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On the Dual of Hilbert Coefficients and Width of the Associated Graded Modules over Artinian Modules

Fatemeh Cheraghi[a], Amir Mafi[a]

[a]*Department of Mathematics, University of Kurdistan, P.O. Box: 416, Sanandaj, Iran*

Abstract. Let $(A, \mathfrak{m})$ be a commutative quasi-local ring with non-zero identity and let M be an Artinian co-Cohen-Macaulay R-module with $\operatorname{Ndim} M = d$. Let $I \subseteq \mathfrak{m}$ be an ideal of R with $\ell(0 :_M I) < \infty$. In this paper, for $0 \leq i \leq d$, we study the dual of Hilbert coefficients $\acute{e}_i(I, M)$ of I relative to M. Also, we prove the dual of Huckaba-Marley's inequality. Moreover, we obtain some consequences of this result.

1. introduction

Throughout this paper, we assume that $(A, \mathfrak{m})$ is a commutative quasi-local ring with non-zero identity and $A/\mathfrak{m}$ is infinite and let M be a non-zero Artinian A-module. Roberts in [12] defined the dual dimension $\operatorname{Ndim} M$ and proved that $\operatorname{Ndim} M$ is equal to the least integer d for which there exists elements $a_1, ..., a_d \in \mathfrak{m}$ such that $\ell(0 :_M (a_1, ..., a_d)) < \infty$ (see also [8]). The sequence $a_1, ..., a_d \in \mathfrak{m}$ is called a system of parameters for M. Matlis in [9] defined that a sequence $x_1, ..., x_n \in \mathfrak{m}$ is an M-cosequence if $0 :_M (x_1, ..., x_{i-1}) \xrightarrow{x_i} 0 :_M (x_1, ..., x_{i-1})$ is surjective for $i = 1, ..., n$. In this case, it should be noted that $0 :_M (x_1, ..., x_n) \neq 0$. The codepth of M, denoted by $\operatorname{width} M$, is defined as the length of a maximal M-cosequence in $\mathfrak{m}$. Then it is always true that $\operatorname{width} M \leq \operatorname{Ndim} M$ (see [11]). When the equality holds, it said that M is co-Cohen-Macaulay. Tang and Zakeri in [16] proved that M is co-Cohen-Macaulay if and only if every system of parameters for M is an M-cosequence (see also [17] and [3]).

For an ideal I of A, Kirby in [7] introduced the following two graded modules dual to the Rees ring and associated graded ring $R(I, M) = \bigoplus_{n=-\infty}^{\infty} R(I, M)_n$, where $R(I, M)_n = M/(0 :_M I^{-n})$ if $n \leq 0$ and $R(I, M)_n = 0$ if $n > 0$, and $G(I, M) = \bigoplus_{n=-\infty}^{\infty} G(I, M)_n$, where $G(I, M)_n = (0 :_M I^{-n+1})/(0 :_M I^{-n})$ if $n \leq 0$ and $G(I, M)_n = 0$ if $n > 0$. He used the two graded modules in the proofs of theorems about the Artin-Rees property and Hilbert polynomials for Artinian modules. For an ideal I of A such that $\ell(0 :_M I) < \infty$, Kirby in [7] proved that, for n sufficiently large, the length $\ell(0 :_M I^n)$ is a polynomial function in n of degree $d = \operatorname{Ndim} M$. From now on, throughout the article, we will denote $d = \operatorname{Ndim} M > 0$. Jorge Perez and Freitas in [6] denoted the dual Hilbert-Samuel function of I by $H_n(I, M) := \ell(0 :_M I^n)$, and the dual Hilbert-Samuel polynomial of I by $P_n(I, M) := \ell(0 :_M I^n)$ for large n. They wrote

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Corresponding author: Amir Mafi
Email addresses: f_cheraghi89@yahoo.com (Fatemeh Cheraghi), a_mafi@ipm.ir (Amir Mafi)

$$\begin{aligned} P_n(I,M) &= \binom{n+d-1}{d}\acute{e}_0(I,M) - \binom{n+d-2}{d-1}\acute{e}_1(I,M) + \ldots + (-1)^d \acute{e}_d(I,M) \\ &= \sum_{i=0}^{d}(-1)^i\binom{n+d-i-1}{d-i}\acute{e}_i(I,M), \end{aligned}$$

where $\acute{e}_i(I,M)$ for $i = 0,1,...,d$ are integers, called the dual Hilbert-Samuel coefficient of I relative to M. The leading coefficient $\acute{e}_0(I,M)$, called the dual Hilbert-Samuel multiplicity of I relative to M.

Sharp and Taherizadeh in [13] defined that an ideal J is a reduction of I relative to M if $J \subseteq I$ and there exists non-negative integer n such that $(0 :_M JI^n) = (0 :_M I^{n+1})$. If J is a reduction of I relative to M and there is no reduction of I relative to M which is strictly contained in J, then it said that J is a minimal reduction of I relative to M. When $A/\mathfrak{m}$ is infinite and $I \subseteq \mathfrak{m}$ is an ideal of A with $\ell(0 :_M I) < \infty$, every reduction of I relative to M contains a minimal reduction of I relative to M and every minimal reduction of I relative to M is generated by a system of parameters for M (see [13, Theorem 6.2]). The reduction number $r_J(I,M)$ of I with respect to J is the smallest $n \in \mathbb{N}$ such that $(0 :_M JI^n) = (0 :_M I^{n+1})$ for some minimal reduction J of I relative to M. We define the dual of Sally module S of I with respect to J as the

$$S = S_J(I,M) = \bigoplus_{n\geq 1}\frac{0 :_M IJ^n}{0 :_M I^{n+1}} = \bigoplus_{n\geq 1} S_n.$$

An element $x \in I\backslash I^2$ is said to be co-superficial of degree one for I with respect to M if and only if there is an integer n_0 such that

$$x(0 :_M I^{n+1}) + (0 :_M I^{n_0}) = (0 :_M I^n) \ for\ all\ n \geq n_0.$$

A sequence $x_1,...,x_s \in I$ is said to be a co-superficial sequence for I with respect to M if for all $i = 1,...,s$ the image of $x_i \in I$ is a co-superficial element with respect to $(0 :_M (x_1,...,x_{i-1}))$. Recall that co-superficial element was introduced in [14].

The objective of this paper is to state known results that link width of $G(I,M)$ with linear relations among the dual Hilbert coefficients $\acute{e}_i(I,M)$ for $0 \leq i \leq d$, specially for $\acute{e}_0(I,M)$ and $\acute{e}_1(I,M)$. We have chosen two fundamental theorems of Huckaba [4] and Huckaba-Marley [5] to illustrate the results in this area. In section two, we introduce the dual of Sally module and state dual of some results about length of its components. In section three, we prove the main facts for Hilbert polynomial of co-Cohen-Macaulay modules with Ndim equal one.
In section four, we provide a simple proof of dual of two theorems of Huckaba [4] and Huckaba-Marley [5]. In section five by using the $\triangle$ operator we prove the main Theorem 5.3 and state a few its consequences.

2. The dual of Sally module

We start this section by the following theorem.

Theorem 2.1. *Let M be a co-Cohen-Macaulay A-module with* $\mathrm{Ndim}\, M = d$*, I be an ideal of A such that $\ell(0 :_M I) < \infty$ and J be a minimal reduction of I relative to M. Let $S = S_J(I,M) = \bigoplus_{n\geq 1}\frac{0:_M IJ^n}{0:_M I^{n+1}} = \bigoplus_{n\geq 1} S_n$ be the dual of Sally module. Then*

$$\ell(S_n) - \ell(S_{n-1}) = \binom{n+d-1}{d-1}\ell(0 :_M J) - \binom{n+d-2}{d-2}\ell(\frac{0 :_M J}{0 :_M I}) - \ell(\frac{0 :_M I^{n+1}}{0 :_M I^n}).$$

Proof. *Since J is generated by a system of parameters for M and every system of parameters for M is M-cosequence, by [1, Proposition 2.2] we have*

$$\ell(0 :_M J^n) = \binom{n+d-1}{d}\ell(0 :_M J), \quad for\ all\ n.$$

Consider the two exact sequences

$$0 \to \frac{0 :_M IJ^{n-1}}{0 :_M I^n} \to \frac{0 :_M J^n}{0 :_M I^n} \to \frac{0 :_M J^n}{0 :_M IJ^{n-1}} \to 0,$$

and

$$0 \to \frac{0 :_M IJ^{n-1}}{0 :_M J^{n-1}} \to \frac{0 :_M J^n}{0 :_M J^{n-1}} \to \frac{0 :_M J^n}{0 :_M IJ^{n-1}} \to 0.$$

Now by using the above two exact sequences we have

$$\begin{aligned}
\ell(0 :_M I^n) &= \ell(0 :_M J^n) - \ell(\frac{0 :_M J^n}{0 :_M I^n}) \\
&= \ell(0 :_M J^n) - \ell(\frac{0 :_M J^n}{0 :_M IJ^{n-1}}) - \ell(\frac{0 :_M IJ^{n-1}}{0 :_M I^n}) \\
&= \ell(0 :_M J^n) - \ell(\frac{0 :_M J^n}{0 :_M J^{n-1}}) + \ell(\frac{0 :_M IJ^{n-1}}{0 :_M J^{n-1}}) - \ell(\frac{0 :_M IJ^{n-1}}{0 :_M I^n}) \\
&= \ell(0 :_M J^{n-1}) + \ell(\frac{0 :_M IJ^{n-1}}{0 :_M J^{n-1}}) - \ell(\frac{0 :_M IJ^{n-1}}{0 :_M I^n}).
\end{aligned}$$

From [1, Lemma 2.1] we have $\ell(\frac{0:_M IJ^{n-1}}{0:_M J^{n-1}}) = \binom{n+d-2}{d-1}\ell(0 :_M I)$ *and* $\ell(\frac{0:_M IJ^{n-1}}{0:_M I^n}) = \ell(S_{n-1})$. *Thus*

$$\ell(0 :_M I^n) = \binom{n+d-2}{d}\ell(0 :_M J) + \binom{n+d-2}{d-1}\ell(0 :_M I) - \ell(S_{n-1})$$

and so

$$\ell(S_n) = -\ell(0 :_M I^{n+1}) + \binom{n+d-1}{d}\ell(0 :_M J) + \binom{n+d-1}{d-1}\ell(0 :_M I).$$

Therefore, we have

$$\begin{aligned}
\ell(S_n) - \ell(S_{n-1}) &= \\
&= \binom{n+d-1}{d-1}\ell(0 :_M J) - \binom{n+d-2}{d-2}\ell(\frac{0 :_M J}{0 :_M I}) - \ell(\frac{0 :_M I^{n+1}}{0 :_M I^n})
\end{aligned}$$

□

From Theorem 2.1 we can conclude that the growth of $\ell(S_n)$ affects the bounding of $\ell(\frac{0:_M I^{n+1}}{0:_M I^n})$.

Lemma 2.2. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$ *and let I be an ideal of A such that $\ell(0 :_M I) < \infty$. Then $S_1 = 0$ if and only if $r(I, M) = 1$.*

Proof. *Since $S_1 = \frac{0:_M JI}{0:_M I^2} = 0$, we have $0 :_M JI = 0 :_M I^2$ and so $r(I, M) = 1$. Conversely, let $r(I, M) = 1$. Then $0 :_M JI = 0 :_M I^2$ and so $S_1 = 0$.* □

Corollary 2.3. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$ *and let I be an ideal of A such that $\ell(0 :_M I) < \infty$. Then $S_1 = 0$ if and only if*

$$\ell(0 :_M I^{n+1}) = \binom{n+d}{d}\ell(0 :_M J) - \binom{n+d-1}{d-1}\ell(\frac{0 :_M J}{0 :_M I}).$$

In this case $G(I, M)$ is co-Cohen-Macaulay.

Proof. By using Lemma 2.2 and [1, Theorem 2.7] the result follows. □

Proposition 2.4. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$ *and let I be an ideal of A such that $\ell(0 :_M I) < \infty$. Let J is a minimal reduction of I and $S = S_J(I, M)$ be the corresponding Sally module of M with respect to I. Then for $n \gg 0$,*

$$H_n(I, M) = \binom{n+d-1}{d}\acute{e}_0 + \binom{n+d-2}{d-1}(\ell(0 :_M I) - \acute{e}_0) - \ell(S_{n-1}).$$

Proof. It is a straightforward conclusion by using the equation

$$H_n(I, M) = \binom{n+d-2}{d}\ell(0 :_M J) + \binom{n+d-2}{d-1}\ell(0 :_M I) - \ell(S_{n-1}).$$

□

The following results immediately obtain by Proposition 2.4.

Corollary 2.5. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$ *and let I be an ideal of A such that $\ell(0 :_M I) < \infty$. If $S_J(I, M) \neq 0$, then the function $\ell(S_n)$ has the growth of a polynomial of degree $d-1$.*

Corollary 2.6. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$ *and let I be an ideal of A such that $\ell(0 :_M I) < \infty$. Let $\acute{s}_0, \acute{s}_1, ..., \acute{s}_{d-1}$ be the coefficients of the Hilbert polynomial of $S_J(I, M)$. Then*

$$\begin{aligned} \acute{e}_1 &= \acute{e}_0 - \ell(0 :_M I) + \acute{s}_0, \\ \acute{e}_{i+1} &= \acute{s}_i \quad for\ \ all\ \ i \geq 1. \end{aligned}$$

Corollary 2.7. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$ *and let I be an ideal of A such that $\ell(0 :_M I) < \infty$. Then the following hold:*

(a) $\ell(0 :_M I) \geq \acute{e}_0 - \acute{e}_1$.
(b) In the case of equality we have $S_J(I, M) = 0$.

Proof. (a) This is clear by using the equality $\ell(0 :_M I) = \acute{e}_0 - \acute{e}_1 + \acute{s}_0$. For (b), we have $\acute{s}_0 = 0$ and so $\ell(S_n)$ is the polynomial of degree $d-2$. Therefore, by Corollary 2.5, we have $S_J(I, M) = 0$. □

Proposition 2.8. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$, *$I$ be an ideal of A such that $\ell(0 :_M I) < \infty$ and J be a minimal reduction of I. Then the length of $(\frac{0:_M IJ^n}{0:_M I^{n+1}})$ and $\ell(\frac{0:_M J}{0:_M I})$ is independent of J.*

Proof. The result immediately follows by using the two equations

$$\ell(\frac{0 :_M J}{0 :_M I}) = \ell(0 :_M J) - \ell(0 :_M I) = \acute{e}_0(I, M) - \ell(0 :_M I),$$

and

$$\ell(S_n) = \binom{n+d-1}{d}\acute{e}_0(I, M) + \binom{n+d-1}{d-1}\ell(0 :_M I) - \ell(0 :_M I^{n+1}).$$

□

3. The dual Hilbert function of 1-dimensional co-Cohen-Macaulay modules

We start this section by the following notations. The dual postulation number of I is defined by

$$n(I, M) = \max\{n \in \mathbb{Z} \mid P_n(I, M) \neq H_n(I, M)\}.$$

Let $f : \mathbb{Z} \to \mathbb{Z}$ be an integer valued function. Then $\triangle^1(f)$ denote the first difference function defined by $\triangle^1(f)(n) = f(n+1) - f(n)$, for all $n \in \mathbb{Z}$. Inductively we define the ith difference function of f by $\triangle^i(f) = \triangle^{i-1}(\triangle^1(f))$.

Theorem 3.1. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = 1$, *$I$ be an ideal of A such that $\ell(0 :_M I) < \infty$ and $J = (a)$ be a minimal reduction of I. Then $\triangle^1(P_n(I, M) - H_n(I, M)) \geq 0$ for all $n \geq 0$.*

Proof. Since $P_n(I, M) = n\acute{e}_0(I, M) - \acute{e}_1(I, M)$, we have

$$\begin{aligned} P_{n+1}(I,M) - H_{n+1}(I,M) &= (n+1)\acute{e}_0(I,M) - \acute{e}_1(I,M) - \ell(0 :_M I^{n+1}) \\ &= n\acute{e}_0(I,M) - \acute{e}_1(I,M) - \ell(\frac{0 :_M JI^n}{0 :_M J}) + \ell(\frac{0 :_M JI^n}{0 :_M I^{n+1}}) \\ &= P_n(I,M) - H_n(I,M) + \ell(\frac{0 :_M JI^n}{0 :_M I^{n+1}}). \end{aligned}$$

Since $\ell(\frac{0:_M JI^n}{0:_M I^{n+1}}) \geq 0$, we conclude that $\triangle^1(P_n(I, M) - H_n(I, M)) \geq 0$, as required. □

Corollary 3.2. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = 1$, *$I$ be an ideal of A such that $\ell(0 :_M I) < \infty$ and $J = (a)$ be a minimal reduction of I. Then for all $n \geq 0$ we have*

$$r_J(I, M) = n(I, M) + 1.$$

Proof. By the proof of Theorem 3.1, we have

$$H_{n+1}(I, M) - P_{n+1}(I, M) = H_n(I, M) - P_n(I, M) - \ell(\frac{0 :_M JI^n}{0 :_M I^{n+1}}).$$

Put $k = n(I, M)$, $r = r_J(I, M)$. Then for all $n \geq r$

$$H_n(I, M) - P_n(I, M) = H_r(I, M) - P_r(I, M).$$

Since $H_n(I, M) = P_n(I, M)$ for all $n \gg 0$, we have $H_r(I, M) = P_r(I, M)$. Therefore $k \leq r - 1$. Now we show that $k \geq r - 1$. Let $n = k + 1$. By using the above equations we have

$$H_{k+2}(I, M) - P_{k+2}(I, M) = H_{k+1}(I, M) - P_{k+1}(I, M) - \ell(\frac{0 :_M JI^{k+1}}{0 :_M I^{k+2}}).$$

Thus $\ell(\frac{0:_M JI^{k+1}}{0:_M I^{k+2}}) = 0$ and so $0 :_M JI^{k+1} = 0 :_M I^{k+2}$. Therefore $r \leq k + 1$ and this completes the proof. □

Theorem 3.3. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = 1$, *$I$ be an ideal of A such that $\ell(0 :_M I) < \infty$ and $J = (a)$ be a minimal reduction of I. Then*

(a) $\acute{e}_1(I, M) = \Sigma_{n\geq 1}\ell(\frac{0:_M JI^{n-1}}{0:_M I^n}) \geq \Sigma_{n\geq 1}\ell(\frac{(0:_M J)+(0:_M I^n)}{(0:_M I^n)})$.

(b) $\acute{e}_1(I, M) = \Sigma_{n\geq 1}\ell(\frac{(0:_M J)+(0:_M I^n)}{(0:_M I^n)})$ *if and only if $G(I, M)$ is co − Cohen − Macaulay.*

Proof. Part (a). For all $m \geq 1$ we consider the exact sequence

$$0 \to (0 :_M I^m) \to (0 :_M JI^{m-1}) \to (\frac{0 :_M JI^{m-1}}{0 :_M I^m}) \to 0.$$

Therefore we have

$$\begin{aligned} \ell(\frac{0 :_M JI^{m-1}}{0 :_M I^m}) &= \ell(0 :_M JI^{m-1}) - \ell(0 :_M I^m) \\ &= \ell(0 :_M J) + \ell(0 :_M JI^{m-1}) - \ell(0 :_M J) - \ell(0 :_M I^m) \\ &= \ell(0 :_M J) + \ell(\frac{0 :_M JI^{m-1}}{0 :_M J}) - \ell(0 :_M I^m) \\ &= \ell(0 :_M J) + \ell(0 :_{\frac{M}{0:_M a}} I^{m-1}) - \ell(0 :_M I^m) \\ &= \ell(0 :_M J) + \ell(0 :_M I^{m-1}) - \ell(0 :_M I^m) \\ &= \ell(0 :_M J) - \ell(\frac{0 :_M I^m}{0 :_M I^{m-1}}). \end{aligned}$$

Hence for all $m \geq 1$,

$$\ell(\frac{0:_M I^m}{0:_M I^{m-1}}) = \ell(0:_M J) - \ell(\frac{0:_M JI^{m-1}}{0:_M I^m}) = \acute{e}_0(I,M) - \ell(\frac{0:_M JI^{m-1}}{0:_M I^m}).$$

Adding the above equation for $m = 1, ..., n$ we obtain

$$\ell(0:_M I^n) = n\acute{e}_0(I,M) - \Sigma_{m=1}^{n}\ell(\frac{0:_M JI^{m-1}}{0:_M I^m}).$$

Taking $n \gg 0$, we get,

$$P_n(I,M) = n\acute{e}_0(I,M) - \acute{e}_1(I,M) = n\acute{e}_0(I,M) - \Sigma_{m=1}^{r_J(I,M)}\ell(\frac{0:_M JI^{m-1}}{0:_M I^m}).$$

Thus $\acute{e}_1(I,M) = \Sigma_{m=1}^{r_J(I,M)}\ell(\frac{0:_M JI^{m-1}}{0:_M I^m})$. Since $(0:_M J)+(0:_M I^m) \subseteq (0:_M JI^{m-1})$, we obtain $\acute{e}_1(I,M) \geq \Sigma_{m=1}^{r_J(I,M)}\ell(\frac{(0:_M J)+(0:_M I^m)}{0:_M I^m})$.

For (b), equality holds if and only if $(0:_M J)+(0:_M I^m) = (0:_M JI^{m-1})$ for all $m \geq 1$. By using [15, Theorem 3.2] this condition is equivalent to $G(I,M)$ is co-Cohen-Macaulay. □

4. The dual of Huckaba-Marley's inequality

We start this section by the following lemma.

Lemma 4.1. *Suppose M is co-Cohen-Macaulay and x is a co-superficial element for I with respect to M. Then*

(1) x is a cosequence element of M.
(2) $x(0:_M I^n) = (0:_M I^{n-1})$ for n sufficiently large.
(3) $P_n(I,\bar{M}) = P_n(I,M) - P_{n-1}(I,M)$ for all n, where $\bar{M} = (0:_M x)$.

Proof. (1) By definition of co-superficial element, there exists a positive integer c such that $x(0:_M I^n) + (0:_M I^c) = (0:_M I^{n-1})$ for n sufficiently large. Now consider the following exact sequence:

$$0 \to 0:_M (I^n, x) \to 0:_M I^n \xrightarrow{x} 0:_M I^n \to \frac{0:_M I^n}{x(0:_M I^n)} \to 0.$$

Hence for large n we have:

$$\begin{aligned}
H_n(I,\bar{M}) &= \ell(0:_M (I^n,x)) \\
&= \ell(\frac{0:_M I^n}{x(0:_M I^n)}) \\
&= \ell(\frac{0:_M I^n}{x(0:_M I^n)+(0:_M I^c)}) + \ell(\frac{x(0:_M I^n)+(0:_M I^c)}{x(0:_M I^n)}) \\
&= \ell(\frac{0:_M I^n}{(0:_M I^{n-1})}) + \ell(\frac{0:_M I^c}{x(0:_M I^n)\cap(0:_M I^c)}). \\
&\leq H_n(I,M) - H_{n-1}(I,M) + \ell(0:_M I^c).
\end{aligned}$$

Since $x(0:_M I^n) \subseteq (0:_M I^{n-1})$ we have $\ell(\frac{0:_M I^n}{x(0:_M I^n)}) \geq \ell(\frac{0:_M I^n}{(0:_M I^{n-1})})$. It therefore follows

$$\begin{aligned}
H_n(I,\bar{M}) &= \ell(0:_M (I^n,x)) \\
&= \ell(\frac{0:_M I^n}{x(0:_M I^n)}) \\
&\geq \ell(\frac{0:_M I^n}{(0:_M I^{n-1})}) \\
&= H_n(I,M) - H_{n-1}(I,M).
\end{aligned}$$

Therefore, we obtain

$$P_n(I,M) - P_{n-1}(I,M) \leq P_n(I,\bar{M}) \leq P_n(I,M) - P_{n-1}(I,M) + \ell(0 :_M I^c)$$

Thus, $\deg P_n(I,\bar{M}) = \deg P_n(I,M) - 1$. Since $\deg P_n(I,M) = \operatorname{Ndim} M$ and $\deg P_n(I,\bar{M}) = \operatorname{Ndim}(0 :_M x)$, we have $\operatorname{Ndim}(0 :_M x) = \operatorname{Ndim} M - 1$. Since M is co-Cohen-Macaulay this implies that x is a cosequence element of M. To prove (2) it is enough to show $(0 :_M I^c) \subset x(0 :_M I^n)$ for n sufficiently large. To see this, by Artin-Rees Lemma [7, Proposition 3], there exists an integer p such that $(0 :_M x) + (0 :_M I^n) = (((0 :_M x) + (0 :_M I^p)) :_M I^{n-p}) \supseteq ((0 :_M x) :_M I^{n-p})$. Therefore we have $x(0 :_M xI^{n-p}) \subseteq x(0 :_M I^n)$. Now since x is a cosequence element i.e., $xM = M$, for $r \in (0 :_M I^{n-p})$ there exists $\acute{m} \in M$ such that $r = x\acute{m}$ and $rI^{n-p} = 0$. So $x\acute{m}I^{n-p} = 0$ and therefore $\acute{m} \in (0 :_M xI^{n-p})$. Consequently $r \in x(0 :_M xI^{n-p})$ and so $x(0 :_M I^n) \supseteq (0 :_M I^{n-p})$. Thus

$$(0 :_M I^{n-p-k}) \subset x(0 :_M I^{n-k}) \subset x(0 :_M I^n).$$

Therefore, for n sufficiently large, $(0 :_M I^c) \subset x(0 :_M I^n)$. Part (3) follows by part (1) and the proof of part (2). $\square$

Proposition 4.2. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$ *and $\underline{x} = x_1, ..., x_k$ is a co-superficial sequence for I. Let $\bar{M} = (0 :_M x)$. Then $\acute{e}_i(I,\bar{M}) = \acute{e}_i(I,M)$ for $0 \leq i \leq d-k$ and $\acute{e}_i(I,\bar{M}) = 0$ for $i > d-k$.*

Proof. It suffices to prove the case $k = 1$. To see this, by Lemma 4.1, we have

$$P_n(I,\bar{M}) = P_n(I,M) - P_{n-1}(I,M) \quad for \quad all \quad n.$$

Since

$$P_n(I,M) = \binom{n+d-1}{d}\acute{e}_0(I,M) - \binom{n+d-2}{d-1}\acute{e}_1(I,M) + ... + (-1)^d\acute{e}_d(I,M),$$

we have

$$P_n(I,\bar{M}) = \sum_{i=0}^{d-1}(-1)^i\{\binom{n+d-i-1}{d-i} - \binom{n+d-i-2}{d-i}\}\acute{e}_i(I,M).$$

Now by the following fact

$$\binom{n+i+1}{k} - \binom{n+i}{k} = \binom{n+i}{k-1},$$

we obtain

$$P_n(I,\bar{M}) = \binom{n+d-2}{d-1}\acute{e}_0(I,M) - \binom{n+d-3}{d-2}\acute{e}_1(I,M) + ... + (-1)^{d-1}\acute{e}_{d-1}(I,M).$$

This completes the proof. $\square$

The following theorem is a dual of [5, Theorem 4.7].

Theorem 4.3. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$*, I be an ideal of A such that $\ell(0 :_M I) < \infty$ and J be a minimal reduction of I. Then*

$$\Sigma_{n\geq 1}\ell(\frac{(0 :_M J) + (0 :_M I^n)}{(0 :_M I^n)}) \leq \acute{e}_1(I,M) \leq \Sigma_{n\geq 1}\ell(\frac{0 :_M JI^{n-1}}{0 :_M I^n}).$$

Proof. We prove this by induction on d. For $d = 1$, it is proved in Theorem 3.3. Now let $d \geq 2$ and $J = (x_1, ..., x_d)$, where $x_1, ..., x_d$ is a co-superficial sequence for I. Let $\bar{M} = (0 :_M x)$. Then $\operatorname{Ndim} \bar{M} = d-1$ and so by Proposition 4.2

$$\acute{e}_i(I,M) = \acute{e}_i(I,\bar{M}) \quad for\ all\ 0 \leq i \leq d-1.$$

Hence by induction hypothesis,

$$\begin{aligned}
\acute{e}_1(I,M) &= \acute{e}_1(I,\bar{M}) \leq \Sigma_{i\geq 1}\ell(\frac{0 :_{\bar{M}} JI^{i-1}}{0 :_{\bar{M}} I^i}) \\
&= \Sigma_{i\geq 1}\ell(\frac{0 :_M (x_1, JI^{i-1})}{0 :_M (x_1, I^i)}) \\
&\leq \Sigma_{i\geq 1}\ell(\frac{0 :_M ((x_1)\cap I^i, JI^{i-1})}{0 :_M I^i}) \\
&\leq \Sigma_{i\geq 1}\ell(\frac{0 :_M JI^{i-1}}{0 :_M I^i}).
\end{aligned}$$

The first inequality yields by the following injective homomorphism:

$$0 :_M (x_1, JI^{i-1})/0 :_M (x_1, I^i) \longrightarrow 0 :_M ((x_1)\cap I^i, JI^{i-1})/0 :_M I^i$$

$$m + 0 :_M (x_1, I^i) \longmapsto m + 0 :_M I^i.$$

Again by using induction hypothesis

$$\begin{aligned}
\acute{e}_1(I,M) &= \acute{e}_1(I,\bar{M}) \geq \Sigma_{i\geq 1}\ell(\frac{0 :_{\bar{M}} J + 0 :_{\bar{M}} I^i}{0 :_{\bar{M}} I^i}) \\
&= \Sigma_{i\geq 1}\ell(\frac{0 :_M (x_1, J) + 0 :_M (x_1, I^i)}{0 :_M (x_1, I^i)}) \\
&= \Sigma_{i\geq 1}\ell(\frac{(0 :_M J)}{(0 :_M J)\cap (0 :_M (x_1, I^i))}) \\
&= \Sigma_{i\geq 1}\ell(\frac{(0 :_M J)}{(0 :_M J)\cap (0 :_M I^i)}) \\
&= \Sigma_{i\geq 1}\ell(\frac{(0 :_M J) + (0 :_M I^i)}{(0 :_M I^i)}).
\end{aligned}$$

This completes the proof. □

Lemma 4.4. *For any $x \in A$, let $\nu(x)$ = the integer i such that $x \in I^i \setminus I^{i+1}$ and $x^* = x + I^{\nu(x)+1}$. If x^* is a $G(I,M)$-cosequence, then*

$$G(I, 0 :_M x) \cong (0 :_{G(I,M)} x^*) \qquad (*)$$

Proof. By [15, Theorem 3.2] x is an M-cosequence and for all $n \geq 0$,

$$0 :_M xI^{n-\nu(x)} = (0 :_M I^n) + (0 :_M x). \qquad (1)$$

On the other hand by [15, Lemma 3.1], (*) is an isomorphism if and only if

$$0 :_M (I^{n+1}, xI^{n-\nu(x)}) = (0 :_M I^n) + (0 :_M (I^{n+1}, x)). \qquad (2)$$

By using the equation (1) and (2) the result follows. □

Theorem 4.5. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$, *$I$ be an ideal of A such that $\ell(0 :_M I) < \infty$ and J be a minimal reduction of I. If* $\operatorname{width} G(I,M) \geq d-1$, *then*

$$\acute{e}_1(I,M) = \Sigma_{n\geq 1}\ell(\frac{0 :_M JI^{n-1}}{0 :_M I^n})$$

.

Proof. If $d = 1$, then by Theorem 3.3 we have the result. Let $d \geq 2$ and $J = (x_1, ..., x_d)$ be a minimal reduction of I such that x_1^* is a $G(I, M)$-cosequence. Then $G(I, 0 :_M x_1) \cong (0 :_{G(I,M)} x_1^*)$ and so width $G(I, \bar{M}) \geq d - 2$. Thus by induction hypothesis we have

$$\begin{aligned} \acute{e}_1(I, M) &= \acute{e}_1(I, \bar{M}) = \Sigma_{i\geq 1}\ell(\frac{0 :_{\bar{M}} JI^{i-1}}{0 :_{\bar{M}} I^i}) \\ &= \Sigma_{i\geq 1}\ell(\frac{0 :_M (x_1, JI^{i-1})}{0 :_M (x_1, I^i)}) \\ &= \Sigma_{i\geq 1}\ell(\frac{0 :_M JI^{i-1}}{0 :_M I^i}). \end{aligned}$$

For the third equality we use [15, Theorem 3.2]. □

Theorem 4.6. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$, *$I$ be an ideal of A such that $\ell(0 :_M I) < \infty$ and J be a minimal reduction of I. If $G(I, M)$ is co-Cohen-Macaulay, then*

$$\acute{e}_1(I, M) = \Sigma_{n\geq 1}\ell(\frac{(0 :_M J) + (0 :_M I^n)}{(0 :_M I^n)}).$$

Proof. Let $J = (x_1, ..., x_d)$ be a minimal reduction of I. Hence $x_1, ..., x_d$ is a M-cosequence and since $G(I, M)$ is co-Cohen-Macaulay we have $(0 :_M J) + (0 :_M I^n) = (0 :_M JI^{n-1})$. Thus $x_1^*, ..., x_d^*$ is a $G(I, M)$-cosequence and so by Lemma 4.4, $G(I, 0 :_M x_1) \cong (0 :_{G(I,M)} x_1^*)$ is co-Cohen-Macaulay. By induction hypothesis

$$\acute{e}_1(I, M) = \acute{e}_1(I, \bar{M}) = \Sigma_{n\geq 1}\ell(\frac{(0 :_{\bar{M}} J) + (0 :_{\bar{M}} I^n)}{0 :_{\bar{M}} I^n}) = \Sigma_{n\geq 1}\ell(\frac{(0 :_M J) + (0 :_M I^n)}{(0 :_M I^n)}).$$

□

Corollary 4.7. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$, *$I$ be an ideal of A such that $\ell(0 :_M I) < \infty$ and J be a minimal reduction of I. If $R(I, M)$ is co-Cohen-Macaulay, then we have*

$$\acute{e}_1(I, M) = \Sigma_{n\geq 1}\ell(\frac{(0 :_M J) + (0 :_M I^n)}{(0 :_M I^n)}).$$

Proof. Since $R(I, M)$ is co-Cohen-Macaulay, by [17, Theorem 4.5] $G(I, M)$ is co-Cohen-Macaulay and $r(I, M) \leq d - 1$. Thus by Theorem 4.6 the result follows. □

The following theorem is a dual of [2, Proposition 3.1].

Theorem 4.8. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$. *Let I be an ideal of A such that $\ell(0 :_M I) < \infty$, J be a minimal reduction of I and $\ell(\frac{0:_M JI^n}{0:_M I^{n+1}}) = 1$. Then for all $n \geq 0$ the following conditions are equivalent:*

(a) *$G(I, M)$ is co-Cohen Macaulay.*
(b) $(0 :_M J) + (0 :_M I^k) = (0 :_M JI^{k-1})$ *for all $k = 1, ..., n$ and* $0 :_M J \nsubseteq 0 :_M I^{n+1}$, $0 :_M JI^{n+1} = 0 :_M I^{n+2}$.

Proof. $(a) \Longrightarrow (b)$. Suppose that $G(I, M)$ is co-Cohen-Macaulay. Then by [15, Theorem 3.2], one has that $(0 :_M J) + (0 :_M I^k) = (0 :_M JI^{k-1})$ *for all k*. In particular $(0 :_M J) + (0 :_M I^{n+1}) = (0 :_M JI^n)$ *and* $0 :_M J \nsubseteq 0 :_M I^{n+1}$ since if $0 :_M J \subseteq 0 :_M I^{n+1}$ we have $(0 :_M I^{n+1}) = (0 :_M JI^n)$ and so $\ell(\frac{0:_M JI^n}{0:_M I^{n+1}}) = 0$, a contradiction. Moreover, from $\ell(\frac{0:_M JI^n}{0:_M I^{n+1}}) = 1$ one concludes that $(\frac{0:_M JI^n}{0:_M I^{n+1}}) \cong \frac{A}{\mathfrak{m}}$. Therefore $\mathfrak{m}(0 :_M JI^n) \subseteq 0 :_M I^{n+1}$ and hence

$$(0 :_M J) \subseteq (0 :_M JI^n) \subseteq (0 :_M I^{n+1}) :_M \mathfrak{m} = 0 :_M I^{n+1}\mathfrak{m} \subseteq 0 :_M I^{n+2}.$$

Therefore $(0 :_M I^{n+2}) = (0 :_M J) + (0 :_M I^{n+2}) = (0 :_M JI^{n+1})$.

$(b) \Longrightarrow (a)$. From the short exact sequence

$$0 \to \frac{(0:_M J) + (0:_M I^{n+1})}{0:_M I^{n+1}} \to \frac{0:_M JI^n}{0:_M I^{n+1}} \to \frac{0:_M JI^n}{(0:_M J) + (0:_M I^{n+1})} \to 0,$$

together with the fact that $\ell(\frac{0:_M JI^n}{0:_M I^{n+1}}) = 1$ and $\frac{(0:_M J)+(0:_M I^{n+1})}{0:_M I^{n+1}} \neq 0$ (as $0:_M J \nsubseteq 0:_M I^{n+1}$) it follows that $(0:_M J) + (0:_M I^{n+1}) = (0:_M JI^n)$. However, $(0:_M JI^{n+1}) = (0:_M I^{n+2})$ implies that $(0:_M J) + (0:_M I^k) = (0:_M JI^{k-1})$ $\quad for\ all\ k \geq n+2$. Hence by [15, Theorem 3.2] we conclude that $G(I, M)$ is co-Cohen Macaulay. □

5. The $\triangle$ operator and Hilbert coefficients

We start this section by the following proposition.

Proposition 5.1. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$*, I be an ideal of A such that $\ell(0:_M I) < \infty$ and $\underline{x} = x_1, ..., x_d$ be a co-superficial sequence for I. Then*

$$\triangle^{d-i}[P_0(I, M)] = (-1)^i \acute{e}_i(I, M).$$

Proof. We prove by induction on d. Let i be an integer such that $1 \leq i \leq d$ and $\bar{M} = (0:_M x)$. If $d = i$, then the result is clear. Suppose $i < d$ and let x be a co-superficial element for I. Then by induction hypothesis

$$\triangle^{d-i}[P_0(I, M)] = \triangle^{(d-1)-i}[P_0(I, \bar{M})] = (-1)^i \acute{e}_i(I, \bar{M}) = (-1)^i \acute{e}_i(I, M).$$

□

Lemma 5.2. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$*, I be an ideal of A such that $\ell(0:_M I) < \infty$. If $x \in I \setminus I^2$, then x^* is a $G(I, M)$-cosequence element if and only if for all $n \geq 1$ we have*

$$x(0:_M I^n) = (0:_M I^{n-1}).$$

Proof. $(\Longrightarrow)$. By [15, Lemma 3.2] x is a M-cosequence element and for all n we have

$$(0:_M I^n) + (0:_M x) = (0:_M xI^{n-1}).$$

Thus we have $x(0:_M I^n) = x(0:_M xI^{n-1})$ and since x is M-cosequence we obtain $x(0:_M xI^{n-1}) = (0:_M I^{n-1})$. Indeed, for $r \in (0:_M I^{n-1})$ there exists $\acute{m} \in M$ such that $r = x\acute{m}$ and $rI^{n-1} = 0$. So $x\acute{m}I^{n-1} = 0$ and therefore $\acute{m} \in (0:_M xI^{n-1})$ consequently $r \in x(0:_M xI^{n-1})$. Conversely let $m \in x(0:_M xI^{n-1})$. Then $m = xt$ such that $txI^{n-1} = 0$. Therefore, $mI^{n-1} = 0$ and so $x(0:_M xI^{n-1}) \subseteq (0:_M I^{n-1})$. Therefore $x(0:_M I^n) = (0:_M I^{n-1})$.

$(\Longleftarrow)$. Suppose $x(0:_M I^n) = (0:_M I^{n-1})$ for all $n \geq 1$. It is clear $(0:_M x) + (0:_M I^n) \subseteq (0:_M xI^{n-1})$. Let $m \in (0:_M xI^{n-1})$ and so $mxI^{n-1} = 0$. Thus $mx \in (0:_M I^{n-1}) = x(0:_M I^n)$ and so there exists $n \in (0:_M I^n)$ such that $mx = nx$. Therefore $m \in (0:_M x) + (0:_M I^n)$ and so $(0:_M x) + (0:_M I^n) = (0:_M xI^{n-1})$. Hence x^* is a $G(I, M)$-cosequence element. □

Theorem 5.3. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$*, I be an ideal of A such that $\ell(0:_M I) < \infty$ and* $\operatorname{width} G(I, M) \geq d-1$*. Then for $0 \leq i \leq d$ and for all non-negative integer n,*

$$(-1)^{d-i}\triangle^i(P_n(I, M) - H_n(I, M)) \geq 0.$$

Proof. It suffices to prove the case $i = d$. To see this, let $h(n) = P_n(I, M) - H_n(I, M)$. Suppose we have $(-1)^{d-i}\triangle^i(h(n)) \geq 0$ for all n and some $i > 0$. Then

$$(-1)^{d-i}\triangle^i(h(n)) = \triangle^1[(-1)^{d-i}\triangle^{i-1}(h(n))] \geq 0$$

for all n. Since $h(n) = 0$ for n sufficiently large, $(-1)^{d-i}\triangle^{i-1}(h(n)) = 0$ for n sufficiently large and so for all n we have

$$(-1)^{d-i}\triangle^{i-1}(h(n)) \leq 0,$$

which gives the theorem for $i-1$. Hence it is enough to prove that

$$\triangle^d(P_n(I,M) - H_n(I,M)) \geq 0 \qquad for \quad all \quad n.$$

We do this by induction on d. For $d = 1$ we proved in Theorem 3.1. Now let $d > 1$, since width $G(I,M) > 0$, there is a cosequence element x^* in $G(I,M)$ such That

$$G(I,\bar{M}) \cong (0 :_{G(I,M)} x^*).$$

Hence width $G(I,\bar{M}) \geq d-2$. So by induction hypothesis we get

$$\triangle^{d-1}(P_n(I,\bar{M}) - H_n(I,\bar{M})) \geq 0 \qquad for \quad all \quad n.$$

Now from the exact sequence

$$0 \to 0 :_M (I^n, x) \to 0 :_M I^n \xrightarrow{x} 0 :_M I^n \to \frac{0 :_M I^n}{x(0 :_M I^n)} \to 0.$$

Therefore $\ell(0 :_M (I^n, x)) = \ell(\frac{0:_M I^n}{x(0:_M I^n)})$. By Lemma 5.2 we have $x(0 :_M I^n) = (0 : I^{n-1})$ for all n and so

$$\begin{aligned}
H_n(I,\bar{M}) &= \ell(0 :_M (I^n, x)) \\
&= \ell(\frac{0 :_M I^n}{x(0 :_M I^n)}) \\
&= \ell(\frac{0 :_M I^n}{0 :_M I^{n-1}}) \\
&= H_n(I,M) - H_{n-1}(I,M) \qquad\qquad for\ all\ n.
\end{aligned}$$

Clearly $P_n(I,\bar{M}) = P_n(I,M) - P_{n-1}(I,M)$ for all n. Hence we get

$$\begin{aligned}
\triangle^d(P_n(I,M) - H_n(I,M)) &= \triangle^{d-1}(\triangle^1(P_n(I,M) - H_n(I,M))) \\
&= \triangle^{d-1}(P_{n-1}(I,\bar{M}) - H_{n-1}(I,\bar{M}))) \geq 0
\end{aligned}$$

for all n. This completes the proof. □

Corollary 5.4. *Let M be a co-Cohen-Macaulay A-module with* Ndim $M = d$, *I be an ideal of A such that $\ell(0 :_M I) < \infty$ and* width $G(I,M) \geq d-1$. *Suppose $P_k(I,M) = H_k(I,M)$ for some integer k. Then $P_n(I,M) = H_n(I,M)$ for all $n \geq k$.*

Proof. By Theorem 5.3, we have

$$(-1)^{d-1}\triangle^1(P_n(I,M) - H_n(I,M)) \geq 0.$$

Hence

$$(-1)^{d-1}(P_{n+1}(I,M) - H_{n+1}(I,M)) \geq (-1)^{d-1}(P_n(I,M) - H_n(I,M))$$

for all n. But since for n sufficiently large $P_n(I,M) - H_n(I,M) = 0$, we get

$$(-1)^{d-1}(P_k(I,M) - H_k(I,M)) \leq (-1)^{d-1}(P_n(I,M) - H_n(I,M)) \leq 0$$

for all $n \geq k$. Thus if $P_k(I,M) - H_k(I,M)) = 0$ then $P_n(I,M) - H_n(I,M)) = 0$ for all $n \geq k$. □

The following result relates $n(I,M)$ to the Hilbert coefficients.

Corollary 5.5. *Let M be a co-Cohen-Macaulay A-module with* Ndim $M = d$, *I be an ideal of A such that $\ell(0 :_M I) < \infty$. For $0 \leq i \leq d-1$ we have the following:*

(a) If $n(I,M) < -i$, then $\acute{e}_j(I,M) = 0$ for $j \geq d-i$.
(b) If width $G(I,M) \geq d-1$, *then the converse of (a) is true.*

Proof. Since $n(I,M) < -i \leq j$ we have $P_j(I,M) = H_j(I,M)$ and so for $-i \leq j \leq 0$, $P_j(I,M) = \sum_{i=0}^{d}(-1)^i \acute{e}_i(I,M)\binom{j+d-i-1}{d-i} = 0$. Now if $j > d-i-1$, we have $\binom{j+d-i-1}{d-i} \neq 0$ and so $\acute{e}_j(I,M) = 0$. For part (b), note that $\acute{e}_j(I,M) = 0$ for $j \geq d-i$ gives $P_{-i}(I,M) = 0 = H_{-i}(I,M)$. Thus by Corollary 5.4 for all $n > -i$, $P_n(I,M) = 0 = H_n(I,M)$ and so $n(I,M) < -i$. □

Lemma 5.6. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$*, I be an ideal of A such that $\ell(0 :_M I) < \infty$. Let $x \in I \setminus I^2$, x^* be a cosequence element in $G(I,M)$. Then*

$$n(I,\bar{M}) = n(I,M) + 1.$$

Proof. By the proof of Theorem 5.3 for all n we have $H_n(I,\bar{M}) = H_n(I,M) - H_{n-1}(I,M)$ and $P_n(I,\bar{M}) = P_n(I,M) - P_{n-1}(I,M)$. Clearly that $H_n(I,\bar{M}) = P_n(I,\bar{M})$ for all $n > n(I,M)+1$, so $n(I,\bar{M}) \leq n(I,M)+1$. Now we show that the case $n(I,\bar{M}) < n(I,M)+1$ is not true. Since in this case we have $H_{n(I,M)+1}(I,\bar{M}) = P_{n(I,M)+1}(I,\bar{M})$ and so $H_{n(I,M)}(I,M) = P_{n(I,M)}(I,M)$, this is a contradiction. Hence $n(I,\bar{M}) = n(I,M)+1$. □

Corollary 5.7. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$*, I be an ideal of A such that $\ell(0 :_M I) < \infty$ and* $\operatorname{width} G(I,M) \geq d-1$*. Let $\acute{e}_0, ..., \acute{e}_d$ be the Hilbert coefficients of I relative to M. Then for $0 \leq i \leq d$*

(a) $\acute{e}_i \geq 0$.
(b) $(-1)^i(\acute{e}_0 - \acute{e}_1 + ... + (-1)^i\acute{e}_i - \ell(0 :_M I)) \geq 0$.

Proof. Recall that if $\underline{x} = x_1, ..., x_k$ is a co-superficial sequence for I, then $\acute{e}_i(I,\bar{M}) = \acute{e}_i(I,M)$ for $0 \leq i \leq d-k$ and $\acute{e}_i(I,\bar{M}) = 0$ for $i > d-k$, where $\bar{M} = (0 :_M x_1, ..., x_k)$. Also by Lemma 5.2 and [6, Proposition 3.4], if $x_1, ..., x_k \in I \setminus I^2$ such that $x_1^*, ..., x_k^*$ form $G(I,M)$-cosequence, then $x_1, ..., x_k$ is a co-superficial sequence for I. To prove statement (a), note that $\acute{e}_0$ is always positive so we may assume $i > 0$. Since $\operatorname{width} G(I,M) \geq d-1$, there exist elements $x_1, ..., x_{d-1} \in I \setminus I^2$ such that $x_1^*, ..., x_{d-1}^*$ form $G(I,M)$-cosequence. Let $\bar{M} = (0 :_M x_1, ..., x_{d-i})$. Then as $i \geq 1$, we have $x_1, ..., x_{d-i}$ also forms cosequence, $\bar{M}$ is co-Cohen-Macaulay of $\operatorname{Ndim} \bar{M} = i$ and $\acute{e}_i(I,\bar{M}) = \acute{e}_i(I,M)$. Furthermore, as $G(I,\bar{M}) \cong (0 :_{G(I,M)} (x_1^*, ..., x_{d-i}^*))$, $\operatorname{width} G(I,\bar{M}) \geq i-1$. Consequently, we can reduce the problem to the case $i = d$; i.e., it is enough to show that the last coefficient is non-negative. But by Theorem 5.3, $(-1)^d(P_0(I,M) - H_0(I,M) \geq 0$ so $(-1)^{2d}\acute{e}_d(I,M) \geq 0$ and hence $\acute{e}_d(I,M) \geq 0$.

For the second statement, note that it is true for $i = 0$, since $\acute{e}_0 = \ell(0 :_M J)$, where J is the minimal reduction of I. So we may assume $i \geq 1$. Also, by the same argument as above we can reduce to the case $i = d$, noting that $\ell(0 :_M I) = \ell(0 :_{\bar{M}} I)$. But by Theorem 5.3 we have $(-1)^d(P_1(I,M) - H_1(I,M)) \geq 0$ and so $(-1)^d(\acute{e}_0 - \acute{e}_1 + ... + (-1)^d\acute{e}_d - \ell(0 :_M I)) \geq 0$. □

Corollary 5.8. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$*, I be an ideal of A such that $\ell(0 :_M I) < \infty$ and* $\operatorname{width} G(I,M) \geq d-1$*. Let $\acute{e}_0, ..., \acute{e}_d$ be the Hilbert coefficients of I relative to M and suppose $\acute{e}_i = 0$ for some $1 \leq i \leq d-1$. Then $\acute{e}_j = 0$ for all $i \leq j \leq d$.*

Proof. It is enough to show $\acute{e}_{i+1} = 0$. Then as in the proof of Corollary 5.7, we can assume that $i = d-1$. Since by assumption $\acute{e}_0 > 0$, we must have that $d > 1$ and so $\operatorname{width} G(I,M) > 0$. Thus there exists $x \in I \setminus I^2$ such that x^* is $G(I,M)$-cosequence. Let $\bar{M} = (0 :_M x)$, then $\bar{M}$ is co-Cohen-Macaulay and $\operatorname{Ndim} \bar{M} = d-1$. Also $\operatorname{width} G(I,\bar{M}) \geq d-2$. Now since $\acute{e}_{d-1}(I,M) = \acute{e}_{d-1}(I,\bar{M}) = 0$ we have that $P_0(I,\bar{M}) = 0 = H_0(I,\bar{M})$. Now by Corollary 5.4, $n(I,\bar{M}) \leq -1$. From Lemma 5.6 we get that $n(I,M) = n(I,\bar{M}) - 1 \leq -2$. By Corollary 5.5, we get that $\acute{e}_d = 0$. □

Lemma 5.9. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$ *and I be an ideal of A such that $\ell(0 :_M I) < \infty$. Then*

(a) $\acute{e}_d(I,M) = \acute{e}_d(I^k,M)$ *for $k \geq 1$.*
(b) $\operatorname{width} G(I^k,M) \geq 1$ *for k sufficiently large.*

Proof. Part (a). Note that for n sufficiently large

$$P_n(I^k, M) = H_n(I^k, M) = H_{kn}(I, M) = P_{kn}(I, M).$$

Thus for all n, we have $P_n(I^k, M) = P_{kn}(I, M)$. So

$$\acute{e}_d(I^k, M) = (-1)^d P_0(I^k, M) = (-1)^d P_0(I, M) = \acute{e}_d(I, M).$$

Part (b). Since $A/\mathfrak{m}$ is infinite, there exists an element $x \in I \setminus I^2$ which is co-superficial for I. Therefore, by Lemma 4.1, we can find a positive integer c such that $x(0 :_M I^n) = (0 :_M I^{n-1})$ for $n \geq c$. If $k \geq c$, then $x^k(0 :_M I^{kn}) = (0 :_M I^{k(n-1)})$ for all $n \geq 1$ and so by Lemma 5.2, $(x^k)^*$ is $G(I, M)$-cosequence. □

The following result is a dual of Narita's Theorem [10].

Corollary 5.10. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d \geq 2$, *$I$ be an ideal of A such that $\ell(0 :_M I) < \infty$. Then $\acute{e}_2(I, M) \geq 0$.*

Proof. By the proof of Corollary 5.7, we can reduce to the case $d = 2$. Choose $k \gg 0$, so by Lemma 5.9 we have $\operatorname{width} G(I^k, M) \geq 1$. Again by Lemma 5.9, we get $\acute{e}_2(I, M) = \acute{e}_2(I^k, M) \geq 0$. □

Lemma 5.11. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$, *$I$ be an ideal of A such that $\ell(0 :_M I) < \infty$ and J be a minimal reduction of I relative to M and there exists $x \in I \setminus I^2$ such that x^* is $G(I, M)$-cosequence. Then*

$$r_J(I, M) = r_J(I, \bar{M})$$

Proof. Let $r = r_J(I, M)$ and $s = r_J(I, \bar{M})$. Suppose $(0 :_{\bar{M}} I^{s+1}) = (0 :_{\bar{M}} JI^s)$. It is equal to

$$(0 :_M I^{s+1}) \cap (0 :_M x) = (0 :_M JI^s) \cap (0 :_M x).$$

Summing with $(0 :_M I^{s+1})$ and by [15, Theorem 3.2], we have

$$(0 :_M I^{s+1}) = (0 :_M JI^s) \cap (0 :_M xI^s).$$

Therefore, $(0 :_M I^{s+1}) = (0 :_M JI^s)$ and so $r \leq s$. Conversely, if $(0 :_M I^{r+1}) = (0 :_M JI^r)$, we have $(0 :_M I^{r+1}) \cap (0 :_M x) = (0 :_M JI^r) \cap (0 :_M x)$ so $(0 :_{\bar{M}} I^{r+1}) = (0 :_{\bar{M}} JI^r)$. Thus $s \leq r$. This completes the proof. □

Theorem 5.12. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$ *and I be an ideal of A such that $\ell(0 :_M I) < \infty$ and* $\operatorname{width} G(I, M) \geq d - 1$. *Then*

$$r(I, M) = n(I, M) + d.$$

Proof. We use induction on d. If $d = 1$ then, by Corollary 3.2, we have the result. Now let $d > 1$. Since $\operatorname{width} G(I, M) \geq d - 1$, there exist elements $x_1, ..., x_{d-1} \in I \setminus I^2$ such that $x_1^*, ..., x_{d-1}^*$ form a $G(I, M)$-cosequence. Let $\bar{M} = (0 :_M x_1)$. Then as x_1 is a cosequence element, $\bar{M}$ is co-Cohen-Macaulay of $\operatorname{Ndim} \bar{M} = d - 1$. Furthermore, as $G(I, \bar{M}) \cong (0 :_{G(I,M)} x_1^*)$ we have $\operatorname{width} G(I, \bar{M}) \geq d - 2$. So by induction hypothesis

$$r(I, \bar{M}) = n(I, \bar{M}) + d - 1.$$

By Lemmas 5.6 and 5.11, we obtain $r(I, M) = n(I, M) + d$. □

Corollary 5.13. *Let M be a co-Cohen-Macaulay A-module with* $\operatorname{Ndim} M = d$, *$I$ be an ideal of A such that $\ell(0 :_M I) < \infty$ and* $\operatorname{width} G(I, M) \geq d - 1$. *Then for $1 \leq i \leq d$*

(1) $\acute{e}_i(I, M) = 0$ *if and only if* $r(I, M) \leq d - 1$.
(2) $r(I, M) = \max\{i \mid \acute{e}_i(I, M) \neq 0\}$ *if* $n(I, M) \leq 0$.

Proof. Part (1), by Theorem 5.12 we have $r(I, M) \leq i - 1$ if and only if $n(I, M) < i - d$. But by Corollaries 5.5 and 5.7, $n(I, M) < i - d$ if and only if $\acute{e}_i(I, M) = 0$. The second part follows from the first part and Corollary 5.5. □

Lemma 5.14. *Let $x_1, ..., x_r \in I^k \setminus I^{k+1}$ such that $x_i^* = x_i + I^{k+1} \neq 0$. Then $x_1^*, ..., x_r^*$ is $G(I, M)$-cosequence if and only if $x_1, ..., x_r$ is M-cosequence and $(0 :_M I^{kn}) + (0 :_M x_1, ..., x_r) = (0 :_M I^{k(n-1)}(x_1, ..., x_r))$ for all $n \geq 1$.*

Proof. By using induction and [15, Theorem 3.2] the result immediately follows. □

Proposition 5.15. *Let M be a co-Cohen-Macaulay A-module with $\operatorname{Ndim} M = d$, I be an ideal of A such that $\ell(0 :_M I) < \infty$. Then $\operatorname{width} G(I^k, M) \geq \operatorname{width} G(I, M)$ for $k \geq 1$.*

Proof. Let $x_1, .., x_r \in I \setminus I^2$ such that $x_1^*, ..., x_r^*$ is a $G(I, M)$-cosequence. Then $(x_1^*)^k, ..., (x_r^*)^k$ is also $G(I, M)$-cosequence and we have $(x_i^*)^k = (x_i^k)^*$ for $i = 1, ..., r$. Thus $(x_1^k)^*, ..., (x_r^k)^*$ is a $G(I, M)$-cosequence and so by Lemma 5.14, we have $x_1^k, ..., x_r^k$ is M-cosequence. Therefore

$$(0 :_M I^{kn}) + (0 :_M x_1^k, ..., x_r^k) = (0 :_M I^{k(n-1)}(x_1^k, ..., x_r^k)).$$

Hence

$$(0 :_M (I^k)^n) + (0 :_M x_1^k, ..., x_r^k) = (0 :_M (I^k)^{n-1}(x_1^k, ..., x_r^k)).$$

Thus $(x_1^k)^*, ..., (x_r^k)^*$ is a $G(I^k, M)$-cosequence. □

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