

## ON CERTAIN PROPERTIES OF A CLASS OF UNIVALENT FUNCTIONS

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**Abstract.** We consider certain properties of the class of univalent functions defined earlier in [4].

### 1. Introduction and preliminaries

Let  $H$  denote the class of functions analytic in the unit disc  $\Delta = \{z : |z| < 1\}$  and let  $A \subset H$  be the class of normalized analytic functions  $f$  in  $\Delta$  such that  $f(0) = f'(0) - 1 = 0$ . Further, let

$$S^*(\beta) = \left\{ f \in A : \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta, 0 \leq \beta < 1, z \in \Delta \right\}$$

denote the class of *starlike functions of order  $\beta$* . We put  $S^* \equiv S^*(0)$  (the class of *starlike functions*). Also, let

$$K = \left\{ f \in A : \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, z \in \Delta \right\}$$

be the class of *convex functions in  $\Delta$* . It is well-known that these classes belong to the class of univalent functions in  $\Delta$  (see, for example [2]).

In [4] the first author has shown that the class of functions defined by the condition

$$(1) \quad \left| f'(z) \left( \frac{z}{f(z)} \right)^{1+\mu} - 1 \right| < \lambda, \quad z \in \Delta,$$

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Received September 3, 1998

2000 *Mathematics Subject Classification.* 30C45.

*Key words and phrases.* Univalent, starlike of order  $\beta$ , convex.

where  $0 < \mu < 1$ ,  $0 < \lambda \leq \frac{1-\mu}{\sqrt{(1-\mu)^2 + \mu^2}}$ , is the subclass of the class  $S^*$ .

In this paper for this class we consider starlikeness of order  $\beta$  and convexity. Also, we give a sufficient condition for  $f \in A$  to be in the class defined by (1).

For our results we need the following lemmas.

**Lemma A** ([3]). *Let  $\omega$  be a nonconstant and analytic function in  $\Delta$  with  $\omega(0) = 0$ . If  $|\omega|$  attains its maximum value on the circle  $|z| = r$  at  $z_0$ , we have  $z_0\omega'(z_0) = k\omega(z_0)$ ,  $k \geq 1$ .*

**Lemma B** ([6]). *Let  $p \in H$ . Then for  $k$ , a complex number satisfying  $\operatorname{Re} k \geq -|k|^2$ , we have*

$$|p(z) + kzp'(z)| < J \quad \text{implies} \quad |p(z)| < \frac{J}{k+1}, \quad z \in \Delta.$$

**Lemma C** ([6]). *Let  $0 < \mu < 1$ ,  $0 < \lambda \leq \frac{1-\mu}{\sqrt{(1-\mu)^2 + \mu^2}}$  and let  $Q$  be a function defined on  $\Delta$  satisfying the condition*

$$Q(z) \prec \frac{\lambda\mu}{1-\mu}z, \quad z \in \Delta.$$

*Suppose that*

$$\beta \equiv \beta(\mu, \lambda) = \frac{1-\mu-\lambda\sqrt{(1-\mu)^2 - \mu^2}}{1-\mu+\mu\lambda}.$$

*If  $p = 1 + p_1z + \dots \in H$  satisfies*

$$Q(z)(\beta + (1-\beta)p(z)) + (1-\beta)(p(z)-1) \prec \lambda z, \quad z \in \Delta,$$

*then  $\operatorname{Re} p(z) > 0$  in  $\Delta$ .*

We note that Lemma C was given in [6] in a slightly different form, but we need it in this form.

**Lemma D** ([7]). *Let  $0 < \lambda_1 < \lambda < 1$  and let  $Q$  be analytic in  $\Delta$  satisfying*

$$Q(z) \prec 1 + \lambda_1 z, \quad Q(0) = 1.$$

*If  $\omega \in H$ ,  $\omega(0) = 0$  and*

$$Q(z)[1 + \omega(z)] \prec 1 + \lambda z,$$

*then*

$$|\omega(z)| \leq \frac{\lambda + \lambda_1}{1 - \lambda_1} = \rho \leq 1, \quad \lambda + 2\lambda_1 \leq 1.$$

## 2. Main results

We consider first the problem of starlikeness of order  $\beta$ .

**Theorem 1.** *Let  $f \in A$  satisfy the condition (1), where  $0 < \mu < 1$  and  $0 < \lambda \leq \frac{1-\mu}{\sqrt{(1-\mu)^2 + \mu^2}}$ . Then  $f \in S^*(\beta)$  where  $\beta$  is defined by (2).*

*Proof.* If we put  $Q(z) = \left(\frac{z}{f(z)}\right)^\mu - 1$ , then we have

$$Q(z) - \frac{1}{\mu} z Q'(z) = \left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) - 1.$$

From this and the condition (1) we get

$$(3) \quad \left| Q(z) - \frac{1}{\mu} z Q'(z) \right| < \lambda.$$

Since  $0 < \mu < 1$  implies  $-\frac{1}{\mu} \geq -\frac{1}{\mu^2}$ , from (3) and Lemma B we obtain that

$$Q(z) \prec \frac{\lambda\mu}{1-\mu} z, \quad z \in \Delta.$$

On the other hand, since

$$\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) - 1 = Q(z)(\beta + (1-\beta)p(z)) + (1-\beta)(p(z) - 1) \prec \lambda z,$$

where  $\frac{zf'(z)}{f(z)} = \beta + (1-\beta)p(z)$ , we have by Lemma C that  $\operatorname{Re} p(z) > 0$ , i.e.  $f \in S^*(\beta)$  where  $\beta$  is defined by (2).

**Theorem 2.** *Let  $f \in A$  and let*

$$(4) \quad \left| \frac{zf'(z)}{f(z)} - \frac{1}{\mu+1} \frac{zf''(z)}{f'(z)} - 1 \right| < \frac{\lambda}{(1+\mu)(1+\lambda)}, \quad z \in \Delta,$$

where  $0 \leq \mu \leq 1$ ,  $0 < \lambda \leq 1$ . Then

$$\left| \left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) - 1 \right| < \lambda.$$

*Proof.* Let's put  $\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) - 1 = \lambda\omega(z)$ . Then  $\omega$  is analytic in  $\Delta$  and  $\omega(0) = 0$ . After logarithmic differentiation, we have

$$(5) \quad \frac{zf'(z)}{f(z)} - \frac{1}{\mu+1} \frac{zf''(z)}{f'(z)} - 1 = -\frac{1}{1+\mu} \frac{\lambda z \omega'(z)}{1 + \lambda \omega(z)}.$$

We want to prove that  $|\omega(z)| < 1$ ,  $z \in \Delta$ . If not, then by Lemma A there exists  $z_0$ ,  $|z_0| < 1$ , such that  $|\omega(z_0)| = 1$  and  $z_0\omega'(z_0) = k\omega(z_0)$ ,  $k \geq 1$ . If we put  $\omega(z_0) = e^{i\theta}$ , then from (5) we have

$$\left| \frac{z_0 f'(z_0)}{f(z_0)} - \frac{1}{\mu + 1} \frac{z_0 f''(z_0)}{f'(z_0)} - 1 \right| = \frac{\lambda}{1 + \mu} \frac{k}{|1 + \lambda e^{i\theta}|} \geq \frac{\lambda}{(1 + \mu)(1 + \lambda)}$$

which is a contradiction to (5). Now we get  $|\omega(z)| < 1$ ,  $z \in \Delta$ , which gives the desired conclusion.

We note that for  $\mu$  and  $\lambda$  given in Theorem 1, the condition (4) implies starlikeness. Especially, for  $\mu = 0$  we have

$$\left| \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} - 1 \right| < \frac{\lambda}{1 + \lambda} \implies \left| \frac{zf'(z)}{f(z)} - 1 \right| < \lambda, \quad z \in \Delta,$$

and for  $\mu = 1$ :

$$\left| \frac{zf'(z)}{f(z)} - \frac{1}{2} \frac{zf''(z)}{f'(z)} - 1 \right| < \frac{\lambda}{2(1 + \lambda)} \implies \left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda, \quad z \in \Delta.$$

(for the class defined by the condition  $\left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda$ ,  $z \in \Delta$ , see the paper [5]).

**Theorem 3.** Let  $f \in A$  satisfy the condition (1) with  $0 \leq \mu < \frac{1}{2}$  and  $0 < \lambda \leq \frac{1-\mu}{1+\mu}$ . Then  $f$  is convex at least in  $|z| < r = \max\{\min\{r_0, r_1\}, r_2\}$ , where

$$(5) \quad r_0 = \frac{\lambda}{1 + \sqrt{1 - \lambda^2}}$$

and  $r_1, r_2$  are the smallest positive roots of the equations

$$(6) \quad \lambda_1 \lambda r^2 - (\lambda_1 + 2\lambda)r + 1 = 0$$

and

$$(7) \quad \lambda_1 r^3 - r^2 - (\lambda_1 + 2r_0)r + 1 = 0,$$

respectively, and where

$$(8) \quad \lambda_1 = \frac{(1 + \mu)\lambda}{1 - \mu - \mu\lambda}.$$

*Proof.* From the condition (1) we have that  $\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) = 1 + \lambda\omega(z)$ , where  $\omega(0) = 0$  and  $|\omega(z)| < 1$ ,  $z \in \Delta$ . From the above equation we obtain

$$(1 + \mu) \left(1 - \frac{zf'(z)}{f(z)}\right) + \frac{zf''(z)}{f'(z)} = \frac{\lambda z\omega'(z)}{1 + \lambda\omega(z)},$$

which implies that

$$(9) \quad \left| \frac{zf''(z)}{f'(z)} \right| \leq (1 + \mu) \left| \frac{zf'(z)}{f(z)} - 1 \right| + \frac{\lambda|z||\omega'(z)|}{1 - \lambda|\omega(z)|}.$$

In Theorem 1 we obtained that

$$\left(\frac{z}{f(z)}\right)^{\mu} \prec 1 + \frac{\mu\lambda}{1 - \mu}z,$$

and since the condition (1) is equivalent to

$$\frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)}\right)^{\mu} \prec 1 + \lambda z,$$

by Lemma D we have

$$(10) \quad \frac{zf'(z)}{f(z)} < 1 + \rho z,$$

where  $\rho = \frac{\lambda + \frac{\mu\lambda}{1-\mu}}{1 - \frac{\mu\lambda}{1-\mu}} = \frac{\lambda}{1-\mu-\mu\lambda}$  if  $\lambda + \frac{2\mu\lambda}{1-\mu} \leq 1$  ( $\iff \lambda \leq \frac{1-\mu}{1+\mu}$ ) and  $0 < \mu < \frac{1}{2}$ . Also

$$(11) \quad |\omega'(z)| \leq \frac{1 - |\omega(z)|^2}{1 - |z|^2} \quad (\text{see, for example [1]}).$$

By using (9), (10) and (11) we have the next estimation

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{(1 + \mu)\lambda|z|}{1 - \mu - \mu\lambda} + \frac{\lambda|z|}{1 - |z|^2} \frac{1 - |\omega(z)|^2}{1 - \lambda|\omega(z)|}, \quad z \in \Delta.$$

If we consider the function  $\Phi(|\omega|) = \frac{1 - |\omega(z)|^2}{1 - \lambda|\omega(z)|}$ , with  $|\omega(z)| \leq |z|$ , then we can conclude that the function  $\Phi$  has the property:



$$(13) \quad \Phi(|\omega|) \leq \begin{cases} \frac{1-|z|^2}{1-\lambda|z|}, & |z| \leq r_0 \\ 2\frac{r_0}{\lambda}, & r_0 \leq |z| < 1 \text{ (also for } |z| < 1) \end{cases},$$

where  $r_0$  is defined by (5). Now, from relation (12), (13) and notation (8) we have that

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq \lambda_1|z| + \frac{\lambda|z|}{1-\lambda|z|} < 1 \quad (|z| \leq r_0),$$

and

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq \lambda_1|z| + \frac{2r_0|z|}{1-|z|^2} < 1 \quad (|z| \leq 1)$$

are satisfied for  $|z| < r$ , where  $r$  is defined in the statement of the theorem.

**Examples.** 1° If we put  $\mu = 0$ ,  $\lambda = 1$ , then we have that the function  $f \in A$  for which  $\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1$  is convex at least in  $|z| < \frac{3-\sqrt{5}}{2}$ .

2° For  $\mu = 1/9$ ,  $\lambda = 4/5$  we have that  $r_0 = 1/2$ ,  $r_1 = \frac{61-\sqrt{1921}}{40} = 0.4292\dots$ ,  $r_2 = 0.4281\dots$  i.e. a function  $f \in A$  which satisfies the condition  $\left| \left( \frac{zf'(z)}{f(z)} \right)^{10/9} f'(z) - 1 \right| < \frac{4}{5}$  in  $\Delta$ , is convex at least in  $|z| < r_1$ .

**Acknowledgement.** The work of the first author was supported by Grant No. 04M03 of MNTRS through Math. Institute SANU.

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