sequences of integers satisfying

$$\tau(n) \leq n-1, \quad \forall n \in \mathbb{N}_0 \quad \text{and} \quad \lim_{n\to\infty} \tau(n) = \infty \tag{1.1}$$

and

$$\sigma(n) \geq n+1, \quad \forall n \in \mathbb{N}, \tag{1.1$'$}$$

respectively. Here, Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$ and ∇ corresponds to the backward difference operator $\nabla x(n) = x(n) - x(n-1)$.

Set

$$w = -\min_{n\geq 0} \tau(n).$$

Clearly, w is a finite positive integer if (1.1) holds.

2010 *Mathematics Subject Classification*. Primary 39A10; Secondary 39A21
Keywords. Difference equations, non-monotone argument, retarded argument, advanced argument, oscillation.
Received: 09 December 2018; Accepted: 26 March 2019
Communicated by Jelena Manojlović
Email address: geaxatz@otenet.gr (George E. Chatzarakis)

Definition 1.1. *By a solution of (E), we mean a sequence of real numbers $(x(n))_{n\geq -w}$ which satisfies (E) for all $n \geq 0$.*

It is clear that, for each choice of real numbers $c_{-w}, c_{-w+1}, ..., c_{-1}, c_0$, there exists a unique solution $(x(n))_{n\geq -w}$ of (E) which satisfies the initial conditions $x(-w) = c_{-w}, x(-w+1) = c_{-w+1}, ..., x(-1) = c_{-1}, x(0) = c_0$. When the initial data is given, we can obtain a unique solution to (E) by using the method of steps.

Definition 1.2. *By a solution of (E′), we mean a sequence of real numbers $(x(n))_{n\geq 0}$ which satisfies (E′) for all $n \geq 1$.*

Definition 1.3. *A solution $(x(n))_{n\geq -w}$ (or $(x(n))_{n\geq 0}$) of (E) (or (E′)) is called **oscillatory**, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be **nonoscillatory**. An equation is oscillatory if all its solutions oscillate.*

In recent years, a considerable effort has been made to investigate oscillatory properties of solutions of difference equations with deviating arguments. See, for example, [1−23] and references cited therein. However, most of these results require for the arguments to be monotone increasing. While this condition is naturally satisfied by a variety of differential equations with variable delays (advances), for difference equations, due to the discrete nature of the arguments, if the arguments are strictly increasing, then the deviations become eventually constant. This is one of main motivations to investigate difference equations with non-monotone arguments. Therefore, an interesting question arising is whether we can state oscillation criteria for (E) (or (E′)), considering the argument $\tau(n)$ (or $\sigma(n)$) to be not necessarily monotone. In the present paper, we achieve this goal by establishing criteria which, up to our knowledge, essentially improve all other known results in the literature.

The paper is organized as follows. First, we present, separately for a retarded and advanced case, some of the related results which motivate the contents of this paper. Next, we establish new suffcient conditions, involving lim sup, for the oscillation of all solutions of (E) and (E′). We base our technique on the proper use of an iterative procedure leading to new inequalities which may replace former ones. To verify the significance of the results, we provide an example along with various comparisons among new and known criteria.

Throughout this paper, we are going to use the following notation:

$$\sum_{i=k}^{k-1} A(i) = 0 \quad \text{and} \quad \prod_{i=k}^{k-1} A(i) = 1,$$

$$\alpha := \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) \quad \text{and} \quad \beta := \liminf_{n\to\infty} \sum_{j=n+1}^{\sigma(n)} q(j)$$

and

$$D(\omega) := \begin{cases} 0, & \text{if } \omega > 1/e \\ \dfrac{1-\omega-\sqrt{1-2\omega-\omega^2}}{2}, & \text{if } \omega \in [0, 1/e] \end{cases}.$$

1.1. RDEs (Chronological review)

In 2008 Chatzarakis, Koplatadze and Stavroulakis [7, 8] proved that, if $\tau(n)$ is nondecreasing and

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1, \tag{1.2}$$

or

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e}, \tag{1.3}$$

then all solutions of (E) are oscillatory.

It is obvious that there is a gap between the conditions (1.2) and (1.3) when the limit

$$\lim_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j)$$

does not exist. How to fill this gap is an interesting problem which has been investigated by several authors. For example, in 2009, Chatzarakis, Philos and Stavroulakis [9] proved that, if

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - D(\alpha), \tag{1.4}$$

then all solutions of (E) are oscillatory.

Now we come to the case that the argument $\tau(n)$ is not necessarily monotone. Set

$$h(n) = \max_{0\le s\le n} \tau(s). \tag{1.5}$$

Clearly, the sequence $h(n)$ is nondecreasing with $\tau(n) \le h(n) \le n-1$ for all $n \ge 0$.

In 2011, Braverman and Karpuz [3] proved that, if

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} > 1, \tag{1.6}$$

then all solutions of (E) are oscillatory, while, in 2014, Stavroulakis [19] improved (1.6) to

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} > 1 - D(\alpha). \tag{1.7}$$

In 2015, Braverman, Chatzarakis and Stavroulakis [2] proved that, if for some $r \in \mathbb{N}$

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) a_r^{-1}\left(h(n), \tau(j)\right) > 1, \tag{1.8}$$

or

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) a_r^{-1}\left(h(n), \tau(j)\right) > 1 - D(\alpha), \tag{1.9}$$

where

$$a_1(n,k) = \prod_{i=k}^{n-1} [1 - p(i)]$$
$$a_{r+1}(n,k) = \prod_{i=k}^{n-1} \left[1 - p(i) a_r^{-1}(i, \tau(i))\right],$$

then all solutions of (E) are oscillatory.

Remark 1.4. *Observe that conditions (1.6) and (1.7) are special cases of (1.8) and (1.9), respectively, when $r = 1$.*

Several improvements were made to the above conditions, see [4, 5, 6, 10, 11] to arrive at the recent forms [6]

$$\limsup_{n\to\infty}\sum_{i=h(n)}^{n}p(i)\exp\left(\sum_{j=\tau(i)}^{h(n)-1}p(j)\prod_{m=\tau(j)}^{j-1}\frac{1}{1-Z_\ell(m)}\right)>1, \tag{1.10}$$

$$\limsup_{n\to\infty}\sum_{i=h(n)}^{n}p(i)\exp\left(\sum_{j=\tau(i)}^{h(n)-1}p(j)\prod_{m=\tau(j)}^{j-1}\frac{1}{1-Z_\ell(m)}\right)>1-D(\alpha) \tag{1.11}$$

and

$$\limsup_{n\to\infty}\sum_{i=h(n)}^{n}p(i)\exp\left(\sum_{j=\tau(i)}^{n}p(j)\prod_{m=\tau(j)}^{j-1}\frac{1}{1-Z_\ell(m)}\right)>\frac{1}{D(\alpha)}-1, \tag{1.12}$$

where

$$Z_\ell(n)=p(n)\left[1+\sum_{i=\tau(n)}^{n-1}p(i)\exp\left(\sum_{j=\tau(i)}^{n-1}p(j)\prod_{m=\tau(j)}^{j-1}\frac{1}{1-Z_{\ell-1}(m)}\right)\right]$$

with

$$Z_0(n)=p(n)\left[1+\sum_{i=\tau(n)}^{n-1}p(i)\exp\left(\lambda_0\sum_{j=\tau(i)}^{n-1}p(j)\right)\right]$$

and λ_0 is the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$.

1.2. ADEs (Chronological review)

In 2012, Chatzarakis and Stavroulakis [12] proved that, if $\sigma(n)$ is nondecreasing and

$$\limsup_{n\to\infty}\sum_{j=n}^{\sigma(n)}q(j)>1, \tag{1.13}$$

or

$$\limsup_{n\to\infty}\sum_{j=n}^{\sigma(n)}q(j)>1-\left(1-\sqrt{1-\beta}\right)^2, \tag{1.14}$$

then all solutions of (E′) are oscillatory.

Now we come to the case that the argument $\sigma(n)$ is not necessarily monotone. Set

$$\rho(n)=\min_{s\geq n}\sigma(s). \tag{1.15}$$

Clearly, the sequence $\rho(n)$ is nondecreasing with $\sigma(n)\geq\rho(n)\geq n+1$ for all $n\geq 1$.

In 2015, Braverman, Chatzarakis and Stavroulakis [2] proved that, if for some $r\in\mathbb{N}$

$$\limsup_{n\to\infty}\sum_{j=n}^{\rho(n)}q(j)b_r^{-1}(\rho(n),\sigma(j))>1, \tag{1.16}$$

or

$$\limsup_{n\to\infty}\sum_{j=n}^{\rho(n)} q(j)b_r^{-1}(\rho(n),\sigma(j)) > 1-D(\beta), \tag{1.17}$$

where

$$\begin{aligned} b_1(n,k) &= \prod_{i=n+1}^{k}[1-q(i)] \\ b_{r+1}(n,k) &= \prod_{i=n+1}^{k}\left[1-q(i)b_r^{-1}(i,\sigma(i))\right], \end{aligned}$$

then all solutions of (E′) are oscillatory.

Several improvements were made to the above conditions, see [4, 5, 6, 10, 11] to arrive at the recent forms [6]

$$\limsup_{n\to\infty}\sum_{i=n}^{\rho(n)} q(i)\exp\left(\sum_{j=\rho(n)+1}^{\sigma(i)} q(j)\prod_{m=j+1}^{\sigma(j)}\frac{1}{1-W_\ell(m)}\right) > 1, \tag{1.18}$$

$$\limsup_{n\to\infty}\sum_{i=n}^{\rho(n)} q(i)\exp\left(\sum_{j=\rho(n)+1}^{\sigma(i)} q(j)\prod_{m=j+1}^{\sigma(j)}\frac{1}{1-W_\ell(m)}\right) > 1-D(\beta), \tag{1.19}$$

$$\limsup_{n\to\infty}\sum_{i=n}^{\rho(n)} q(i)\exp\left(\sum_{j=n}^{\sigma(i)} q(j)\prod_{m=j+1}^{\sigma(j)}\frac{1}{1-W_\ell(m)}\right) > \frac{1}{D(\beta)}-1, \tag{1.20}$$

where

$$W_\ell(n) = q(n)\left[1+\sum_{i=n+1}^{\sigma(n)} q(i)\exp\left(\sum_{j=n+1}^{\sigma(i)} q(j)\prod_{m=j+1}^{\sigma(j)}\frac{1}{1-W_{\ell-1}(m)}\right)\right]$$

with

$$W_0(n) = q(n)\left[1+\sum_{i=n+1}^{\sigma(n)} q(i)\exp\left(\lambda_0\sum_{j=n+1}^{\sigma(i)} q(j)\right)\right]$$

and λ_0 is the smaller root of the transcendental equation $\lambda = e^{\beta\lambda}$.

2. MAIN RESULTS

2.1. RDEs

The proofs of our main results are essentially based on the following lemmas.

Lemma 2.1. *[11] Assume that $h(n)$ is defined by (1.5). If $\alpha > 0$ then*

$$\liminf_{n\to\infty}\sum_{j=h(n)}^{n-1} p(j) = \liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1} p(j) = \alpha. \tag{2.1}$$

Lemma 2.2. *[4, 9] Assume that $h(n)$ is defined by (1.5) and $x(n)$ is an eventually positive solution of (E). If $0 < \alpha \le 1/e$ then*

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(h(n))} \ge D(\alpha) \tag{2.2}$$

and

$$\liminf_{n\to\infty} \frac{x(h(n))}{x(n)} \ge \lambda_0, \tag{2.3}$$

where λ_0 is the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$.

Theorem 2.3. *Assume that $h(n)$ is defined by (1.5). If for some $\xi \in \mathbb{N}$*

$$\limsup_{n\to\infty} \sum_{k=h(n)}^{n} p(k) \exp\left(\sum_{\ell=\tau(k)}^{h(n)-1} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j) \prod_{u=\tau(j)}^{j-1} \frac{1}{1-d_\xi(u)} \right)\right) > 1, \tag{2.4}$$

where

$$d_\xi(n) = p(n) \left[1 + \sum_{k=\tau(n)}^{n-1} p(k) \exp\left(\sum_{\ell=\tau(k)}^{n-1} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j) \prod_{u=\tau(j)}^{j-1} \frac{1}{1-d_{\xi-1}(u)} \right)\right)\right] \tag{2.5}$$

with

$$d_0(n) = p(n) \left[1 + \sum_{k=\tau(n)}^{n-1} p(k) \exp\left(\sum_{\ell=\tau(k)}^{n-1} p(\ell) \exp\left(\lambda_0 \sum_{j=\tau(\ell)}^{\ell-1} p(j) \right)\right)\right]$$

and λ_0 is the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$, then all solutions of (E) are oscillatory.

Proof. Assume, for the sake of contradiction, that $(x(n))_{n\ge -w}$ is a nonoscillatory solution of (E). Then it is either eventually positive or eventually negative. As $(-x(n))_{n\ge -w}$ is also a solution of (E), we may restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_1 \ge -w$ be an integer such that $x(n) > 0$ for all $n \ge n_1$. Then, there exists $n_2 \ge n_1$ such that $x(\tau(n)) > 0$, $\forall n \ge n_2$. In view of this, Eq.(E) becomes

$$\Delta x(n) = -p(n)x(\tau(n)) \le 0, \quad \forall n \ge n_2,$$

which means that the sequence $(x(n))$ is eventually nonincreasing.

Dividing (E) by $x(n)$ and summing up from $\tau(n)$ to $n-1$, we take

$$\sum_{j=\tau(n)}^{n-1} \frac{\Delta x(j)}{x(j)} = -\sum_{j=\tau(n)}^{n-1} p(j) \frac{x(\tau(j))}{x(j)}. \tag{2.6}$$

However, since $e^x \ge x+1$, $x \in \mathbb{R}$ we have

$$\sum_{j=\tau(n)}^{n-1} \frac{\Delta x(j)}{x(j)} = \sum_{j=\tau(n)}^{n-1} \left(\frac{x(j+1)}{x(j)} - 1 \right) = \sum_{j=\tau(n)}^{n-1} \left[\exp\left(\ln \frac{x(j+1)}{x(j)} \right) - 1 \right] \ge \sum_{j=\tau(n)}^{n-1} \ln \frac{x(j+1)}{x(j)} = \ln \frac{x(n)}{x(\tau(n))}. \tag{2.7}$$

Combining (E), (2.6) and (2.7), we obtain

$$\Delta x(n) + p(n)x(n) \exp\left(\sum_{j=\tau(n)}^{n-1} p(j) \frac{x(\tau(j))}{x(j)} \right) \le 0. \tag{2.8}$$

Dividing (2.8) by $x(n)$ and summing up from $\tau(k)$ to $n-1$, we take

$$\sum_{\ell=\tau(k)}^{n-1} \frac{\Delta x(j)}{x(j)} \leq -\sum_{\ell=\tau(k)}^{n-1} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\frac{x(\tau(j))}{x(j)}\right)$$

which, in view of (2.7), gives

$$\ln \frac{x(n)}{x(\tau(k))} \leq -\sum_{\ell=\tau(k)}^{n-1} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\frac{x(\tau(j))}{x(j)}\right)$$

or

$$x(\tau(k)) \geq x(n) \exp\left(\sum_{\ell=\tau(k)}^{n-1} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\frac{x(\tau(j))}{x(j)}\right)\right). \tag{2.9}$$

Summing up (E) from $\tau(n)$ to $n-1$, we have

$$x(n) - x(\tau(n)) + \sum_{k=\tau(n)}^{n-1} p(k)x\left(\tau(k)\right) = 0. \tag{2.10}$$

Combining (2.9) and (2.10), we obtain

$$x(n) - x(\tau(n)) + x(n)\sum_{k=\tau(n)}^{n-1} p(k) \exp\left(\sum_{\ell=\tau(k)}^{n-1} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\frac{x(\tau(j))}{x(j)}\right)\right) \leq 0.$$

Multiplying the last inequality by $p(n)$, we get

$$x(n)p(n) - p(n)x(\tau(n)) + p(n)x(n)\sum_{k=\tau(n)}^{n-1} p(k) \exp\left(\sum_{\ell=\tau(k)}^{n-1} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\frac{x(\tau(j))}{x(j)}\right)\right) \leq 0$$

which, in view of (E), becomes

$$\Delta x(n) + p(n)x(n) + p(n)x(n)\sum_{k=\tau(n)}^{n-1} p(k) \exp\left(\sum_{\ell=\tau(k)}^{n-1} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\frac{x(\tau(j))}{x(j)}\right)\right) \leq 0.$$

Since $\tau(j) \leq h(j)$, clearly

$$\Delta x(n) + p(n)x(n) + p(n)x(n)\sum_{k=\tau(n)}^{n-1} p(k) \exp\left(\sum_{\ell=\tau(k)}^{n-1} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\frac{x(h(j))}{x(j)}\right)\right) \leq 0.$$

Taking into account the fact that (2.3) of Lemma 2.2 is satisfied, the last inequality becomes

$$\Delta x(n) + p(n)x(n) + p(n)x(n)\sum_{k=\tau(n)}^{n-1} p(k) \exp\left(\sum_{\ell=\tau(k)}^{n-1} p(\ell) \exp\left((\lambda_0 - \epsilon)\sum_{j=\tau(\ell)}^{\ell-1} p(j)\right)\right) \leq 0.$$

Thus

$$\Delta x(n) + p(n)\left[1 + \sum_{k=\tau(n)}^{n-1} p(k) \exp\left(\sum_{\ell=\tau(k)}^{n-1} p(\ell) \exp\left((\lambda_0 - \epsilon)\sum_{j=\tau(\ell)}^{\ell-1} p(j)\right)\right)\right] x(n) \leq 0,$$

or

$$\Delta x(n) + d_0(n,\epsilon)x(n) \leq 0, \tag{2.11}$$

with

$$d_0(n,\epsilon) = p(n)\left[1 + \sum_{k=\tau(n)}^{n-1} p(k)\exp\left(\sum_{\ell=\tau(k)}^{n-1} p(\ell)\exp\left((\lambda_0-\epsilon)\sum_{j=\tau(\ell)}^{\ell-1} p(j)\right)\right)\right].$$

Applying the discrete Grönwall inequality in (2.11), we obtain

$$x(m) > x(n)\prod_{u=m}^{n-1}\frac{1}{1-d_0(u,\epsilon)}, \quad n \geq m.$$

Thus

$$x(\tau(j)) > x(j)\prod_{u=\tau(j)}^{j-1}\frac{1}{1-d_0(u,\epsilon)}, \quad \text{for all } n \geq n(\epsilon). \tag{2.12}$$

Combining the inequalities (2.9) and (2.12) we obtain

$$x(\tau(k)) \geq x(n)\exp\left(\sum_{\ell=\tau(k)}^{n-1} p(\ell)\exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\prod_{u=\tau(j)}^{j-1}\frac{1}{1-d_0(u,\epsilon)}\right)\right). \tag{2.13}$$

In view of this, (2.10) becomes

$$x(n) - x(\tau(n)) + x(n)\sum_{k=\tau(n)}^{n-1} p(k)\exp\left(\sum_{\ell=\tau(k)}^{n-1} p(\ell)\exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\prod_{u=\tau(j)}^{j-1}\frac{1}{1-d_0(u,\epsilon)}\right)\right) \leq 0.$$

Multiplying the last inequality by $p(n)$, we get

$$p(n)x(n) - p(n)x(\tau(n)) + p(n)x(n)\sum\nolimits_{k=\tau(n)}^{n-1} p(k)\exp\left(\sum_{\ell=\tau(k)}^{n-1} p(\ell)\exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\prod_{u=\tau(j)}^{j-1}\frac{1}{1-d_0(u,\epsilon)}\right)\right) \leq 0$$

which, in view of (E), becomes

$$\Delta x(n) + p(n)x(n) + p(n)x(n)\sum_{k=\tau(n)}^{n-1} p(k)\exp\left(\sum_{\ell=\tau(k)}^{n-1} p(\ell)\exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\prod_{u=\tau(j)}^{j-1}\frac{1}{1-d_0(u,\epsilon)}\right)\right) \leq 0$$

or

$$\Delta x(n) + p(n)\left[1 + \sum_{k=\tau(n)}^{n-1} p(k)\exp\left(\sum_{\ell=\tau(k)}^{n-1} p(\ell)\exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\prod_{u=\tau(j)}^{j-1}\frac{1}{1-d_0(u,\epsilon)}\right)\right)\right]x(n) \leq 0.$$

Hence, for sufficiently large n

$$\Delta x(n) + d_1(n,\epsilon)x(n) \leq 0$$

where

$$d_1(n,\epsilon) = p(n)\left[1 + \sum_{k=\tau(n)}^{n-1} p(k)\exp\left(\sum_{\ell=\tau(k)}^{n-1} p(\ell)\exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\prod_{u=\tau(j)}^{j-1}\frac{1}{1-d_0(u,\epsilon)}\right)\right)\right].$$

By induction, we can build inequalities on $\Delta x(n)$ progressively higher indices $d_\xi(n,\epsilon)$, $\xi \in \mathbb{N}$. In general, for sufficiently large n, the positive solution $x(n)$ satisfies the inequality

$$\Delta x(n) + d_\xi(n,\epsilon)x(n) \leq 0 \tag{2.14}$$

where

$$d_\xi(n,\epsilon) = p(n)\left[1 + \sum_{k=\tau(n)}^{n-1} p(k)\exp\left(\sum_{\ell=\tau(k)}^{n-1} p(\ell)\exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\prod_{u=\tau(j)}^{j-1}\frac{1}{1-d_{\xi-1}(u,\epsilon)}\right)\right)\right]$$

and

$$x(\tau(k)) \geq x(h(n))\exp\left(\sum_{\ell=\tau(k)}^{h(n)-1} p(\ell)\exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\prod_{u=\tau(j)}^{j-1}\frac{1}{1-d_\xi(u,\epsilon)}\right)\right). \tag{2.15}$$

Summing up (E) from $h(n)$ to n, we have

$$x(n+1) - x(h(n)) + \sum_{k=h(n)}^{n} p(k)x(\tau(k)) = 0. \tag{2.16}$$

Combining (2.15) and (2.16), we have, for all sufficiently large n,

$$x(n+1) - x(h(n)) + x(h(n))\sum_{k=h(n)}^{n} p(k)\exp\left(\sum_{\ell=\tau(k)}^{h(n)-1} p(\ell)\exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\prod_{u=\tau(j)}^{j-1}\frac{1}{1-d_\xi(u,\epsilon)}\right)\right) \leq 0. \tag{2.17}$$

The inequality is valid if we omit $x(n+1) > 0$ in the left-hand side.Thus, as $x(h(n)) > 0$, for all sufficiently large n it holds

$$\sum_{k=h(n)}^{n} p(k)\exp\left(\sum_{\ell=\tau(k)}^{h(n)-1} p(\ell)\exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\prod_{u=\tau(j)}^{j-1}\frac{1}{1-d_\xi(u,\epsilon)}\right)\right) < 1,$$

from which by letting $n \to \infty$, we have

$$\limsup_{n\to\infty}\sum_{k=h(n)}^{n} p(k)\exp\left(\sum_{\ell=\tau(k)}^{h(n)-1} p(\ell)\exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\prod_{u=\tau(j)}^{j-1}\frac{1}{1-d_\xi(u,\epsilon)}\right)\right) \leq 1.$$

Since ϵ may be taken arbitrarily small, this inequality contradicts (2.4).

The proof of the theorem is complete. □

Theorem 2.4. *Assume that $h(n)$ is defined by (1.5) and $0 < \alpha \leq 1/e$. If for some $\xi \in \mathbb{N}$*

$$\limsup_{n\to\infty}\sum_{k=h(n)}^{n} p(k)\exp\left(\sum_{\ell=\tau(k)}^{h(n)-1} p(\ell)\exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\prod_{u=\tau(j)}^{j-1}\frac{1}{1-d_\xi(u)}\right)\right) > 1 - D(\alpha), \tag{2.18}$$

where $d_\xi(n)$ is defined by (2.5), then all solutions of (E) are oscillatory.

Proof. Assume, for the sake of contradiction, that $(x(n))_{n\geq -w}$ is an eventually positive solution of (E). Then, as in the proof of Theorem 2.3, for sufficiently large n, (2.17) is satisfied, i.e.,

$$x(n+1) - x(h(n)) + x(h(n))\sum_{k=h(n)}^{n} p(k)\exp\left(\sum_{\ell=\tau(k)}^{h(n)-1} p(\ell)\exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j)\prod_{u=\tau(j)}^{j-1}\frac{1}{1-d_\xi(u,\epsilon)}\right)\right) \leq 0.$$

That is,

$$\sum_{k=h(n)}^{n} p(k) \exp\left(\sum_{\ell=\tau(k)}^{h(n)-1} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j) \prod_{u=\tau(j)}^{j-1} \frac{1}{1-d_\xi(u,\epsilon)}\right)\right) \le 1 - \frac{x(n+1)}{x(h(n))},$$

which gives

$$\limsup_{n\to\infty} \sum_{k=h(n)}^{n} p(k) \exp\left(\sum_{\ell=\tau(k)}^{h(n)-1} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j) \prod_{u=\tau(j)}^{j-1} \frac{1}{1-d_\xi(u,\epsilon)}\right)\right) \le 1 - \liminf_{n\to\infty} \frac{x(n+1)}{x(h(n))}.$$

By Lemma 2.2, inequality (2.2) holds. So the last inequality leads to

$$\limsup_{n\to\infty} \sum_{k=h(n)}^{n} p(k) \exp\left(\sum_{\ell=\tau(k)}^{h(n)-1} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j) \prod_{u=\tau(j)}^{j-1} \frac{1}{1-d_\xi(u,\epsilon)}\right)\right) \le 1 - D(\alpha).$$

Since ϵ may be taken arbitrarily small, this inequality contradicts (2.18).

The proof of the theorem is complete. □

Theorem 2.5. *Assume that $h(n)$ is defined by (1.5) and $0 < \alpha \le 1/e$. If for some $\xi \in \mathbb{N}$*

$$\limsup_{n\to\infty} \sum_{k=h(n)}^{n} p(k) \exp\left(\sum_{\ell=\tau(k)}^{n} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j) \prod_{u=\tau(j)}^{j-1} \frac{1}{1-d_\xi(u)}\right)\right) > \frac{1}{D(\alpha)} - 1, \tag{2.19}$$

where $d_\xi(n)$ is defined by (2.5), then all solutions of (E) are oscillatory.

Proof. Assume, for the sake of contradiction, that $(x(n))_{n\ge -w}$ is an eventually solution of (E). Then, as in the proof of Theorem 2.3, for sufficiently large n, (2.15) is satisfied. Therefore

$$x(\tau(k)) \ge x(n+1) \exp\left(\sum_{\ell=\tau(k)}^{n} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j) \prod_{u=\tau(j)}^{j-1} \frac{1}{1-d_\xi(u,\epsilon)}\right)\right). \tag{2.20}$$

Combining (2.16) and (2.20), we have

$$x(n+1) - x(h(n)) + \sum_{k=h(n)}^{n} p(k)x(n+1) \exp\left(\sum_{\ell=\tau(k)}^{n} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j) \prod_{u=\tau(j)}^{j-1} \frac{1}{1-d_\xi(u,\epsilon)}\right)\right) \le 0.$$

Thus, for all sufficiently large n it holds

$$\sum_{k=h(n)}^{n} p(k) \exp\left(\sum_{\ell=\tau(k)}^{n} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j) \prod_{u=\tau(j)}^{j-1} \frac{1}{1-d_\xi(u,\epsilon)}\right)\right) \le \frac{x(h(n))}{x(n+1)} - 1.$$

Letting $n \to \infty$, we take

$$\limsup_{n\to\infty} \sum_{k=h(n)}^{n} p(k) \exp\left(\sum_{\ell=\tau(k)}^{n} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j) \prod_{u=\tau(j)}^{j-1} \frac{1}{1-d_\xi(u,\epsilon)}\right)\right) \le \limsup_{n\to\infty} \frac{x(h(n))}{x(n+1)} - 1,$$

which, in view of (2.2), gives

$$\limsup_{n\to\infty} \sum_{k=h(n)}^{n} p(k) \exp\left(\sum_{\ell=\tau(k)}^{n} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j) \prod_{u=\tau(j)}^{j-1} \frac{1}{1-d_\xi(u,\epsilon)}\right)\right) \le \frac{1}{D(\alpha)} - 1.$$

Since ϵ may be taken arbitrarily small, this inequality contradicts (2.19).

The proof of the theorem is complete. □

Remark 2.6. *If $d_\xi(n,\epsilon) \geq 1$ then (2.14) guarantees that all solutions of (E) are oscillatory. In fact, (2.14) gives*

$$\Delta x(n) + x(n) \leq 0$$

which means that $x(n+1) \leq 0$. This contradics $x(n) > 0$ for all $n \geq n_1$. Thus, in Theorems 2.3, 2.4 and 2.5 we consider only the case $d_\xi(n) < 1$. Another conclusion, that can be draw from the above, is that if at some point through the iterative process, we get a value of ξ, for which $d_\xi(n) \geq 1$, then the process terminates, since in any case, all solutions of (E) will be oscillatory. The value of ξ, that is the number of iterations, obviously, depends on the coefficient $p(n)$ and the form of the non-monotone argument $\tau(n)$.

2.2. ADEs

Similar lemmas for the (dual) advanced difference equation (E′), easily, can be derived. The proof of these lemmas are omitted, since they are quite similar to those of the corresponding lemmas, for the retarded equation.

Lemma 2.7. *Assume that $\rho(n)$ is defined by (1.15). If $\beta > 0$ then*

$$\liminf_{n\to\infty} \sum_{j=n+1}^{\rho(n)} q(j) = \liminf_{n\to\infty} \sum_{j=n+1}^{\sigma(n)} q(j) = \beta.$$

Lemma 2.8. *Assume that $\rho(n)$ is defined by (1.15), $0 < \beta \leq 1/e$ and $x(n)$ is an eventually positive solution of (E′). Then*

$$\liminf_{n\to\infty} \frac{x(n-1)}{x(\rho(n))} \geq D(\beta)$$

and

$$\liminf_{n\to\infty} \frac{x(\rho(n))}{x(n)} \geq \lambda_0$$

where λ_0 is the smaller root of the transcendental equation $\lambda = e^{\beta\lambda}$.

Based on Lemmas 2.7 and 2.8, we derive new sufficient oscillation conditions, involving lim sup, which essentially improve all previously known results in the literature.

Theorem 2.9. *Assume that $\rho(n)$ is defined by (1.15). If for some $\xi \in \mathbb{N}$*

$$\limsup_{n\to\infty} \sum_{k=n}^{\rho(n)} q(k) \exp\left(\sum_{\ell=\rho(n)+1}^{\sigma(k)} q(\ell) \exp\left(\sum_{j=\ell+1}^{\sigma(\ell)} q(j) \prod_{u=j+1}^{\sigma(j)} \frac{1}{1-g_\xi(u)} \right)\right) > 1, \tag{2.21}$$

where

$$g_\xi(n) = q(n)\left[1 + \sum_{k=n+1}^{\sigma(n)} q(k) \exp\left(\sum_{\ell=n+1}^{\sigma(k)} q(\ell) \exp\left(\sum_{j=\ell+1}^{\sigma(\ell)} q(j) \prod_{u=j+1}^{\sigma(j)} \frac{1}{1-g_{\xi-1}(u)} \right)\right)\right] \tag{2.22}$$

with

$$g_0(n) = q(n)\left[1 + \sum_{k=n+1}^{\sigma(n)} q(k) \exp\left(\sum_{\ell=n+1}^{\sigma(k)} q(\ell) \exp\left(\lambda_0 \sum_{j=\ell+1}^{\sigma(\ell)} q(j) \right)\right)\right]$$

and λ_0 is the smaller root of the transcendental equation $\lambda = e^{\beta\lambda}$, then all solutions of (E′) are oscillatory.

Theorem 2.10. *Assume that $\rho(n)$ is defined by (1.15) and $0 < \beta \leq 1/e$. If for some $\xi \in \mathbb{N}$*

$$\limsup_{n\to\infty} \sum_{k=n}^{\rho(n)} q(k) \exp\left(\sum_{\ell=\rho(n)+1}^{\sigma(k)} q(\ell) \exp\left(\sum_{j=\ell+1}^{\sigma(\ell)} q(j) \prod_{u=j+1}^{\sigma(j)} \frac{1}{1-g_\xi(u)} \right)\right) > 1 - D(\beta), \tag{2.23}$$

where $g_\xi(n)$ is defined by (2.22), then all solutions of (E′) are oscillatory.

Theorem 2.11. *Assume that $\rho(n)$ is defined by (1.15) and $0 < \beta \leq 1/e$. If for some $\xi \in \mathbb{N}$*

$$\limsup_{n\to\infty} \sum_{k=n}^{\rho(n)} q(k) \exp\left(\sum_{\ell=n}^{\sigma(k)} q(\ell) \exp\left(\sum_{j=\ell+1}^{\sigma(\ell)} q(j) \prod_{u=j+1}^{\sigma(j)} \frac{1}{1-g_\xi(u)} \right)\right) > \frac{1}{D(\beta)} - 1, \tag{2.24}$$

where $g_\xi(n)$ is defined by (2.22), then all solutions of (E′) are oscillatory.

Remark 2.12. *Similar comments as those in Remark 2.6, can be made for Theorems 2.9, 2.10 and 2.11, concerning equation (E′).*

3. AN EXAMPLE AND COMMENTS

In this section, an example illustrates cases when the results of the present paper imply oscillation while previously known results fail. The example not only illustrates the significance of main results, but also serves to indicate the high degree of improvement, compared to the previous oscillation criteria in the literature. All the calculations were made in Matlab.

Example 3.1. *Consider the retarded difference equation*

$$\Delta x(n) + \frac{12}{125} x(\tau(n)) = 0, \quad n \in \mathbb{N}_0, \tag{3.1}$$

with (see Fig. 1, (a))

$$\tau(n) = \begin{cases} n-1, & \text{if } n = 5\mu \\ n-6, & \text{if } n = 5\mu+1 \\ n-2, & \text{if } n = 5\mu+2 \\ n-6, & \text{if } n = 5\mu+3 \\ n-3, & \text{if } n = 5\mu+4 \end{cases}$$

where $\mu \in \mathbb{N}_0$ and $\mathbb{N}_0$ is the set of nonnegative integers.

By (1.5), we see (Fig. 1, (b)) that

$$h(n) = \begin{cases} n-1, & \text{if } n = 5\mu \\ n-2, & \text{if } n = 5\mu+1 \\ n-2, & \text{if } n = 5\mu+2 \\ n-3, & \text{if } n = 5\mu+3 \\ n-3, & \text{if } n = 5\mu+4 \end{cases}.$$

It is easy to see that

$$\alpha = \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) = \liminf_{\mu\to\infty} \sum_{j=5\mu-1}^{5\mu-1} \frac{12}{125} = 0.096$$

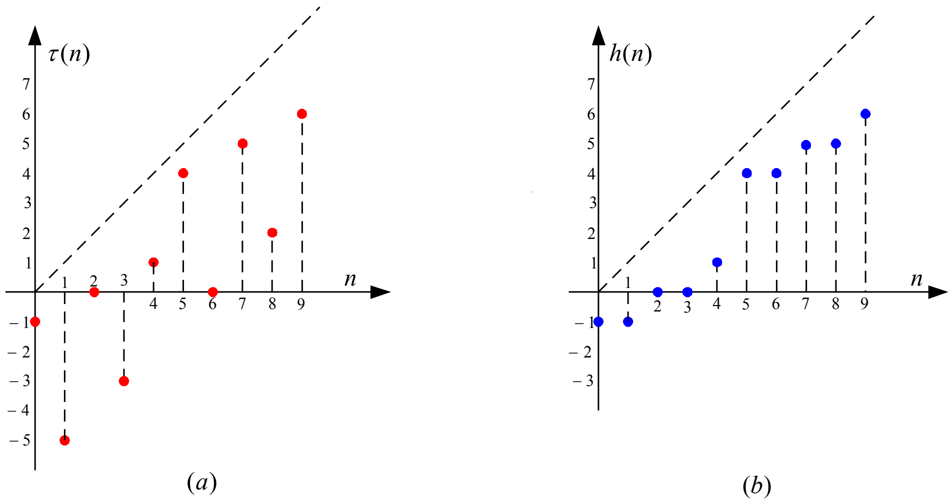

Figure 1: The graphs of $\tau(n)$ and $h(n)$

and therefore, the smaller root of $e^{0.096\lambda} = \lambda$ *is* $\lambda_0 = 1.11274$.

Observe that the function $F_\xi : \mathbb{N}_0 \to \mathbb{R}_+$ *defined as*

$$F_\xi(n) = \sum_{k=h(n)}^{n} p(k) \exp\left(\sum_{\ell=\tau(k)}^{h(n)-1} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j) \prod_{u=\tau(j)}^{j-1} \frac{1}{1-d_\xi(u)} \right) \right)$$

attains its maximum at $n = 5\mu + 4$, $\mu \in \mathbb{N}_0$, *for every* $\xi \in \mathbb{N}$. *Specifically,*

$$F_1(n) = \sum_{k=h(n)}^{n} p(k) \exp\left(\sum_{\ell=\tau(k)}^{h(n)-1} p(\ell) \exp\left(\sum_{j=\tau(\ell)}^{\ell-1} p(j) \prod_{u=\tau(j)}^{j-1} \frac{1}{1-d_1(u)} \right) \right)$$

where

$$d_1(u) = p(u) \left[1 + \sum_{i=\tau(u)}^{u-1} p(i) \exp\left(\sum_{v=\tau(i)}^{u-1} p(v) \exp\left(\sum_{\omega=\tau(v)}^{v-1} p(\omega) \prod_{\varphi=\tau(\omega)}^{\omega-1} \frac{1}{1-d_0(\varphi)} \right) \right) \right]$$

with

$$d_0(\varphi) = p(\varphi) \left[1 + \sum_{\theta=\tau(\varphi)}^{\varphi-1} p(\theta) \exp\left(\sum_{\psi=\tau(\theta)}^{\varphi-1} p(\psi) \exp\left(\lambda_0 \sum_{\zeta=\tau(\psi)}^{\psi-1} p(\zeta) \right) \right) \right].$$

By using an algorithm on MATLAB software, we obtain

$$F_1(5\mu + 4) \simeq 1.0876$$

and therefore

$$\limsup_{n\to\infty} F_1(n) \simeq 1.0876 > 1.$$

That is, condition (2.4) of Theorem 2.3 is satisfied for $\xi = 1$. *Therefore, all solutions of equation (3.1) are oscillatory.*

Observe, however, that

Condition	***Value***	***Conclusion***
(1.2)	$= 0.384 < 1$	*is not satisfied*
(1.3)	$= 0.096 < 1/e$	"
(1.4)	$= 0.384 < 1 - D(\beta) \simeq 0.9948$	"
(1.6)	$\simeq 0.5218 < 1$	"
(1.7)	$\simeq 0.5218 < 1 - D(\beta) \simeq 0.9948$	"
(1.8) *for* $r = 1$	$\simeq 0.5218 < 1$	"
(1.9) *for* $r = 1$	$\simeq 0.5218 < 1 - D(\beta) \simeq 0.9948$	"
(1.10) *for* $\ell = 1$	$\simeq 0.8499 < 1$	"
(1.11) *for* $\ell = 1$	$\simeq 0.8499 < 1 - D(\beta) \simeq 0.9948$	"
(1.12) *for* $\ell = 1$	$\simeq 2.8049 < 1/D(\beta) - 1 \simeq 194.06805$	"

That is, none of conditions (1.2), (1.3), (1.4), (1.6) $\equiv$ (1.8) *(for* $r = 1$*)*, (1.7) $\equiv$ (1.9) *(for* $r = 1$*)*, (1.10) *(for* $\ell = 1$*)*, (1.11) *(for* $\ell = 1$*) and* (1.12) *(for* $\ell = 1$*) is satisfied.*

Comment. *It is worth noting that the improvement of condition* (2.4) *to the corresponding condition* (1.2) *is significant, approximately 183.23%, if we compare the values on the left-side of these conditions. Also, the improvement compared to conditions* (1.6) $\equiv$ (1.8) *(for* $r = 1$*) and* (1.10) *(for* $\ell = 1$*) is very satisfactory, around 108.43% and 27.97%, respectively.*

Finally, observe that the conditions (1.8) − (1.12) *do not lead to oscillation for the first iteration. On the contrary, condition* (2.4) *is satisfied from the first iteration. This means that our condition is better and much faster than* (1.8) − (1.12).

Remark 3.2. *Similarly, one can construct examples, illustrating the other main results, in the paper.*

Acknowledgment

The author would like to thank the anonymous referee for the constructive remarks which greatly improved the paper.

References

[1] P. G. Asteris and G. E. Chatzarakis, Oscillation tests for difference equations with non-monotone arguments, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, **24** (2017), no. 4, 287–302.

[2] E. Braverman, G. E. Chatzarakis and I. P. Stavroulakis, Iterative oscillation tests for difference equations with several non-monotone arguments, *J. Difference Equ. Appl.*, **21**(2015), no. 9, 854–874.

[3] E. Braverman and B. Karpuz, On oscillation of differential and difference equations with non-monotone delays, *Appl. Math. Comput.*, **218** (2011), 3880–3887.

[4] G. E. Chatzarakis and I. Jadlovská, Oscillations in deviating difference equations using an iterative technique, *J. Inequal. Appl.*, 2017, Paper No. **173**, 24 pp.

[5] G. E. Chatzarakis and I. Jadlovská, Improved iterative oscillation tests for first-order deviating difference equations, *Int. J. Difference Equ.*, **12** (2017), no. 2, 185–210.

[6] G. E. Chatzarakis and I. Jadlovská, Oscillations of deviating difference equations using an iterative method, *Medit. J. Math.*, **16** (2019), no. 1, 16:16.

[7] G. E. Chatzarakis, R. Koplatadze and I. P. Stavroulakis, Oscillation criteria of first order linear difference equations with delay argument, *Nonlinear Anal.*, **68** (2008), 994–1005.

[8] G. E. Chatzarakis, R. Koplatadze, and I. P. Stavroulakis, Optimal oscillation criteria for first order difference equations with delay argument, *Pacific J. Math.*, **235** (2008), 15–33.

[9] G.E. Chatzarakis, Ch. G. Philos and I.P. Stavroulakis, An oscillation criterion for linear difference equations with general delay argument, *Portugal. Math.*, **66** (2009), 513–533.

[10] G.E. Chatzarakis, I. K. Purnaras and I.P. Stavroulakis, Oscillation of retarded difference equations with a non-monotone argument, *J. Difference Equ. Appl.*, **23** (2017), no. 8, 1354–1377.

[11] G. E. Chatzarakis and L. Shaikhet, Oscillation criteria for difference equations with non-monotone arguments, *Adv. Difference Equ.*, 2017, Paper No. 62, 16pp.

[12] G.E. Chatzarakis and I. P. Stavroulakis, Oscillations of difference equations with general advanced argument, *Cent. Eur. J. Math.,* **10** (2012), 807–823.

[13] M.-P. Chen and J. S. Yu, Oscillations of delay difference equations with variable coefficients, *In Proceedings of the First International Conference on Difference Equations,* Gordon and Breach, London 1994, 105–114.

[14] L. H. Erbe and B. G. Zhang, Oscillation of discrete analogues of delay equations, *Differential Integral Equations* **2** (1989), 300–309.

[15] I. Györi and G. Ladas, Linearized oscillations for equations with piecewise constant arguments, *Differential Integral Equations,* **2** (1989), 123–131.

[16] G. Ladas, Ch. G. Philos, and Y. G. Sficas, Sharp conditions for the oscillation of delay difference equations, *J. Appl. Math. Simulation* **2** (1989), 101–111.

[17] G. Ladas, Explicit conditions for the oscillation of difference equations, *J. Math. Anal. Appl.,* **153** (1990), 276–287.

[18] X. Li and D. Zhu, Oscillation of advanced difference equations with variable coefficients, *Ann. Differential Equations,* **18** (2002), 254–263.

[19] I. P. Stavroulakis, Oscillation criteria for delay and difference equations with non-monotone arguments, *Appl. Math. Comput.,* **226** (2014), 661–672.

[20] X. H. Tang and J. S. Yu, Oscillation of delay difference equations, *Comput. Math. Appl.,* **37** (1999), 11–20.

[21] X. H. Tang and R. Y. Zhang, New oscillation criteria for delay difference equations, *Comput. Math. Appl.,* **42** (2001), 1319–1330.

[22] W. Yan, Q. Meng and J. Yan, Oscillation criteria for difference equation of variable delays, *DCDIS Proceedings* **13A** (2006), 641–647.

[23] B. G. Zhang and C. J. Tian, Nonexistence and existence of positive solutions for difference equations with unbounded delay, *Comput. Math. Appl.,* **36** (1998), 1–8.