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A STUDY OF A NEW CLASS OF IDEALS IN SEMIRING

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Abstract. In this paper we would like to introduce a new class of ideals in semirings. Then defining a new form of regularity, compatible with this new class of ideals, our aim is to explore the possibilities of establishing a new ideal theory in semirings, going alongside the existing literature of semiring theory.

1. Introduction

In this paper by a semiring we mean a non-empty set S together with two binary operations '$+$' and '$\cdot$' (usually denoted by juxtaposition) such that $(S, +)$ is a commutative semigroup and $(S, \cdot)$ is a semigroup, which are connected by ring-like distributivity. An additively cancellative semiring is called a halfring. An inversive semiring S is a semiring in which $(S, +)$ is an inversive semigroup, i.e. for each $a \in S$ there is a unique element $a' \in S$ such that $a + a' + a = a$ and $a' + a + a' = a'$ [7]. It is well-known [5] that in an inversive semiring S, we have $(ab)' = a'b = ab'$ and $(a + b)' = a' + b'$. A semiring S is called E-inversive, if for every $a \in S$, there exists $x \in S$ such that $a+x \in E^+(S)$, where $E^+(S)$ denotes the set of all additive idempotents of a semiring S. The set of all multiplicative idempotents of a semiring S is denoted by $E^0(S)$. An element $s \in S$ is called a zeroid element of S if $s + a = a$, for some $a \in S$. We denote by $Z(S)$, the set of all zeroid elements of S. If S is a semiring with zero 0, then $0 \in Z(S)$. If $Z(S) \neq \emptyset$ and $Z(S)$ is a proper subset of S, then S is called non-zeroic [2]. The zero element of S, denoted by 0, is called an absorbing zero if $a \cdot 0 = 0 \cdot a = 0$ for all $a \in S$.

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A k-ideal [3] I of a semirnig S is an ideal of S such that if $a \in I$ and $x \in S$ and $a + x \in I$, then $x \in I$. An h-ideal [6] J of a semiring S, is an ideal of S such that, if $x, z, \in S$ with $i_1, i_2 \in J$ such that $x + i_1 + z = i_2 + z$ implies $x \in J$. An ideal I of a semiring is called full if $E^+(S) \subseteq I$. A subsemiring H of a direct product of two semirings S and T is called a subdirect product of S and T if the two projection mappings $\pi_1 : H \to S$ given by $\pi_1(s,t) = s$ and $\pi_2 : H \to T$ given by $\pi_2(s,t) = t$ are surjective. Throughout the paper N stands for the set of natural numbers.

2. p-ideals of a semiring

It is well-known that a ring R contains only one additive idempotent, namely the zero element. In a semiring S with additive idempotents, the set $E^+(S)$ forms an ideal of S, which is not necessarily a k-ideal of S. Another generalization of zero element in the theory of semirings is the concept of zeroid, $Z(S)$ of a semiring S. Clearly, $Z(R) = \{0\}$ for any ring R. We point out that $Z(S)$ is a k-ideal of S. In fact, it is the smallest h-ideal of S. Now, let us consider the set $P^+(S) = \{x \in S : nx = (n+1)x \text{ for some } n \in N\}$ which consists of some additively periodic elements of S. Clearly, $P^+(R) = \{0\}$ for any ring R. We note that $P^+(S)$ is an ideal of S, which is not necessarily a k-ideal, but $P^+(S)$ satisfies the following property:

Proposition 2.1. *In a semiring S, let $a \in P^+(S)$ such that for some $b \in S$ and some $n \in N$, $a + nb = (n+1)b$ holds. Then $b \in P^+(S)$.*

Proof. We must have $ma = (m+1)a$ for some $m \in N$, whence $ma + mnb = m(n+1)b$ implies $(m+1)a + mnb + nb = m(n+1)b + nb$, so that $(m+1)(n+1)b = (mn+m+n)b$ gives that $b \in P^+(S)$. □

This motivates the following definition.

Definition 2.2. An ideal I of a semiring S is called a p-ideal if for some $x \in S$, $n \in N$, $nx + a = (n+1)x$ and $a \in I$ implies $x \in I$.

In particular, if S is inversive, then the definition boils down to the following: if for some $x \in S$, $a + x = 2x$, $a \in I$ then $x \in I$.

We observe that in any halfring, every ideal is p-ideal. But all p-ideals are not k-ideals, as the ideal $I = 3Z_0^+ \setminus \{3\}$ is not a k-ideal, in the halfring Z_0^+ of all positive integers with zero. We also note that k-ideals are not p-ideals in general. Indeed, in the semiring $(Z^+, \max, \min)$, $I_n = \{1, 2, \ldots, , n\}$ is a k-ideal for any $n \in Z^+$ but not a p-ideal. We now present few natural examples of p-ideals in different classes of semirings.

Proposition 2.3. *In an inversive semiring S, $E^+(S)$ is a p-ideal. In fact, any full ideal of S is a p-ideal.*

Proof. In fact, for any $e \in E^+(S)$, $e+x = 2x$ implies $e+(x+x') = 2x+x' = x$ whence $x \in E^+(S)$ as $x + x' \in E^+(S)$ for all $x \in S$. □

Proposition 2.4. *In an E-inversive semiring S every full k-ideal is a p-ideal.*

Proof. Let I be a full k-ideal of S, i.e. $E^+(S) \subseteq I$. For some $a \in I$, let $a + nx = (n+1)x$, $n \in N$. Clearly there exists $y \in S$ such that $nx + y = e \in E^+(S)$ (as S is E-inversive). So, $a+(nx+y) = (nx+y)+x$ i.e. $a+e = x+e$ whence $x + e \in I$, as $a + e \in I$, as I is full; but I is also a k-ideal of S, whence $x \in I$. Consequently, I is a p-ideal of S. □

Proposition 2.5. *In a non-zeroic semiring S, the zeroid $Z(S)$ is a p-ideal of S.*

Proof. Let $x \in S$ such that $a+nx = (n+1)x$ for some $a \in Z(S)$ and $n \in N$. Since $a \in Z(S)$, $a + y = y$ for some $y \in S$. Hence, $a + nx + y = (n+1)x + y$ so that $y + nx = nx + x + y$ which implies $x \in Z(S)$. Hence $Z(S)$ is a p-ideal. □

Proposition 2.6. *In an inversive semiring S an ideal I is a p-ideal if and only if $I = I + E^+(S)$.*

Proof. For any p-ideal I, $a + (a + e) = 2(a + e)$ for all $a \in I$, $e \in E^+(S)$ implies $a + e \in I$ so that $I + E^+(S) \subseteq I$. Again, $a = a + (a + a')$ implies $I \subseteq I+E^+(S)$ since $a+a' \in E^+(S)$ for all $a \in S$. Conversely, suppose I is an ideal of S satisfying $I = I + E^+(S)$. Then for some $A \in I$ $a + x = 2x$ holds, implying that $a+(x+x') = 2x+x'$ i.e. $a+(x+x') = x$ i.e. $x \in I+E^+(S) = I$, whence I is a p-ideal of S. □

Corollary 2.7. *In an inversive semiring S, sum of any two p-ideals is also a p-ideal.*

Proof. Follows trivially from Proposition 2.6 □

Proposition 2.8. *For two p-ideals I, J of a semiring S, $I \cap J$ is also so.*

Proof. Straight forward. □

We observe that this result can be extended to arbitrary family of p-ideals of S. We point out that for any two left (right) p-ideals I and J, defined as usual, $I \cap J$ is a left (right) p-ideal provided that $I \cap J \neq \emptyset$. Now, in order to search the smallest p-ideal containing a given ideal, we introduce the following:

Definition 2.9. For any subsemiring R of a semiring S, we define

$$\hat{R} = \{x \in S : a + nx = (n+1)x \text{ for some } n \in N, a \in R\}.$$

Proposition 2.10. *For any two ideals I, J of S we see that $\hat{I}$ is a p-ideal of S such that $\hat{\hat{I}} = \hat{I}$, $I \subset \hat{I}$, if $I \subseteq J$ then $\hat{I} \subset \hat{J}$; indeed, $\hat{I}$ is the smallest p-ideal of S containing I.*

Proof. Let I be an ideal of a semiring S. Then defining $\hat{I}$ as in 2.9 it is plain to see that $I \subseteq \hat{I}$. Let $x, y \in \hat{I}$; let $a + nx = (n+1)x$ for some $a \in I$ and $n \in N$ and $b + my = (m+1)y$ for some $b \in I$ and $m \in N$. Then, $(a+b) + k(x+y) = (k+1)(x+y)$ where $a + b \in I$, $k = \max(n, m) \in N$, whence we have $x + y \in \hat{I}$. Again, for some $s \in S$ it is easy to see that $xs \in \hat{I}$ and $sx \in \hat{I}$ hold. Thus $I \subseteq \hat{I}$ indicates $\hat{I} \subseteq \hat{\hat{I}}$. To prove the reverse inclusion, let $x \in \hat{\hat{I}}$. Then there exists some $a \in \hat{I}$ such that $a + nx = (n+1)x$, for some $n \in N$. Again, as $a \in \hat{I}$, there exists $b \in I$ such that $b + ma = (m+1)a$ for some $m \in N$. Now through some calculations it can be shown that $b + rx = (r+1)x$, where $b \in I$, $r = mn + m + n \in N$, which indicates that $\hat{\hat{I}} \subseteq \hat{I}$ whence $\hat{I} = \hat{\hat{I}}$.

It is a routine matter to check that for two ideals I, J of S with $I \subseteq J$ we have $\hat{I} \subseteq \hat{J}$. Also, we observe that if I is a p-ideal then $I = \hat{I}$. Indeed, for $x \in \hat{I}$ there exists $a \in I$ such that $a + nx = (n+1)x$, showing that $x \in I$ so that $\hat{I} \subseteq I$. □

We point out that in case of an inversive semiring S, for an ideal I of S we have $\hat{I} = \{x \in S | a + x = 2x \text{ for some } a \in I\}$. It is worth noticing that in a semiring S with absorbing zero 0, we have $\hat{\{0\}} = P^+(S)$.

Corollary 2.11. *In an inversive semiring S, we have $\hat{I} = I + E^+(S)$ for any ideal I.*

Proof. We see that $I + E^+(S) \subseteq \hat{I}$ follows, since for any $a \in I$, $a + e \in \hat{I}$ for any $e \in E^+(S)$. Indeed, $a + (e + a) = 2(a + e)$ shows that $a + e \in \hat{I}$. Now, let $x \in \hat{I}$. Then for some $a \in I$, $a + x = 2x$ implies $a + x + x' = 2x + x' = x$ so that $x \in I + E^+(S)$, as $x + x' \in E^+(S)$ for all $x \in S$. □

Proposition 2.12. *Let a semiring S be a subdirect product of a distributive lattice D and a ring R. Then I is a full ideal of S if and only if I is a p-ideal of S.*

Proof. Let S be a subdirect product of a distributive lattice D and a ring R. Then as in [1], it can be easily seen that S is an inversive semiring, whence any full ideal of S is a p-ideal of S (by 2.3). Conversely, let I be a p-ideal of the semiring S. As $I \neq \{0\}$, there exists some $a \in I$, so that $a = (\gamma, r)$ for some $\gamma \in D$ and $r \in R$. Now, idempotents of S are of the form $(\alpha, 0)$ for each $\alpha \in D$. As I is an ideal of S, $(\gamma, r)(\alpha, 0) = (\gamma\alpha, 0) \in I$. We see that

$(\gamma\alpha, 0) + (\alpha, 0) = (\gamma\alpha + \alpha, 0) = (\alpha, 0) = 2(\alpha, 0)$ which implies $(\alpha, 0) \in I$ for all $\alpha \in D$. Hence I is a full ideal of S. $\square$

3. p-regular semirings

In this section we define p-regularity of a semiring S and try to obtain several characterizations of p-regularity of S in connection with p-ideals of S.

Definition 3.1. A semiring S is called p-regular, if for each $a \in S$ there exists some $b \in S$ such that

$$na + aba = (n+1)a \text{ for some } n \in N. \tag{3.1}$$

If S is an inversive semiring this relation reduces to

$$a + aba = 2a. \tag{3.2}$$

We point out that any multiplicative regular semiring S [i.e. in which $(S, \cdot)$ is regular semigroup] is p-regular, but the converse is not necessarily true, as it is evident from the following example. Moreover, this example shows that even there exists semirings satisfying (3.1), but not (3.2).

Example 3.2. We define a relation ρ on Z^+ as follows: $m\rho n$ if and only if, either $m = n$ or $m, n > 6$ and 3 devides $(m - n)$, for all $m, n \in Z^+$. It is a routine matter to verify that ρ, as defined above, is a congruence on Z^+. We have the congruence classes as follows: $1\rho = \{1\}$, $2\rho = \{2\}$, $3\rho = \{3\}$, $4\rho = \{4\}$, $5\rho = \{5\}$, $6\rho = \{6\}$, $7\rho = \{3n+7 | n \in Z_0^+\}$, $8\rho = \{3n+8 | n \in Z_0^+\}$, $9\rho = \{3n + 9 | n \in Z_0^+\}$. Then we consider $S = Z^+/\rho$; for convenience, we write $a\rho$ as $\overline{a}$, for all $a = 1, 2, \ldots, 9$. We see that $S = \{x \in Z^+ | x < 7\} \cup \{\overline{7}, \overline{8}, \overline{9}\}$. Under usual addition and multiplication of classes it can be easily seen that S is a semiring. Now we observe that $\overline{1} \cdot \overline{1} \cdot \overline{1} = \overline{1}$, $3\,\overline{2} + \overline{2} \cdot \overline{2} \cdot \overline{2} = 4\,\overline{2}$; $3\,\overline{3} + \overline{3} \cdot \overline{1} \cdot \overline{3} = 4\,\overline{3}$; $\overline{4} + \overline{4} \cdot \overline{1} \cdot \overline{4} = 2\,\overline{4}$; $\overline{5} + \overline{5} \cdot \overline{2} \cdot \overline{5} = 2\,\overline{5}$; $\overline{6} + \overline{6} \cdot \overline{1} \cdot \overline{6} = 2\,\overline{6}$; $\overline{7} \cdot \overline{1} \cdot \overline{7} = \overline{7}$; $\overline{8} \cdot \overline{2} \cdot \overline{8} = \overline{8}$; $\overline{9} \cdot \overline{1} \cdot \overline{9} = \overline{9}$.

Since $2x2 = 4x \geq 4 > 2$ so $\overline{2} \cdot \overline{x} \cdot \overline{2} \neq \overline{2}\ \forall \overline{x} \in S$. Also, as $2 + 2x2 = 2 + 4x \geq 6 > 4 = 2(2)$, we have $\overline{2} + \overline{2} \cdot \overline{x} \cdot \overline{2} \neq 2\overline{2}\ \forall \overline{x} \in S$. Similar is the case for $\overline{3}$. Again, $4x4 = 16x > 4$ indicates $\overline{4} \cdot \overline{x} \cdot \overline{4} \neq \overline{4}\ \forall \overline{x} \in S$. Hence for each $a \in S$, there exists some $b \in S$ such that $na + aba = (n+1)a$ but $a + aba = 2a$ does not hold for all $a \in S$ (eg. $\overline{2} \in S$) and $aba = a$ does not hold for all $a \in S$ (eg. $\overline{4} \in S$). Now, it is also interesting to see that $2S = \{\overline{2}, \overline{4}, \overline{6}, \overline{7}, \overline{8}, \overline{9}\}$ is an ideal of S which is not a p-ideal as $\overline{2} + \overline{5} = 2\overline{5}$ but $\overline{5} \notin 2S$. We assert that $\widehat{2S} = S$. Indeed, $\overline{7} + 7\,\overline{1} = 8\,\overline{1}$, $\overline{6} + 2\,\overline{3} = 3\,\overline{3}$,

$\overline{2}+2\,\overline{5}=3\,\overline{5}$, shows that $\overline{1},\overline{3},\overline{5}\in\widehat{2S}$ whence $\widehat{2S}=S$. But $\overline{x}+\overline{1}=2\,\overline{1}$ has the only solution $\overline{x}=\overline{1}$ and also $\overline{x}+\overline{3}=2\,\overline{3}$ has the only solution $\overline{x}=\overline{3}$ in S. Therefore, $\{x\in S|a+x=2x \text{ for some } a\in 2S\}\neq\widehat{2S}=\{x\in S|a+nx=(n+1)x \text{ for some } n\in N \text{ and } a\in 2S\}$. □

It is well-known that the zero element of a ring is regular (in the sense of usual literature of ring theory). We point out that $P^+(S)$ is p-regular for any semiring S with an absorbing zero.

Proposition 3.3. *A sufficient condition for $Z(S)$ of a non-zeroic semiring S to be p-regular is $Z(S)\subseteq E^0(S)$.*

Proof. Indeed, in a non-zeroic semiring S, satisfying the given condition, $b+bbb=2b$ for any $b\in Z(S)$, proving our claim. □

However the following examples show that the condition is not necessary.

Example 3.4. a) Any additively idempotent semiring with an absorbing zero is p-regular. b) Any additively idempotent semiring with zero multiplication is p-regular. c) Inclines [4] are p-regular. The following semiring S is not p-regular but the corresponding zeroid $Z(S)$ is p-regular.

d) A subdirect product S of a distributive lattice D and a non-regular ring is a semiring which is not p-regular. But, $Z(S)=D\times\{0\}$, showing that $Z(S)$ is p-regular.

Proposition 3.5. *In a p-regular semiring S, every ideal of S is p-regular.*

Proof. Let I be an ideal of a p-regular semiring S. Let $a\in I$. Then there exists some $b\in S$ such that $na+aba=(n+1)a$ for some $n\in N$. Now, since I is an ideal we have $bab\in I$, whence $n(n+1)a+ababa=n^2a+naba+ababa=n^a+(na+aba)ba=n^2a+(n+1)aba=n(na+aba)+aba=n(n+1)a+aba=n^2a+(na+aba)=n^2a+(n+1)a=(n^2+n+1)a$ which shows that I is p-regular. □

Proposition 3.6. *In a p-regular semiring S, if for some $a,b\in S$ we have $na+aba=(n+1)a$, for some $n\in N$, then*

$$ma+a(bab)a=(m+1)a \text{ for some } m\in N \text{ and} \tag{3.3}$$

$$m'(bab)+(bab)a(bab)=(m'+1)bab \text{ for some } m'\in N. \tag{3.4}$$

Proof. We observe that (3.3) follows from Proposition 3.5 where $m=n^2+n\in N$. Now, to justify the condition (3.4) we see that $na+aba=(n+1)a$, so we have $nbab+babab=(n+1)bab$ which implies $n(n+1)bab+(n+1)babab=$

$(n+1)^2bab$, i.e. $(n^2+n)bab + bab((n+1)a)b = (n^2+2n+1)bab$, i.e. $n^2bab + nbab + bab(na+aba)b = (n^2+2n+1)bab$, i.e. $n^2bab + nbab + nbabab + (bab)a(bab) = (n^2+2n+1)bab$, i.e. $nbab + n(nbab + babab) + (bab)a(bab) = (n^2+2n+1)bab$, i.e. $nbab + nb(na+aba)b + (bab)a(bab) = (n^2+2n+1)bab$, i.e. $nbab + n(n+1)bab + (bab)a(bab) = (n^2+2n+1)bab$, i.e. $(n^2+2n)bab + (bab)a(bab) = (n^2+2n+1)bab$, i.e. $m'(bab) + (bab)a(bab) = (m'+1)bab$, where $m' = n^2+2n \in N$. $\square$

Definition 3.7. In a semiring S an element e is called p-idempotent if $ne + e^2 = (n+1)e$ for some $n \in N$.

In the case of an inversive semiring S this definition reduces to $e + e^2 = 2e$. Clearly any multiplicatively idempotent element of S is p-idempotent. However, the converse fails as can be seen from the following:

In Example 3.2, $\overline{4}$ is p-idempotent but not multiplicatively idempotent.

Proposition 3.8. *In a semiring S, p-idempotents are p-regular.*

Proof. Let $e \in S$ be a p-idempotent, i.e. $ne + e^2 = (n+1)e$ for some $n \in N$. Then we have $ne^2 + e^3 = (n+1)e^2$ i.e. $n(ne+e^2) + e^3 = n(ne+e^2) + e^2$ i.e. $n(n+1)e + e^3 = n(n+1)e + e^2$ i.e. $(n^2+n)e + e^3 = n^2e + (ne+e^2) = n^2e + (n+1)e$ i.e. $(n^2+n)e + eee = (n^2+n+1)e$ whence e is p-regular. $\square$

Theorem 3.9. *A semiring S with 1_S is p-regular if and only if for every right p-ideal A and left p-ideal B we have $A \cap B = \widehat{AB}$.*

Proof. Let S be a p-regular semiring. Obviously $AB \subseteq A$ and $AB \subseteq B$ so that $\widehat{AB} \subseteq \hat{A} = A$ and $\widehat{AB} \subseteq \hat{B} = B$. Hence $\widehat{AB} \subseteq A \cap B$. Conversely, let $a \in A \cap B$, then $na + aba = (n+1)a$, for some $b \in S$, $n \in N$. Now, $aba \in AB$ whence $a \in \widehat{AB}$ and consequently, $A \cap B \subseteq \widehat{AB}$, so that $A \cap B = \widehat{AB}$.

Let us now assume that the given condition holds. Then for any $a \in S$, $a \in \widehat{aS} \cap \widehat{Sa} = \widehat{\widehat{aS}\widehat{Sa}}$ implies that there exists some $xy \in \widehat{aS}\widehat{Sa}$, such that

$$xy + na = (n+1)a \text{ for some } n \in N. \tag{i}$$

We also have $x \in \widehat{aS}$ and $y \in \widehat{Sa}$ so that

$$ar_1 + mx = (m+1)x \tag{ii}$$

$$r_2a + ky = (k+1)y \tag{iii}$$

for some $r_1, r_2 \in S$ and $m, k \in N$. Now, from (ii) we have $ar_1y + mxy = (m+1)xy$, whence $ar_1y + (mn+m+n)a = (mn+m+n+1)a \ldots$ [by (i)], i.e.

$$ar_1y + (mn+m+n)a = (mn+m+n+1)a \quad \text{[by (i)]}. \tag{iv}$$

Again, from (iii) we have $ar_1r_2a + kar_1y = (k+1)ar_1y$ i.e.

$$ar_1r_2a+kar_1y+k(mn+m+n)a = (k+1)ar_1y+k(mn+m+n)a \quad \text{[by (iv)]}$$

from which we can show that $pa + ar_1r_2a = (p+1)a$ where $p = kmn + km + kn + k + mn + m + n \in N$, i.e. $pa + ara = (p+1)a$ where $r = r_1r_2 \in S$. Since a was chosen arbitrarily we conclude that S is p-regular. $\square$

Definition 3.10. A p-ideal B of a semiring S is called idempotent if $B = \hat{B}^2$.

Theorem 3.11. *A multiplicatively commutative semiring S is p-regular if and only if every p-ideal is idempotent.*

Proof. Let S be a p-regular semiring which is multiplicatively commutative. Then by Theorem 3.9 with $A = B$, we have $A \cap A = \widehat{AA}$ i.e. $A = \hat{A}^2$ and hence A is an idempotent. Conversely, let A, B be any two p-ideals of S. Using idempotence of p-ideals of S we get $A \cap B = \widehat{(A \cap B)}^2$. Let $a \in \widehat{(A \cap B)}^2$, so that there exists some $b \in (A \cap B)^2$ such that $b + na = (n+1)a$ for some $n \in N$; let $b = b_1b_2$, where $b_1, b_2 \in AB$; then we have $b_1b_2 + na = (n+1)a$ whence $a \in \widehat{AB}$ as $b_1b_2 \in AB$, so that $A \cap B \subseteq \widehat{AB}$. Again, $AB \subseteq A$ and $AB \subseteq B$ shows that $AB \subseteq A \cap B$, i.e. $\widehat{AB} \subseteq \widehat{A \cap B} = A \cap B$. Consequently, $\widehat{AB} = A \cap B$. Hence the p-regularity of S follows from 3.9. $\square$

Theorem 3.12. *In a multiplicatively commutative semiring S with 1_S, the condition $\widehat{Sab} = \widehat{Sa} \cap \widehat{Sb}$, $a, b \in S$ is equivalent to p-regularity of S, where $\widehat{Sa}$ is the p-ideal generated by a.*

Proof. In a semiring with given conditions we may have $\widehat{Sa^2} = \widehat{Sa} \cap \widehat{Sa}$ for any $a \in S$, i.e. $\widehat{Sa^2} = \widehat{Sa}$, whence we have $a \in \widehat{Sa^2}$, so that, for some $s \in S$, $sa^2 + na = (n+1)a$, $n \in N$; i.e. $na + asa = (n+1)a$, whence S is p-regular. Conversely, in a p-regular semiring S by 3.9 we have $\widehat{Sa} \cap \widehat{Sb} = \widehat{\widehat{Sa}\widehat{Sb}}$. Now we observe that $Sab = Sa1_Sb \subseteq Sa\,Sb \subseteq \widehat{Sa}\widehat{Sb}$, so that $\widehat{Sab} \subseteq \widehat{\widehat{Sa}\widehat{Sb}}$; conversely, let $y \in \widehat{\widehat{Sa}\widehat{Sb}}$, then through some calculations essentially similar to that of 3.9 we can show that $y \in \widehat{Sab}$, i.e. we have $\widehat{\widehat{Sa}\widehat{Sb}} \subseteq \widehat{Sab}$, whence the theorem follows. $\square$

Definition 3.13. A p-ideal I of a semiring S is said to be semiprime if and only if $I = \widehat{\sqrt{I}}$, where $\sqrt{I} = \{a \in S | a^n \in I \text{ for some } n \in N\}$.

Lemma 3.14. *In a commutative p-regular semiring S, every p-ideal I satisfies $I = \sqrt{I}$.*

Proof. Since $I \subseteq \sqrt{I}$ is trivially true, it suffices to show the reverse inclusion only. Let $0 \neq a \in \sqrt{I}$. Then $a^n \in I$, for some $n \in N$. Now, as S is p-regular, there exists some $b \in S$ such that $ma + aba = (m+1)a$ for some $m \in N$, i.e. $ma + a^2b = (m+1)a$, i.e. $ma^{n-1} + a^nb = (m+1)a^{n-1}$, $m \in N$. Now, $a^n \in I$ implies $a^nb \in I$ and as$I$ is a p-ideal, so $a^{n-1} \in I$. Repeating this process enough number of times, ultimately, we have $a \in I$. Consequently, $\sqrt{I} \subseteq I$, so that $I = \sqrt{I}$. □

Theorem 3.15. *A commutative semiring S is p-regular if and only if every p-ideal of S is semiprime.*

Proof. The condition is necessary by Lemma 3.14. Going in the other direction, let us assume that S is a commutative semiring in which every p-ideal I is semiprime, i.e. $I = \sqrt{I}$. Now, for any $0 \neq a \in S$ we consider the p-ideal $\widehat{Sa^2}$. As we know that $a^3 \in \widehat{Sa^2}$ and every p-ideal is semiprime, we have $a \in \sqrt{Sa^2} \subseteq \sqrt{\widehat{Sa^2}} = \widehat{Sa^2}$; so that for some $s \in S$ we have $sa^2 + pa = (p+1)a$ for some $p \in N$, i.e. $pa + asa = (p+1)a$ for some $p \in N$, showing the p-regularity of the semiring S. □

Theorem 3.16. *If a semiring S with 1_S is p-regular, then we have for $a \in S$, $\widehat{Sa} = \widehat{Se}$, where e is a p-idempotent of S, i.e. every principal left p-ideal is generated by a p-idempotent.*

Proof. For any $a \in S$, there exists $b \in S$ such that $na + aba = (n+1)a$ for some $n \in N$. Then $nba + baba = (n+1)ba$ for some $n \in N$, shows that ba is a p-idempotent. Let $ba = e$, so that we have $ne + e^2 = (n+1)e$ for some $n \in N$. Let $p \in \widehat{Se}$; then there exists $r_1 \in S$ such that $r_1e + mp = (m+1)p$ for some $m \in N$, i.e. $r_1ba + mp = (m+1)p$, i.e. $r_2a + mp = (m+1)p$ for $r_2 = r_1b \in S$, so that $p \in \widehat{Sa}$, whence $\widehat{Se} \subseteq \widehat{Sa}$. To prove the reverse inclusion, let $t \in \widehat{Sa}$, i.e.

$$\text{(i)} \qquad ra + kt = (k+1)t, \text{ for some } r \in S \text{ and } k \in N.$$

Now, $na+aba = (n+1)a$ implies $na+ae = (n+1)a$, i.e. $nra+rae = (n+1)ra$, implies $nra+nkt+rae = n(k+1)t+rae$, i.e. $(n+1)ra+nkt = rae+(nk+n)t$, i.e. $(n+1)ra+(n+1)kt+nkt = se+(nk+n)t+(n+1)kt$ for $s = ra \in S$, i.e. $(n+1)(k+1)t+nkt = se+(2nk+n+k)t$ [from (i)], i.e. $(2nk+n+k+1)t = se + (2nk+n+k)t$ which implies that $t \in \widehat{Se}$, i.e. $\widehat{Sa} \subseteq \widehat{Se}$. Consequently we have $\widehat{Sa} = \widehat{Se}$. □

In what follows we shall study that class of inversive semirings S for which $E^+(S)$ is a sublattice of S and we shall see that the above condition in 3.16 becomes necessary and sufficient for this class of semirings [cf. Theorem 3.18]. Let R be such a semiring. We note that for every $a \in R$ and for every $e \in E^+(R)$, $a + ae = a$; indeed, $a + a' + ae = a + a' + ae + ae = (a + a') + ae + (ae)' = (a + a') + ae + a'e = (a + a') + (a + a')e = a + a'$, whence $a + a + a' + ae = a + a'$ proves our claim. Now we proceed to prove the following:

Theorem 3.17. *In the semiring R with 1_R, if for every $a \in R$ there exists some p-idempotent $e \in R$ satisfying $\widehat{Ra} = \widehat{Re}$, then R is p-regular.*

Proof. Since

(i) $$\widehat{Ra} = \widehat{Re}$$

we get $a \in \widehat{Re}$ i.e. $re + a = 2a$ for some $r \in R$, i.e. $re^2 + ae = 2ae$, so that $re^2 + ae + re + a = 2ae + 2a$ whence $2ae + 2a + a' = r(e^2 + e) + ae + a + a'$ i.e.

(ii)
$$2ae+a = 2re+ae+2a+a'+a' = 2(re+a)+2a'+ae = 4a+2a'+ae = 2a+ae.$$

Again, from (i) as $e \in \widehat{Ra}$ we have $ba + e = 2e$ for some $b \in S$ i.e. $aba + ae = 2ae$ whence $a + aba + ae = 2ae + a = 2a + ae$ [by (ii)] i.e. $a + aba + ae + ae' = 2a + ae + ae'$ i.e. $a + aba + a(e + e') = 2a + a(e + e')$ i.e. $a + aba = 2a$ as $a + a(e + e') = a$ since $e + e' \in E^+(R)$ which is a sublattice of R. Hence R is p-regular. □

Combining the previous two theorems we have,

Theorem 3.18. *The semiring R with 1_R is p-regular if and only if every principal left p-ideal of R is generated by p-idempotent of R.*

We now prove the following lemma, which is instrumental in proving the next theorem.

Lemma 3.19. *In the semiring R an element $e \in R$ is a p-idempotent if and only if $e \in E^\circ(R)$.*

Proof. We see that $e^2 + (e^2)' = 2e^2 + 2ee'$ [since $e^2 = 2e^2 + (e^2)'$] $= e^2 + ee' + e'e + (e')^2$ [since $(e')^2 = e^2$] $= (e + e')^2 = e + e'$, so that $e^2 + e + e' = e^2 + e^2 + (e^2)' = e^2$ as S is inversive, which implies $e^2 = (e^2 + e) + e' = 2e + e' = e$. □

Theorem 3.20. *In the semiring R with 1_R, the sum of any two principal left p-ideals is again a principal left p-ideal.*

Proof. Let us consider the left p-ideal $\widehat{Ra} + \widehat{Rb}$. By Theorem 3.16 we have $\widehat{Ra} = \widehat{Re}$ with $e^2 = e$ for some p-idempotent $e \in R$ (by 3.19). We claim that $\widehat{Re} + \widehat{Rb} = \widehat{Re} + \widehat{Rc}$ for $c = b(1_R + e')$. Indeed, for some $x, y \in R$, $xe + yb = xe + y(b + b(e + e'))$ [since $E^+(R)$ is a sublattice of R] $= (x + yb)e + y(be' + b) = (x + yb)e + yb(1_R + e') = (x + yb)e + yc \in Re + Rc$, which implies $\widehat{Re} + \widehat{Rb} \subseteq \widehat{Re} + \widehat{Rc}$. Again, we see that $xe + yc = xe + yb(1_R + e') = xe + yb + ybe' = xe + yb1'_R e + yb = (x + yb1'_R)e + yb \in Re + Rb$, which indicates that $\widehat{Re} + \widehat{Rc} \subseteq \widehat{Re} + \widehat{Rb}$ whence we have $\widehat{Re} + \widehat{Rb} = \widehat{Re} + \widehat{Rc}$.

Now, we have $\widehat{Rc} = \widehat{Rf}$ for some $f = f^2$ by 3.16 and 3.19. This gives $f = xc$ for some $x \in R$. Now, $fe = xce = xb(1_R + e')e = xb(e + e'e) = xb(e^2 + e'e) = xb(e + e') \in E^+(R)$.

Now, let $g = (1_R + e')f \in Rf$. We see that $f = f^2 = xcf = xb(1_R + e')f = xbg \in Rg$. Therefore $Rg = Rf$ so that $\widehat{Rg} = \widehat{Rf}$ whence we have

$$\text{(i)} \qquad \widehat{Ra} + \widehat{Rb} = \widehat{Re} + \widehat{Rg}$$

Now, we assert that $\widehat{Re} + \widehat{Rg} = \widehat{R(e+g)}$. In fact, $R(e+g) \subseteq Re + Rg$ implies $\widehat{R(e+g)} \subseteq \widehat{Re} + \widehat{Rg}$. To show the reverse inclusion, we observe that $eg = e(1_R + e')f = (e + ee')f = (e^2 + ee')f = (e + e')ef \in E^+(R)$. Now, $e = e + e(eg) = e^2 + eg = e(e + g) \in R(e + g)$ and $g = g + g(ge) = g^2 + ge = g(e + g) \in R(e + g)$. Hence, $Re + Rg \subseteq R(e + g)$ implies that $\widehat{Re} + \widehat{Rg} \subseteq \widehat{R(e+g)}$. Consequently, $\widehat{Re} + \widehat{Rg} = \widehat{R(e+g)}$, whence from (i) we get $\widehat{Ra} + \widehat{Rb} = \widehat{R(e+g)}$. Hence, the theorem follows. $\square$

We would like to point out that the results proved for left p-ideals have their obvious duals in right p-ideals also.

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