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A SUFFICIENT CONDITION FOR UNIVALENCY FOR FUNCTIONS WITH INTEGRAL REPRESENTATION

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Abstract. In this paper we prove the analyticity and the univalence of the functions which are defined by means of integral operators. In particular cases we find some known results.

1. Introduction

We denote by $U_r = \{ z \in C : |z| < r \}$ the disk of z-plane, where $r \in (0,1]$, $U_1 = U$ and $I = [0, \infty)$.

Let A be the class of functions f analytic in U such that $f(0) = 0$, $f'(0) = 1$. Let S denote the class of function $f \in A$, f univalent in U. The usual subclasses of S consisting of starlike functions and α-convex functions will be denoted by S^* respectively M_α.

Definition 1.1. *([2])Let $f \in A$, $f(z)f'(z) \neq 0$ for $0 < |z| < 1$ and let $\alpha \geq 0$. We denote by*

$$(1) \quad M(\alpha, f) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha(\frac{zf''(z)}{f'(z)} + 1)$$

If $ReM(\alpha, f) > 0$ in U, then f is said to be an α- convex function ($f \in M_\alpha$).

Theorem 1.1. *([2]). The function $f \in M_\alpha$ if and only if there exists a function $g \in S^*$ such that*

$$(2) \quad f(z) = \left(\frac{1}{\alpha}\int_0^z \frac{g^{\frac{1}{\alpha}}(u)}{u}du\right)^\alpha$$

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Definition 1.2. *([5]) Let $f \in A$. We said that $f \in S^*(a,b)$ if*

$$(3) \quad \left|\frac{zf'(z)}{f(z)} - a\right| < b, \qquad |z| < 1,$$

where

$$(4) \quad a \in C,\ Rea \geq b,\ |a-1| < b.$$

Theorem 1.2. *([1]) Let $f \in A$. If for all $z \in U$*

$$(5) \quad (1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

then the function f is univalent in U.

2. Preliminaries

Theorem 2.1. *([4]) Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$, $a_1(t) \neq 0$ be analytic in U_r for all $t \in I$, locally absolutely continuous in I and locally uniform with respect to U_r. For almost all $t \in I$ suppose*

$$(6) \quad z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \quad \forall z \in U_r,$$

where $p(z,t)$ is analytic in U and satisfies the condition $Rep(z,t) > 0$ for all $z \in U$, $t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r, then for each $t \in I$ the function $L(z,t)$ has an analytic and univalent extension to the whole disk U.

3. Main results

Theorem 3.1. *Let f, $g \in A$ and let the numbers $\alpha \in C$, $\beta > 0$, such that $|\alpha - \beta| < \beta$. If*

$$(7) \quad \frac{1-|z|^{2\beta}}{\beta}\left|(\alpha-1)\frac{zg'(z)}{g(z)} + \frac{zf''(z)}{f'(z)} + 1 - \beta\right| \leq 1, \qquad \forall z \in U,$$

then the function

$$(8) \quad H(z) = \left(\alpha\int_0^z g^{\alpha-1}(u)f'(u)du\right)^{1/\alpha}$$

is analytic and univalent in U, where the principal branch is intended.

Proof. Let us prove that there exists a real number $r \in (0,1]$ such that the function $L : U_r \times I \to C$ defined formally by

$$(9) \quad L(z,t) =$$

$$= \left[\alpha \int_0^{e^{-t}z} g^{\alpha-1}(u)f'(u)du + \frac{\alpha}{\beta}(e^{(2\beta-1)t} - e^{-t})zg^{\alpha-1}(e^{-t}z)f'(e^{-t}z)\right]^{1/\alpha}$$

is analytic in U_r for all $t \in I$.

Since $g \in A$, the function $h(z) = \frac{g(z)}{z}$ is analytic in U and $h(0) = 1$. Then there is a disk U_{r_1}, $0 < r_1 \le 1$, in which $h(z) \neq 0$ for any $z \in U_{r_1}$ and we choose the uniform branch of $(h(z))^{\alpha-1}$ equal to 1 at the origin, denoted by h_1.

For the function

$$h_2(z,t) = \alpha \int_0^{e^{-t}z} u^{\alpha-1}h_1(u)f'(u)du$$

we have $h_2(z,t) = z^\alpha h_3(z,t)$ and is easy to see that h_3 is also analytic in U_{r_1}. The function

$$h_4(z,t) = h_3(z,t) + \frac{\alpha}{\beta}(e^{(2\beta-1)t} - e^{-t})e^{-(\alpha-1)t}h_1(e^{-t}z)f'(e^{-t}z)$$

is analytic in U_{r_1} and we get

$$h_4(0,t) = e^{(2\beta-\alpha)t}\left[\frac{\alpha}{\beta} + (1 - \frac{\alpha}{\beta})e^{-2\beta t}\right]$$

Let us prove that $h_4(0,t) \neq 0$ for any $t \in I$. We have $h_4(0,0) = 1$. Assume now that there exists $t_0 > 0$ such that $h_4(0,t_0) = 0$. Then $e^{2\beta t_0} = (\alpha-\beta)/\alpha$ and since $1 - \beta/\alpha$ is a real number only in the case $\alpha \in R$, from $|\alpha - \beta| < \beta$ we get $1 - \beta/\alpha < 1$. It follows that $e^{2\beta t_0} < 1$ and in view of $\beta > 0$, $t_0 > 0$ this inequality is imposible. Therefore, there is a disk U_{r_2}, $r_2 \in (0, r_1]$ in which $h_4(z,t) \neq 0$ for all $t \in I$. Then we can choose an uniform branch of $[h_4(z,t)]^{1/\alpha}$ analytic in U_{r_2} denoted by $h_5(z,t)$, which is equal to

$$a_1(t) = e^{(\frac{2\beta}{\alpha}-1)t}\left[\frac{\alpha}{\beta} + (1 - \frac{\alpha}{\beta})e^{-2\beta t}\right]^{1/\alpha}$$

at the origin and for $a_1(t)$ we fix the principal branch ($a_1(0) = 1$).

From this considerations it results that the relation (9) may be written as

$$L(z,t) = zh_5(z,t) = a_1(t)z + a_2(t)z^2 + \dots$$

and then the function $L(z,t)$ is analytic in U_{r_2}.
Since $|\alpha - \beta| < \beta$ is equivalent with $Re 2\beta/\alpha > 1$ it results that $\lim_{t\to\infty} |a_1(t)| = \infty$. We saw also that $a_1(t) \neq 0$ for all $t \in I$.
It is easy to prove that $L(z,t)$ is locally absolutely continous in I, locally uniformly with respect to U_{r_3} and that $\{L(z,t)/a_1(t)\}$ is a normal family in U_{r_3}, $r_3 \in (0, r_2]$. It follows that the function $p(z,t)$ defined by (6) is analytic in U_r, $r \in (0, r_3]$, for all $t \geq 0$.
In order to prove that the function $p(z,t)$ has an analytic extension with positive real part in U, for all $t \in I$, it is sufficient to prove that the function $w(z,t)$ defined in U_r by

$$w(z,t) = \frac{p(z,t) - 1}{p(z,t) + 1}$$

can be continued analytically in U and $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$.
After computation we obtain

$$(10) \quad w(z,t) = \frac{1 - e^{-2\beta t}}{\beta} \left[(\alpha - 1) \frac{e^{-t} z g'(e^{-t} z)}{g(e^{-t} z)} + \frac{e^{-t} z f''(e^{-t} z)}{f'(e^{-t} z)} + 1 - \beta \right]$$

From (7) we deduce that the function $w(z,t)$ is analytic in the unit disk U.
We have $w(z,0) = 0$ and for $z = 0$, $t > 0$ since $|\alpha - \beta| < \beta$ we get

$$|w(0,t)| = \left| \frac{1 - e^{-2\beta t}}{\beta} (\alpha - \beta) \right| < \frac{|\alpha - \beta|}{\beta} < 1.$$

Let us denote $u = e^{-t} e^{i\theta}$. Then $|u| = e^{-t}$ and taking into account the relation (7) we have

$$|w(e^{i\theta}, t)| = \frac{1 - |u|^{2\beta}}{\beta} \left| (\alpha - 1) \frac{u g'(u)}{g(u)} + \frac{u f''(u)}{f'(u)} + 1 - \beta \right| \leq 1$$

Using the maximum principle for all $z \in U \setminus \{0\}$ and $t > 0$ we conclude that $|w(z,t)| < 1$ and finally we have $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$.
From Theorem 2.1 it results that the function

$$L(z,0) = \left(\alpha \int_0^z g^{\alpha-1}(u) f'(u) du \right)^{1/\alpha}$$

is analytic and univalent in U and then the function H defined by (8) is analytic and univalent in U.
For particular choices of f and g we get the following

Corollary 3.1. *Let $f \in A$ and let $\alpha \in C$, $\beta > 0$, $|\alpha - \beta| < \beta$. If*

$$(11) \quad \frac{1-|z|^{2\beta}}{\beta}\left|\frac{zf''(z)}{f'(z)} + \alpha - \beta\right| \leq 1 \qquad \forall z \in U,$$

then the function

$$(12) \quad F(z) = \left(\alpha \int_0^z u^{\alpha-1} f'(u)du\right)^{1/\alpha}$$

is analytic and univalent in U.

Proof. If we take $g(z) = z$, from (7) we obtain the relation (11).

Corollary 3.2. *Let $f \in A$ and let $\alpha \geq 1$. If*

$$(13) \quad (1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \qquad \forall z \in U,$$

then the function F defined by (12) is analytic and univalent in U.

Proof. It is easy to see that the function $\varphi : (0,\infty) \longrightarrow R$, $\varphi(x) = (1-a^{2x})/x$, $0 < a < 1$ is a decreasing function.
If $\beta \geq 1$ we have

$$(14) \quad \frac{1-|z|^{2\beta}}{\beta} \leq 1 - |z|^2$$

Then , from $\beta \geq 1$, if the inequality

$$(15) \quad (1-|z|^2)\left|\frac{zf''(z)}{f'(z)} + \alpha - \beta\right| \leq 1$$

is true, from (14) it results that (11) is also true and then, from corollary 3.1, the function F defined by (12) is analytic and univalent in U. In particular case $\alpha = \beta$, from (15) we get (13) and we observe that the condition (13) is just Becker's univalence criterion, but here the conclusion of corollary 3.2 gives us not only the univalence of the function f $(\alpha = 1)$, but also the univalence of the function F defined by (12).
For the function $f \in A$, $f(z) = z$, from theorem 3.1 we get the following

Theorem 3.2. *Let $g \in A$ and let $\alpha \in C$, $\beta > 0$, $|\alpha - \beta| < \beta$. If*

$$(16) \quad \frac{1-|z|^{2\beta}}{\beta}\left|(\alpha-1)\frac{zg'(z)}{g(z)} + 1 - \beta\right| \leq 1, \qquad \forall z \in U,$$

then the function

$$(17) \quad G(z) = \left(\alpha \int_0^z g^{\alpha-1}(u)du\right)^{1/\alpha}$$

is analytic and univalent in U.

Remark. For $\beta = 1$, from theorem 3.2 we find a result from paper [3].

Corollary 3.3. *Let $g \in A$, $\alpha \in C$, $\beta > 0$, $|\alpha - \beta| < \beta$. If*

$$\left|\frac{zg'(z)}{g(z)} - \frac{\beta - 1}{\alpha - 1}\right| \leq \frac{\beta}{|\alpha - 1|}, \qquad \forall z \in U,$$

then the function G defined by (17) is analytic and univalent in U.

Corollary 3.4. *Let $\alpha \in C$, $\beta > 0$ and let $g \in S^*(a,b)$, where $a = (\beta - 1)/(\alpha - 1)$, $b = \beta/|\alpha - 1|$. If one of the hypothesis*

$$i) \quad |\alpha - \beta| < \beta \quad for \quad \beta \in (0, -1 + \sqrt{2}]$$

ii) $|\alpha - \beta| < \beta$ *and* $|\alpha - 1| < (1 - \beta)/\beta Re(1 - \alpha)$, *for* $\beta \in (1 - \sqrt{2}, 1/2)$

is true, then the function G defined by (17) is analytic and univalent in U.

Proof. For this choise of a and b we must test if the conditions (4) are satisfied. Since $|\alpha - \beta| < \beta$ we get immediately $|a - 1| < b$ and the condition $Rea \geq b$ take place only in the case $\beta < 1/2$.
For the function $f \in A$, $f'(z) = \frac{g(z)}{z}$, from theorem 3.1 we get the following

Theorem 3.3. *Let $g \in A$ and let $\alpha \in C$, $\beta > 0$, $|\alpha - \beta| < \beta$. If*

$$(18) \quad \frac{1 - |z|^{2\beta}}{\beta}\left|\alpha\frac{zg'(z)}{g(z)} - \beta\right| \leq 1, \quad \forall z \in U,$$

then the function

$$(19) \quad G(z) = \left(\alpha \int_0^z \frac{g^\alpha(u)}{u} du\right)^{1/\alpha}$$

is analytic and univalent in U.

The operator (19) is just the integral operator introduced by Prof. P. T. Mocanu in the integral representation of α-convex functions.

Corollary 3.5. *Let $g \in A$, $\alpha \in C$, $\beta > 0$, $|\alpha - \beta| < \beta$. If*

$$\left|\frac{zg'(z)}{g(z)} - \frac{\beta}{\alpha}\right| \leq \frac{\beta}{|\alpha|}, \qquad \forall z \in U,$$

then the function G defined by (19) is analytic and univalent in U.

Remark . Let $\beta > 0$, $\alpha \in (0,\ 2\beta)$ and let $g \in S^*(\frac{\beta}{\alpha}, \frac{\beta}{\alpha})$. Then the function G defined by (19) is analytic and univalent in U.
Indeed, if we consider $a = \beta/\alpha$ and $b = \beta/|\alpha|$, the conditions (4) are satisfied for $\alpha \in (0,\ 2\beta)$.
If in theorem 3.1 we take $f \equiv g$, we have

Corollary 3.6. *Let $f \in A$, $\gamma \in C$, $\beta > 0$, $Re\gamma > 1/(2\beta)$. If*

(20) $\mid M(\ \gamma,\ f\ ) - \beta\gamma \mid \leq \beta|\gamma|$

for all $z \in U$, then the function f is univalent in U.

Proof. For $\gamma = 1/\alpha$, from $|\alpha - \beta| < \beta$ we get $Re\gamma > 1/(2\beta)$ and

$$\frac{1 - |z|^{2\beta}}{\beta}\left|\frac{zf''(z)}{f'(z)} + (\alpha - 1)\frac{zf'(z)}{f(z)} + 1 - \beta\right| =$$

$$= \frac{1 - |z|^{2\beta}}{\beta|\gamma|}\left|\gamma\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1 - \gamma)\frac{zf'(z)}{f(z)} - \beta\gamma\right|$$

If the condition (20) is true it follows, from theorem 3.1 that the function f is univalent in U.
Remark. For γ a real number, $\gamma > 1/(2\beta)$, where $\beta > 0$, the condition (20) implies $ReM(\gamma,\ f) > 0$ and from Theorem 1.1 we get that f is a γ-convex function.

References

[1] J. Becker, *Löwnersce Differentialgleichung und quasi-konform fortsetzbare schlichte Funktionen* , J. Reine Angew. Math., 255(1972), 23-43.

[2] P. T. Mocanu, *Une propriété de convexité généralisées dans la théoriede la representation conforme* , Mathematica (Cluj), 11(34),(1969), 127-133.

[3] N.N. Pascu, I. Radomir, *On the univalence of an integral* , Studia (Mathematica), XXXVI, 1(1991), 23-26.

[4] Ch. Pommerenke, *Über die Subordination analytischer Funktionen* , J. Reine Angew. Math., 218(1965), 159-173.

[5] P. Rotaru, *Subclasses of starlike functions* , Mathematica 29(52),(1987), 183-191.

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