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SOME PROPERTIES OF ISTRATESCU'S MEASURE OF NONCOMPACTNESS

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Dedicated to professor Dušan Adamović on the occasion of his 70^{th} birthday

Abstract. In this note we present some properties of Istratescu's measure of noncompactness on metric linear spaces. As applications of these results we prove that Kuratowski's and Istratescu's measures of noncompactness of the unit ball in ℓ^p and L^p $(0 < p < 1)$ spaces equal to 2.

Introduction

The theory of measures of noncompactness has many applications in Topology, Functional analysis and Operator theory (see [1], [5] and [9]).

Let (X, d) be a metric space, $x \in X$, $A \subseteq X$ and $r > 0$. By $\mathcal{P}(X)$ we denote the set of all subsets of X, by $\operatorname{diam}(A)$ the diameter of the set A and by $B(x,r)$ the closed ball $\{y \in X : d(x,y) \leq r\}$. A is a $r-$ discrete set, if $d(x,y) \geq r$ for any $x, y \in A$.

Definition. *Let (X, d) be a metric space. A measure of noncompactness on X is an arbitrary function $\phi : \mathcal{P}(X) \to [0, \infty]$ which satisfies the following conditions:*

1) $\phi(A) = \infty$ if and only if A is an unbounded set;
2) $\phi(A) = \phi(\overline{A})$;
3) $\phi(A) = 0$ if and only if A is a totally bounded set;
4) from $A \subseteq B$ it follows $\phi(A) \leq \phi(B)$;

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5) if X is complete, and if $\{B_n\}_{n\in N}$ is a sequence of closed subsets of X such that $B_{n+1} \subseteq B_n$ for each $n \in N$ and $\lim_{n\to\infty} \phi(B_n) = 0$, then $K = \bigcap_{n\in N} B_n$ is a nonempty compact set.

The most important examples of measures of noncompactness are:

1) Kuratowski's function

$\alpha(A) = \inf\{r > 0 \mid A \subseteq \bigcup_{i=1}^{n} S_i,\ S_i \subseteq X,\ \mathrm{diam}(S_i) < r\ 1 \leq i \leq n,\ n \in N\}$;

2) Hausdorff's function

$\chi(A) = \inf\{\varepsilon > 0\ :\ A$ has a finite $\varepsilon-$ net in $X\}$;

3) inner function of Hausdorff

$\chi_i(A) = \inf\{\varepsilon > 0\ :\ A$ has a finite $\varepsilon-$ net in $A\}$;

4) function of Istratescu

$I(A) = \inf\{\varepsilon > 0\ :\ A$ contains no infinite $\varepsilon-$ discrete set in $A\}$.

Relations between this functions are given by following inequality, which are obtain by Danes [6]:

$$\chi(A) \leq \chi_i(A) \leq I(A) \leq \alpha(A) \leq 2\chi(A).$$

Proposition 0. *(see Rolewicz [10]) Let X be a metric linear space. Then there exists a metric d on X which is equivalent to the original metric on X such that function $|.\|: X \to [0,+\infty)$ defined by $|x\| = d(x,0)$ has following properties:*

1) $|x\| = 0$ if and only if $x = 0$;

2) $|x\| = |-x\|$;

3) $|x+y\| \leq |x\| + |y\|$;

4) $0 < |\alpha| < |\beta|$ implies $|\alpha x\| < |\beta x\|$.

The mapping $|.\|$ is said to be an $F-$ *norm* or *paranorm*. If there exists a number p, $0 < p \leq 1$, such that $|tx\| = |t|^p|x\|$ for any scalar t and $x \in X$ it is said that $|.\|$ is a *p-norm* and X is a *p-normed* space.

Let X be a Hausdorff topological vector space. A set $A \subseteq X$ is *bounded* if for each neighborhood of zero U there is a scalar α such that $A \subseteq \alpha U$. The space X is *locally bounded* if it contains a bounded neighborhood of zero. X is a locally bounded space if and only if X is metrizable and p-normable.

The following lemma was proved in [4].

Lemma. *If X is locally bounded Hausdorff topological vector space, $Q \subseteq X$ a bounded subset, $|.\|$ a p-norm on X and β an arbitrary scalar then*

$$\alpha(\beta Q) = |\beta|^p \alpha(Q).$$

The corresponding result for Hausdorff measure of noncompactness was obtained in [3].

Results

Proposition 1. *If Q, Q_1 and Q_2 are bounded subsets of arbitrary metric linear space X and $x \in X$, then:*

1) $I(Q_1 + Q_2) \leq I(Q_1) + I(Q_2)$;
2) $I(x + Q) = I(Q)$.

Proof.
1) Let $\varepsilon > 0$ be an arbitrary positive real number and the sequence $\{z_i\}$ an $[I(Q_1) + I(Q_2) + \varepsilon]-$ discrete set in $Q_1 + Q_2$, where $z_i = x_i + y_i$, $x_i \in Q_1$ and $y_i \in Q_2$. Then for all $i \neq j$:

$$I(Q_1 + Q_2) - \varepsilon \leq |z_i - z_j|| \leq |x_i - x_j|| + |y_i - y_j||.$$

An arbitrary $I(Q) + \varepsilon-$ discrete set in $\{x_i\}$ is a finite set and so, one of its point, say x_1^1, is a cluster point for sequence $\{x_i\}$ and there is a subsequence of $\{x_i\}$ which tends to x_1^1. We denote this sequence by x_i^1, $i = 2, 3, \cdots$. We can assume that all members of the sequence $\{x_i^1\}$ satisfy the condition $|x_i^1 - x_1^1|| \leq I(Q_1) + \varepsilon$. We can apply this method to the sequence $\{x_i^1\}$ and so we obtain a point x_1^2 and a subsequence $\{x_i^2\}$ of $\{x_i^1\}$ which satisfies $|x_i^2 - x_1^2|| \leq I(Q_1) + \varepsilon$ for any i. Further, by induction, we obtain sequences $\{x_i^3\}$, $\{x_i^4\}$,... Let $u_i = x_i^i$. From $|u_i - u_j|| \leq I(Q_1) + \varepsilon$, it follows that there exists a subsequence $\{v_i\} \subseteq \{y_i\}$ which is $I(Q_1 + Q_2) - I(Q_1) - 2\varepsilon-$ discrete in set Q_2. Hence, $I(Q_1 + Q_2) - I(Q_1) - 2\varepsilon \leq I(Q_2)$, for any $\varepsilon > 0$, which implies $I(Q_1 + Q_2) \leq I(Q_1) + I(Q_2)$.
2) From 1) follows $I(x + Q) \leq I(\{x\}) + I(Q) = I(Q)$, which implies
$I(Q) = I(-x + x + Q) \leq I(x + Q)$. Hence, $I(x + Q) = I(Q)$.

When X is a normed space statement 1) from Proposition 1 was obtained by Nina A. Yerzakova (see [1]). This result is the solution of Danes's conjecture from [6].

Proposition 2. *If X is locally bounded Hausdorff topological vector space, $Q \subseteq X$ a bounded subset, $|.||$ a p-norm on X and α an arbitrary scalar then*

$$I(\alpha Q) = |\alpha|^p I(Q)$$

for some p, $0 < p \leq 1$.

Proof. Let $\beta \neq 0$. From

$$|\beta x - \beta y|| = |\beta||^p |x - y||$$

follows that for every finite $\varepsilon-$ discrete subset of Q, there exists one $|\beta|^p\varepsilon-$ discrete subset of βQ and for every finite $|\beta|^p\varepsilon-$ discrete subset of βQ, there exists one $\varepsilon-$ discrete subset of Q, which implies that $I(\beta Q) = |\beta|^p I(Q)$.

Applications

A real (or complex) sequence $\{x_n\}$ belongs to ℓ^p $(0 < p < 1)$ if $\sum\limits_{i=1}^{\infty} |x_n|^p < \infty$. ℓ^p $(0 < p < 1)$ is complete locally bounded metric linear space which is not normable. This space is p-normable with p-norm defined by

$$|\{x_n\}|| = \sum_{i=1}^{\infty} |x_n|^p < \infty.$$

Distance between two elements of this space are defined by $d(x,y) = |x-y||$.

I. Jovanović and V. Rakočević [8] proved that $\chi(B(0,1)) = 1$ in ℓ^p spaces $(0 < p < 1)$. An extension of this result to the class of locally bounded spaces is given in [3].

Let $(X, \mathcal{A}, \mu)$ be a measure space, such that $\mu(X) < \infty$. A measurable real (or complex) function f defined on X belongs to $L^p(X, \mathcal{A}, \mu)$ $(0 < p < 1)$ if $\int_X |f|^p d\mu < \infty$. $L^p(X, \mathcal{A}, \mu)$ $(0 < p < 1)$ is a complete locally bounded metric linear space which is not normable. This space is p-normable with p-norm defined by

$$|f|| = \int_X |f|^p d\mu < \infty.$$

Distance between two elements of this space are defined by $d(x,y) = |x-y||$. By L^p we denote the space $L^p(X, \mathcal{A}, \mu)$, where μ is non atomic measure.

Furi and Vignoli [7] proved that $\alpha(B(0,1)) = 2$ in infinite dimensional normed spaces.

Now, we prove that in ℓ^p and L^p $(0 < p < 1)$ spaces

$$\alpha(B(0,1)) = I(B(0,1)) = 2.$$

This fact is not true for $p \geq 1$. For example in L^2 spaces $I(B(0,1)) = 2^{\frac{1}{2}}$ (see [1]).

Proposition 3. *In ℓ^p space $(0 < p < 1)$ is $I(B(0,1)) = 2$.*

Proof. From triangle inequality it follows $\operatorname{diam}(B(0,1)) \leq 2$ which implies $I(B(0,1)) \leq 2$. Let $\{e_i | i \in \mathcal{N}\} \subseteq \ell^p$ be the standard bases of ℓ^p. Since this set is $2-$discret we have $I(B(0,1)) \geq 2$. So $I(B(0,1)) = 2$.

Proposition 4. *In L^p $(0 < p < 1)$ space, the equality $I(B(0,1)) = 2$ holds.*

Proof. From Danes's inequality, it follows $I(B(0,1)) \leq 2\chi(B(0,1)) = 2$. Let $\{A_i | i \in \mathcal{N}\} \subseteq \mathcal{A}$ be a sequence of mesurable sets such that $\mu(A_i) = 2^{-i}\mu(X)$. Such sequence exists beacause μ is non atomic. Let

$$t_n = \mu(A_n)^{-\frac{1}{p}}.$$

Let $\{f_i | i \in \mathcal{N}\} \subseteq L^p$ be a sequence of functions defined by

$$f_i(x) = \begin{cases} t_n, & \text{for } x \in A_n \\ 0, & \text{for } x \in A_n^c. \end{cases}$$

Since $\{f_i\}_{i \in \mathcal{N}} \subseteq L^p$ is infinite and $d(f_i, f_j) = 2$ for $i \neq j$ we have $I(B(0,1)) \geq 2$. So $I(B(0,1)) = 2$.

Proposition 5. *In ℓ^p and L^p $(0 < p < 1)$ spaces, the equality $\alpha(B(0,1)) = 2$ holds.*

Proof. From Danes's inequality and propositions 3,4 follows

$$2 = I(B(0,1)) \leq \alpha(B(0,1)) \leq 2\chi(B(0,1)) = 2.$$

Corollary. *Let X be either ℓ^p or L^p for some p $(0 < p < 1)$. Then, for any $x_0 \in X$ and $r > 0$, the equality $\alpha(B(x_0, r)) = 2r$ holds.*

Proof. For $r > 0$ conditions $|x\| \leq 1$ and $|r^{\frac{1}{p}} x\| \leq r$ are equivalent, which implies $B(0,r) = r^{\frac{1}{p}} B(0,1)$. So,

$$\begin{aligned} &\alpha(B(x_0, r)) = \alpha(x_0 + B(0,r)) = \alpha(B(0,r)) = \\ &= \alpha(r^{\frac{1}{p}} B(0,1)) = r\alpha(B(0,1)) = 2r. \end{aligned}$$

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