



Cospectrality Graphs of Smith Graphs

Dragoš M. Cvetković^a, Vesna P. Todorčević^b

^aMathematical Institute SANU, Belgrade, Serbia

^bFaculty of Organizational Sciences, and Mathematical Institute SANU, Belgrade, Serbia

Abstract. Graphs whose spectrum belongs to the interval $[-2, 2]$ are called Smith graphs. The structure of a Smith graph with a given spectrum depends on a system of Diophantine linear algebraic equations. We have established in [1] several properties of this system and showed how it can be simplified and effectively applied. In this way a spectral theory of Smith graphs has been outlined. In the present paper we introduce cospectrality graphs for Smith graphs and study their properties through examples and theoretical consideration. The new notion is used in proving theorems on cospectrality of Smith graphs. In this way one can avoid the use of the mentioned system of Diophantine linear algebraic equations.

1. Introduction

Let G be a simple graph on n vertices (or of order n), and adjacency matrix A . The characteristic polynomial of A (equal to $\det(xI - A)$) is also called the *characteristic polynomial* of G . The eigenvalues and the spectrum of A (which consists of n eigenvalues) are called the *eigenvalues* and the *spectrum* of G , respectively. Since A is real and symmetric, its eigenvalues are real. The eigenvalues of G (in non-increasing order) are denoted by $\lambda_1, \dots, \lambda_n$. In particular, λ_1 , as the largest eigenvalue of G , will be called the *spectral radius* (or *index*) of G . For general information on spectra of graphs see, for example, [2].

The spectrum of G (as a multiset or family of reals) will be denoted by \widehat{G} . The disjoint union of graphs G_1 and G_2 will be denoted by $G_1 + G_2$, while the union of their spectra (i.e. the spectrum of $G_1 + G_2$) will be denoted by $\widehat{G}_1 + \widehat{G}_2$; in addition, kG ($k\widehat{G}$) stands for the union of k copies of G (resp. \widehat{G}).

We say that two (non-isomorphic) graphs are *cospectral* if their spectra coincide. They are also called *cospectral mates*. On the other hand, we say that a graph is determined by its spectrum if it is a unique graph having this spectrum. In the literature (see, for example, [6]) the abbreviation DS (non-DS) is used to indicate that some graph is determined (resp. non-determined) by its spectrum. Many results on spectral characterizations can be found in [6]. For early results see [2].

The *cospectral equivalence class* of a graph G is the set of all graphs cospectral to G (including G itself).

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Email addresses: ecvetkod@etf.rs (Dragoš M. Cvetković), vesnat@fon.rs (Vesna P. Todorčević)

We consider the class of graphs whose spectral radius is at most 2. This class includes, for example, the graphs whose each component is either a path or a cycle.

All graphs with the spectral radius at most 2 have been constructed by J.H. Smith [7]. Therefore these graphs are usually called the *Smith graphs*. Eigenvalues of these graphs have been determined in [3]. All eigenvalues are of the form $2 \cos \frac{p}{q}\pi$, where p, q are integers and $q \neq 0$.

A path (cycle) on n vertices will be denoted by P_n (resp. C_n).

A connected graph with index ≤ 2 is either a cycle C_n ($n = 3, 4, \dots$), or a path P_n ($n = 1, 2, \dots$), or one of the graphs depicted in Fig. 1 (see [7]). Note that W_1 coincide with the star $K_{1,4}$, while Z_1 with P_3 . In addition, the graphs C_n, W_n, T_4, T_5 , and T_6 are connected graphs with index equal to 2; all other graphs, namely, P_n, Z_n, T_1, T_2 and T_3 are the induced subgraphs of these graphs (so the index of each of them is less than 2). The graph Z_n is called a *snake* while W_n is a *double snake*. The trees T_1, T_2, T_3, T_4, T_5 , and T_6 will be called *exceptional Smith graphs*.

The spectrum of each of these graphs can be found (in an explicit form) in [3].

A Smith graph has connected Smith graphs as components.

We denote the set of all Smith graphs by \mathcal{S}^* ; the set of those which are bipartite, so odd cycles are excluded, will be denoted by \mathcal{S} .

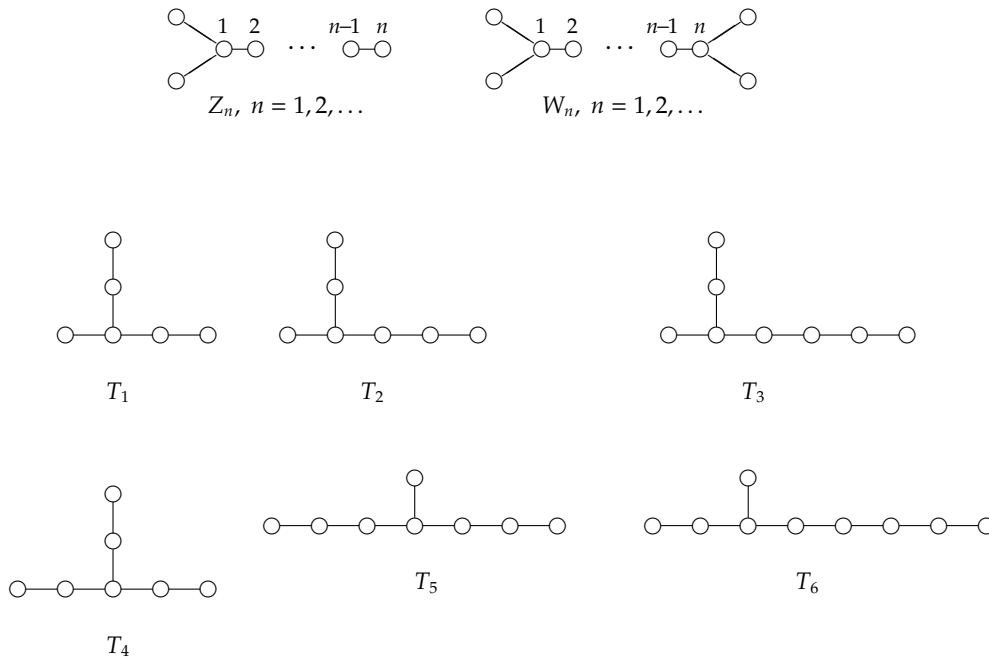


Figure 1: Some of the Smith graphs

Let G be any graph each component of which belongs to \mathcal{S}^* , we can write

$$G = \sum_{H \in \mathcal{S}^*} r(H)H, \tag{1}$$

where $r(H) \geq 0$ is a repetition factor (tells how many times H is appearing as a component in G).

The repetition factor $r(S_i)$ of some of the graph $S_i \in \mathcal{S}^*$ for any relevant index i will be denoted by s_i . So we have non-negative integers

$$p_1, p_2, p_3, \dots, z_2, z_3, \dots, w_1, w_2, w_3, \dots, t_1, t_2, t_3, t_4, t_5, t_6.$$

We have omitted z_1 since $Z_1 = P_3$ and the variable p_3 is relevant. We shall use c_2, c_3, \dots , for repetition factors of the even cycles C_4, C_6, \dots .

For non-bipartite graphs from \mathcal{S}^* we have to introduce variables o_3, o_5, o_7, \dots counting the numbers of odd cycles C_3, C_5, C_7, \dots .

For a given graph $G \in \mathcal{S}^*$ the above variables which do not vanish, together with their values, are called *parameters* of G . Parameters of a graph indicate the actual number of components of particular types present in G .

From the spectrum of a Smith graph the sub-multiset belonging to odd cycles can be recognized, so we can ignore odd cycles, and the spectrum of the remaining bipartite graph can be calculated [3]. Therefore, only bipartite Smith graphs will be considered in the sequel.

The paper [1] describes a system of Diophantine linear algebraic equations whose solution yields parameters of all Smith graphs with given spectrum.

The paper [3] has given foundations of spectral theory of Smith graphs. The paper [1] can be considered as a continuation of the research initiated in [3] and further extended in [5] and [4]. See also [4] for references to some related results on spectral determination of Smith graphs. In this paper we continue investigations from [1].

The rest of the paper is organized as follows. Section 2 contains some preliminary results. Cospectrality and quasi-cospectrality graphs are introduced in Section 3 as new tools for handling Smith graphs. The structure of the cospectrality graph is investigated in Section 4.

2. Preliminary results

Let $H \in \mathcal{S}$. Let

$$\widehat{H} = \sigma_0 \widehat{C}_4 + \sum_{i=1}^m \sigma_i \widehat{P}_i,$$

be the canonical representation (as defined in [1]) of the spectrum \widehat{H} of the bipartite Smith graph H . Here $\sigma_0, \sigma_1, \sigma_2, \dots, \sigma_m$ are integers with $\sigma_0 \geq 0$. This representation always exists and is unique. The expression

$$\sigma_0 C_4 + \sum_{i=1}^m \sigma_i P_i,$$

is called *canonical representation* of H . It defines a graph if $\sigma_0, \sigma_1, \sigma_2, \dots, \sigma_m$ are non-negative, otherwise it is just a formal expression. In the first case H is cospectral to its canonical representation but not necessarily isomorphic.

If all quantities σ_i are non-negative, the graph H is called a *Smith graph of type A*, otherwise it is of *type B*. Let I (resp. J) be the set of indices i for which σ_i in a graph of type B is negative (resp. positive).

Obviously, cospectral Smith graphs are of the same type.

Let $P_H = \sum_{i \in I} |\sigma_i| P_i$. Components of the graph P_H are paths whose spectra appear with a negative sign in the canonical representation of the spectrum of H . The graph P_H is called the *basis* of H . The basis of a graph of type A is empty. If we add components from its basis to a graph of type B, it becomes a graph of type A.

The graph $K_H = \sigma_0 C_4 + \sum_{i \in J} \sigma_i P_i$ is called the *kernel* of H .

Following [1] we shall consider the corresponding component transformations:

$$\begin{aligned}
 (\gamma_1) \quad & W_n \rightleftharpoons C_4 + P_n, & (\delta_1) \\
 (\gamma_2) \quad & Z_n + P_n \rightleftharpoons P_{2n+1} + P_1, & (\delta_2) \\
 (\gamma_3) \quad & C_{2n} + 2P_1 \rightleftharpoons C_4 + 2P_{n-1}, n \geq 3 & (\delta_3) \\
 (\gamma_4) \quad & T_1 + P_5 + P_3 \rightleftharpoons P_{11} + P_2 + P_1, & (\delta_4) \\
 (\gamma_5) \quad & T_2 + P_8 + P_5 \rightleftharpoons P_{17} + P_2 + P_1, & (\delta_5) \\
 (\gamma_6) \quad & T_3 + P_{14} + P_9 + P_5 \rightleftharpoons P_{29} + P_4 + P_2 + P_1, & (\delta_6) \\
 (\gamma_7) \quad & T_4 + P_1 \rightleftharpoons C_4 + 2P_2, & (\delta_7) \\
 (\gamma_8) \quad & T_5 + P_1 \rightleftharpoons C_4 + P_3 + P_2, & (\delta_8) \\
 (\gamma_9) \quad & T_6 + P_1 \rightleftharpoons C_4 + P_4 + P_2. & (\delta_9)
 \end{aligned} \tag{2}$$

They are of the form $A \rightarrow B$ or $B \rightarrow A$ meaning that in a graph the group of components A is replaced with the group of components B or vice versa. Transformations (2) are called G -transformations. Those of the form $A \rightarrow B$ are denoted by $\gamma_1, \gamma_2, \dots, \gamma_9$ and are called C -transformations. For each C -transformation $A \rightarrow B$ we define the corresponding opposite transformation $B \rightarrow A$, also denoted by $A \leftarrow B$. Transformations $A \leftarrow B$ are called D -transformations and are denoted by $\delta_1, \delta_2, \dots, \delta_9$.

Graphs C_4, P_1, P_2, \dots , appearing in canonical representations of bipartite Smith graphs, are called *basic* graphs. All other connected bipartite Smith graphs are called *non-basic* graphs. Non-basic graphs are of two types. Graphs $W_n, (n = 1, 2, \dots), C_{2k}, (k = 3, 4, \dots), T_4, T_5, T_6$ are non-basic graphs of type I while graphs $Z_n, (n = 2, 3, \dots), T_1, T_2, T_3$ are non-basic graphs of type II. Note that non-basic graphs of type I have spectral radius equal to 2 while for those of type II spectral radius is less than 2.

G -transformations $\gamma_1, \gamma_2, \gamma_3$ and their opposite transformations $\delta_1, \delta_2, \delta_3$ are not unique since they depend on the index n of the involved non-basic graphs W_n, Z_n, C_{2n} . If we want to specify this index in the name of the G -transformation, we shall use superscripts (for example, γ_1^n or δ_2^n).

Application of any G -transformation does not change the spectrum of the corresponding graph. Moreover, we have the following theorem from [1].

Theorem 2.1. *Let H_1 and H_2 be Smith graphs with corresponding bases P_{H_1} and P_{H_2} . If graphs H_1 and H_2 are cospectral, then the graph $H_1 + P_{H_1}$ can be transformed into $H_2 + P_{H_2}$ by a finite number of G -transformations.*

Example 2.1. The cospectral equivalence class of graph $W_1 + T_4$ consists of the following seven graphs: $W_1 + T_4, P_1 + C_6 + W_1, P_1 + C_4 + T_4, P_2 + C_4 + W_2, 2P_2 + 2C_4, 2W_2$ and $C_6 + C_4 + 2P_1$. This was proved in [4] using extended system of equations and in [1] using condensed system of equations. We shall now prove the statement using Theorem 2.1.

Indeed, graph $W_1 + T_4$ has 12 vertices, and we have: $\widehat{W}_1 + \widehat{T}_4 = 2\widehat{C}_4 + 2\widehat{P}_2$.

The seven graphs in question are represented in Fig. 2 in a special manner. For each of these graphs we can easily establish by inspection which G -transformations are applicable. After applying a G -transformation another graph from the set is obtained. Possible G -transformations are indicated at Fig.2 by arrows with the corresponding transformation names. The claim on seven cospectral graphs is now evident. ■

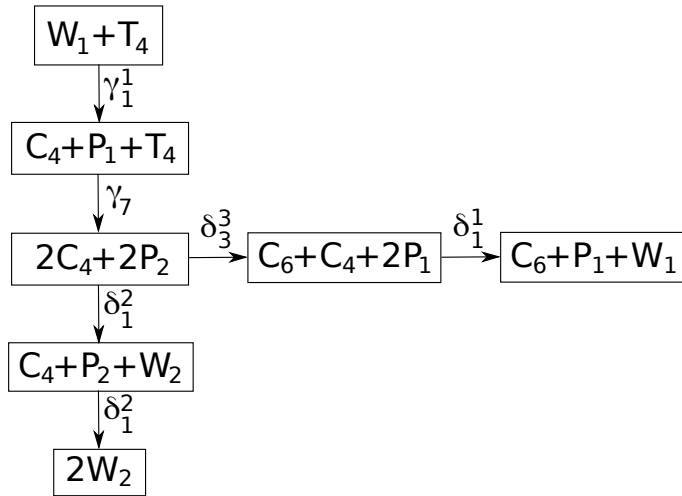


Figure 2: The cospectral equivalence class of graph $W_1 + T_4$

3. Cospectrality graphs and quasi-cospectrality graphs

Example from the previous section can be generalized.

For any A -type graph G we define its *cospectrality graph* $C(G)$ in the following way. Vertices of $C(G)$ are all graphs cospectral with G , i.e. the set of vertices of $C(G)$ is the cospectral equivalence class of G . Two vertices x and y are adjacent if there exists a G -transformation transforming one to another. Of course, if x can be transformed into y by a G -transformation, then y can be transformed into x by the opposite transformation. Hence, $C(G)$ is an undirected graph without multiple edges or loops. By Theorem 2.1. graph $C(G)$ is connected.

The following proposition is obvious.

Proposition 3.1. *If $G, H \in \mathcal{S}$ are cospectral graphs of type A , then $C(G) = C(H)$.*

The following lemma is useful.

Lemma 3.2. *Let G be a bipartite Smith graph of type A . The numbers of non-basic Smith graphs, contained as components in graphs corresponding to adjacent vertices in $C(G)$, differ by 1.*

Proof. Any G -transformation changes the number of non-basic graphs by 1. ■

Theorem 3.3. *For any A type Smith graph G , the cospectrality graph $C(G)$ is bipartite.*

Proof. By Lemma 3.2. graphs associated to adjacent vertices of $C(G)$ contain the number of non-basic graphs of different parity. Hence, $C(G)$ can properly be colored by two colors. ■

A cospectrality graph is not always a tree. For example, $C(T_5 + T_6 + 2P_1)$ contains a quadrangle induced by vertices

$$V_1 = T_5 + T_6 + 2P_1, V_2 = C_4 + P_3 + P_2 + T_6 + P_1, V_3 = T_5 + P_1 + C_4 + P_4 + P_2, V_4 = 2C_4 + P_4 + P_3 + 2P_2.$$

V_2 and V_3 are obtained from V_1 by γ_8 and γ_9 respectively while V_4 is obtained from V_2 or V_3 by γ_9 and γ_8 respectively.

For any B -type graph G we define its *quasi-cospectrality graph* $QC(G)$ as $QC(G) = C(G + P_G)$, i.e. as the cospectrality graph of the kernel of G .

Although all graphs cospectral to the kernel $G + P_G$ are contained as vertices in $QC(G)$, only vertices which contain the basis P_G give rise to a graph cospectral to G .

A condensed version of $QC(T_5 + T_6)$ is given in Fig. 2 of [1]. We have $T_5 + T_6 = 2C_4 + P_4 + P_3 + 2P_2 - 2P_1$. The kernel $2C_4 + P_4 + P_3 + 2P_2$ of $T_5 + T_6$ is located in the center of the figure. Two D -transformations are necessary to obtain graphs which contain the basis $2P_1$ starting from the kernel and only such vertices give rise to graphs cospectral to $T_5 + T_6$.

The graph $QC(T_5 + T_6) = C(T_5 + T_6 + 2P_1)$ is given here in Fig. 3 with all details.

We see that G -transformation δ_1 is used with various non-basic graphs $(\delta_1^1, \delta_1^2, \delta_1^3, \delta_1^4)$.

In Fig. 3 Smith graphs are presented as disjoint unions of connected Smith graphs where the symbol $+$, denoting the disjoint union, is omitted. This gives the idea that a Smith graph can be thought as a family of symbols representing its components. G -transformations are then just replacements of some symbol groups with other symbol groups.

We see from Fig. 3 that there are 15 graphs cospectral to $T_5 + T_6 + 2P_1$ including $T_5 + T_6 + 2P_1$ itself.

In fact, the following theorem has been proved.

Theorem 3.4. *The only cospectral mates of the graph $T_5 + T_6 + 2P_1$ are 14 graphs represented in Fig. 3.*

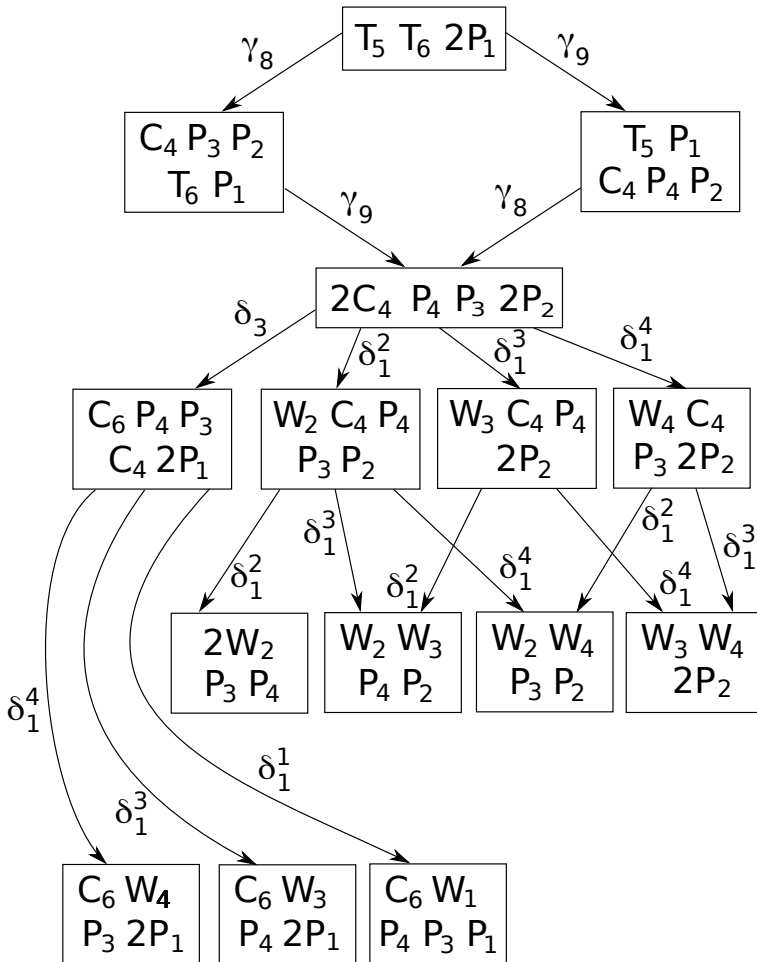


Figure 3: Cospectrality graph of the graph $T_5 + T_6 + 2P_1$

Using G -transformations we can easily prove the following theorem.

Theorem 3.5. *The only cospectral mates of the graph $Z_n + W_n$ are the following four graphs: $Z_n + C_4 + P_n$, $C_4 + P_1 + P_{2n+1}$, $W_1 + P_{2n+1}$ and $W_{2n+1} + P_1$.*

Proof. The only G -transformation applicable at the graph $Z_n + W_n$ is γ_1 giving rise to the graph $Z_n + C_4 + P_n$. Now δ_1 reproduces the previous graph while γ_2 yields $C_4 + P_1 + P_{2n+1}$. Applying now δ_1 in two different ways we get graphs $W_1 + P_{2n+1}$ and $W_{2n+1} + P_1$. We cannot obtain new graphs any more since applying opposite transformation of those used leads to previous graphs. ■

Theorem 3.5 and other similar results can be proved using system of Diophantine linear algebraic equations but the approach with cospectrality graphs and G -transformations is obviously more effective. In particular, cospectrality graphs can be used in finding all Smith graphs with the given spectrum, thus avoiding the use of system of Diophantine linear algebraic equations.

One can easily construct sets with arbitrarily many cospectral Smith graphs.

Example 3.1. Graphs

$$(n - k)(C_4 + P_1) + kW_1, k = 0, 1, \dots, n$$

are non-isomorphic and cospectral. We have $C(nC_4 + nP_1) = C(nW_1) = P_{n+1}$. ■

If involved graphs are considered as labeled graphs, the G -transformation δ_1 can be applied in several different ways. However, since the resulting graphs are isomorphic, we shall consider all such applications of δ_1 as the same one.

Example 3.1. gives rise to the following theorem.

Theorem 3.6. *Given a positive integer n , there exist n mutually non-isomorphic cospectral Smith graphs.*

4. The structure of a cospectrality graph

Consider the cospectrality graph $C(G)$ of a bipartite Smith graph G of type A.

The vertex v_0 representing the canonical representation of G is called the c -center of $C(G)$.

For any vertex v of $C(G)$ we define $H(v)$ to be the graph which is represented by v . The rank $rankH$ of a Smith graph H is the number of non-basic components of H . We have $rankH = b_1 + b_2$ where b_1, b_2 denote the number of non-basic graphs of types I and II respectively.

Numbers of non-basic graphs can be expressed in terms of graph parameters:

$$b_1 = w_1 + w_2 + \dots + c_3 + c_4 + \dots + t_4 + t_5 + t_6, b_2 = z_2 + z_3 + \dots + t_1 + t_2 + t_3.$$

Vertices of $C(G)$ are partitioned into *layers* according to ranks of corresponding graphs. Layer k contains vertices v such that $rankH(v) = k$. The largest rank of a vertex in $C(G)$ is called the c -radius of $C(G)$. The vertices with largest rank are called *peripheral* vertices. Their rank is equal to the c -radius. Applying a D -transformation on a vertex enhances its rank while C -transformations diminish the rank. Using C -transformations we are approaching the c -center while by D -transformations we go from c -center to peripheral vertices.

Note that notions of center and radius in cospectrality graphs (c -center and c -radius) and in general graphs are differently defined. As an illustration see Example 3.1.

For further consideration we need the following equations [1] for parameters and coefficients of canonical representation

$$(F_0) = (w_1 + w_2 + w_3 + \dots) + (c_2 + c_3 + \dots) + t_4 + t_5 + t_6 = \sigma_0,$$

$$(F_1) = p_1 + w_1 + (z_2 + z_3 + \dots) - 2(c_3 + c_4 + \dots) + t_1 + t_2 + t_3 - t_4 - t_5 - t_6 = \sigma_1.$$

We immediately obtain $b_1 + c_2 = \sigma_0$ and $b_2 + p_1 + w_1 - 2(c_3 + c_4 + \dots) - t_4 - t_5 - t_6 = \sigma_1$.

Now the following proposition is immediate.

Proposition 4.1. *The number of non-basic components of type I of a graph $H \in \mathcal{S}$ is at most equal to the coefficient σ_0 in its canonical representation.*

Some information on the number b_2 of non-basic components of type II can be obtained from equations (F_0) and (F_1) . However, for a precise estimation of b_2 coefficients σ_i with higher i are relevant. In particular, coefficients $\sigma_{11}, \sigma_{17}, \sigma_{29}$ are relevant (cf., D -transformations $\delta_4, \delta_5, \delta_6$).

It would be interesting to obtain some (upper) bounds on the number of vertices of the cospectrality graph $C(G)$.

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