FILOMAT (Niš) **13** (1999), 31–40

TREES AND THE KUREPA HYPOTHESIS FOR LEFT FACTORIAL

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Abstract. We construct certain trees. From their properties we establish the connections between $!n$ and $n!$. We give a necessary and sufficient condition for primality of a number and two equivalents of Kurepa's hypothesis for left factorial also. We define a sequence $D_i(n)$ and another equivalent of Kurepa's hypothesis based on proporties of this sequence.

1. Introduction

Using Picture 1 we introduce some terms which will be used. Nodes are denoted by $A, ..., F$. The Node C is processor for the node E and the node E is the successor for node C. On the Picture 1 the tree has four levels, and the start is denoted by the zero level. The Processor and the successor are on different neighbor levels and make a line (C is not the processor for F although they are on different neighbor levels).

Dj. Kurepa in [Dj K1] defined the left factorial for a natural number n (denoted with $!n$) with $!n = 0! + 1! + 2! + \cdots + (n-2)! + (n-1)!$. He also formulated hypothesis, later named Kurepa's hypothesis for left factorial (KH):

$$(1) \qquad (!n, n!) = 2, \quad n \in N, \; n > 1,$$

where $(!n, n!)$ is the greates common divisor for $!n$ and $n!$.
The inverse proposition for (1) is proven for every prime number p, $p > 2$:

Received April 28, 1998
2000 *Mathematics Subject Classification.* Primary 11A05; Secondary 05C05.
Key words and phrases. Kurepa's hypothesis, left factorial, trees, prime numbers.

$$!p \not\equiv 0 \pmod{p}.$$

The left factorial in complex field is defined with:

$$!z = \int_0^{\infty} e^{-x} \frac{x^z - 1}{x - 1} dx$$

for a complex number z ($Re\, z > 0$).
It's proven that $!(z+1) = \Gamma(z+1) + !z$, where $\Gamma(z)$ denotes the gamma - function.

J. Stanković and M. Žižović proved the next statements in [JS] and [SŽ]:

$$\sum_{i=0}^{n} !i = !(n-1)\, n + 1, \quad n \geq 1,$$

$$2 \sum_{i=0}^{n-1} (!i) i = !n + !(n-2)(n-1)n, \quad n \geq 2,$$

L. Carlitz among other proved the next statement in [LC]:

$$\sum_{k=0}^{n-1} !k k^m = Q_m(n) = \sum_{k=0}^{m} (k!)\, S(m,k)\, R_m(n),$$

where $S(m,k)$ are Stirling's numbers of the second type and $R_m(n) = \sum\limits_{k=0}^{n-1} \binom{k}{m} !k$.

J. Stanković , M. Žižović in [SŽ] and Z. Šami in [ZŠ1] proved the next equivalents of KH, in this order:

For every prime number p $\sum\limits_{k=0}^{p-1} (-1)^k (k+1)(k+2) \cdot \dots \cdot (p-1) \not\equiv 0 \pmod{p}$.

For every number $n > 2$ $\left((!n), \sum\limits_{k=2}^{n-1} (!k) \right) = 2$.

Z. Šami in [ZŠ2] and [ZŠ3] proved that

$$y_n = f^{(n)}(0) \Leftrightarrow y_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k!,$$

where $f(x)$ is the function defined by $f(x) = \dfrac{e^{-x}}{1-x}$.

The Sequence $u_m(x)$, $m \in Z$, is defined in the following way:

$$u_m(x) = \begin{cases} e^x \int\limits_0^x dt_1 \int\limits_0^{t_1} dt_2 \cdots \int\limits_0^{t_{m-1}} f(t_m)dt_m, \ m > 0 \\ e^x f^{(-m)}(x), \ m \leq 0 \end{cases}.$$

Using the definition of the sequence $u_m(x)$, the sequence $u_{n,m}$, $m, m \in N$, is defined as:

$$u_{n,m}(x) = u_m^{(n)}(0)\,,$$

and it is shown that following four statements are equivalent KH:

$$\begin{aligned} (\exists k)\, k \geq p \ \wedge u_{k,2} &\not\equiv 1 \pmod p, \text{ for prime number } p \geq 3, \\ u_{p-1,2} &\not\equiv 0 \pmod p, \text{ for prime number } p \geq 3, \\ u_{p-2,2} &\not\equiv 0 \pmod p, \text{ for prime number } p \geq 3, \\ u_{p+1,2} &\not\equiv p+1 \pmod p, \text{ for prime number } \ p \geq 3\,. \end{aligned}$$

Dj. Kurepa in [Dj K2] proved that following asymptotic relations are valid:

$$\lim_{x\to\infty} \frac{!x}{\Gamma(x)} = 1\,, \quad \lim_{x\to\infty} \frac{!x}{\Gamma(x+1)} = 0\,.$$

D. V. Slavić in [DVS] proved that for a complex number z the following holds:

$$!z = -\frac{\pi}{e} \cot g\pi z + \frac{1}{e}\left(\sum_{n=1}^{\infty} \frac{1}{(n!)n} + C\right) + \sum_{n=0}^{\infty} \Gamma(z-n)\,,$$

C is Euler cons.

G. V. Milovanović defined the following sequence in [GM]:

$$S_t = t! \sum_{i=0}^{t} \frac{(-1)^i}{i!}\,, \ (i \geq 0)\,, \text{ e. t.}$$

$$S_t = tS_{t-1} + (-1)^t\,, \text{ for } S_0 = 1\,.$$

Using the definition of the sequence S_t, the next function is defined:

$$K_m(n) = \sum_{t=0}^{n-1} \binom{m+n}{t+m+1} S_t\,,$$

and its properties are investgated.

In [AP] the next sequence of integer numbers $\{d_n\}$, is defined by the following recurrent formula:

$$d_1 = -1\,,\ d_n = -(n+1)d_{n-1} - 1 \text{ for every natural number } n\,.$$

Numbers of this sequence $\{d_n\}$ are $d_1 = -1$, $d_2 = 2$, $d_3 = -9$, $d_4 = 44$, $d_5 = -265, \ldots$. The following is notified:

$$y_{n+1} = S_{n+1} = d_n \text{ for } n = 2k\,,\ k = 1,2,3,..$$
$$y_{n+1} = S_{n+1} = -d_n \text{ for } n = 2k-1\,,\ k = 1,2,3,..$$

It is proved elementary that for every prime number p:

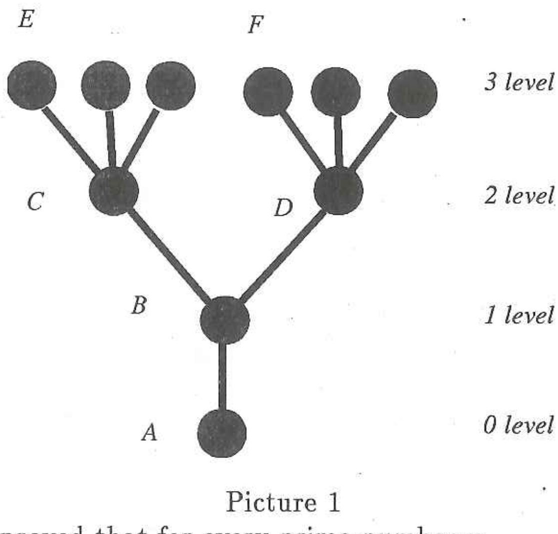
$$!p \equiv -d_{p-2} \pmod p \text{ for } d_{p-2} \in \{d_n\}.$$

Denote with

$$Ne(p) = (p-2)! + (p-4)! + (p-6)! + \cdots + 3! + 1! \text{ and}$$
$$Pa(p) = (p-1)! + (p-3)! + (p-5)! + \cdots + 4! + 2! + 0!.$$

Define two sequences of integer numbers

$$f_1 = -1,\ f_{n+1} = (2n-1)2nf_n - 1\,,$$
$$g_1 = 1,\ g_{n+1} = 2n(2n+1)g_n + 1 \text{ for } n > 1\,.$$



Picture 1

In [AP] it is proved that for every prime number p

$$g_{\frac{p-1}{2}} \equiv Ne(p) \pmod p \wedge f_{\frac{p+1}{2}} \equiv -Pa(p) \pmod p.$$

In [RG] (R. Guy) Kurepa's hypothesis is given as one of the unsolved problems is the Theory of numbers under number $B44$.

2. Connection between $!n$ i $n!$

Denote with $T(A)$ the tree with the initial node A (zero - level node Picture 1). Let X be the node of the i-th level of a tree $T(A)$. Denote with $T_i(X)$ the subtree with the initial node X.

(2) A tree $T(A)$ is generated in a following way : Zero - level has one node. A tree has n levels (last is $n-1$ level). If the node is in the k-th level then it has $k+1$ successors, where $0 \leq k < n-1$. A number of nodes in this tree is:

$$!n = \sum_{i=0}^{n-1} i!$$

Let X be an element of the i-th level. Then for the tree $T(Y) = T_i(X) \cup \{Y\}$, $Y \notin T(A)$, we obtain:

1) $T(Y)$ has n-i levels,
2) the elements of j-th level ($j > 0$) from $T(Y)$ are elements from the $j+i-1$-th level in $T(A)$.

The number of nodes in $i!$ trees $T(Y)$ is $!n + \sum_{j=2}^{i-1}(j-1)j!$. For $i = n-2$ we obtain that the number of nodes in $(n-2)!$ trees is equal to $!n + \sum_{j=2}^{n-3}(j-1)!j!$. The number of nodes in the tree $T_{n-2}(X)$ is $n-1+1 = n$. The number of nodes in $(n-2)!$ trees $T_{n-2}(X)$ is $n(n-2)!$. The number of nodes in $(n-2)!$ trees $T_{n-2}(X)$, using the number of nodes in the tree $T(Y)$, is equal $!n + \left(\sum_{j=2}^{n-3}(j-1)j!\right) - (n-2)!$. On the basis of previous, we have:

$$(3) \qquad !n - (n-2)! + \sum_{i=2}^{n-3}(i-1)i! = n(n-2)!$$

Using (3) one can obtain:

Theorem 1. *For every natural number $n > 2$:*

$$n! = !n + \sum_{i=2}^{n-1}(i-1)i!.$$

Corollary 1. *For every natural number $n > 3$:*

$$\sum_{i=1}^{n-3} i\,i! \equiv -(n-1)! - 1 - n(n-2)(n-2)! \pmod{n}!$$

Corollary 2. *A natural number $n > 3$ is a prime number iff:*

$$\sum_{i=1}^{n-3} i\,i! \equiv 0 \pmod{n}.$$

Proof. Using Corollary 1 and Wilson's theorem.

Corollary 3. *For every natural number $n > 2$:*

$$!n \equiv 0 \pmod{n} \Leftrightarrow \sum_{i=2}^{n-1} (i-1)i! \equiv 0 \pmod{n}.$$

Remark. In [JS] Corollary 3 is proved using the gamma function.
Let p be a prime number. Then:

$$!p = \sum_{i=0}^{p-1} i! = 1 + (p-1)! + \sum_{i=0}^{p-3} i! + \sum_{i=1}^{p-3} i\,i!.$$

Using the last relation and Corollary 2, we obtain:

$$\begin{aligned} &!p \equiv !(p-3) - (p-3)(p-3)! + 1 && \pmod{p} \\ \Leftrightarrow &!p \equiv !(p-4) - (p-4)(p-4)! - 2(p-3)(p-3)! + 2 && \pmod{p} \\ &\vdots \\ \Leftrightarrow &!p \equiv !(p-k) + k - 2 - \sum_{i=1}^{k-2} i(p-k+i-1)(p-k+i-1)! && \pmod{p} \end{aligned}$$

From this last relation we get next equivalent KH:

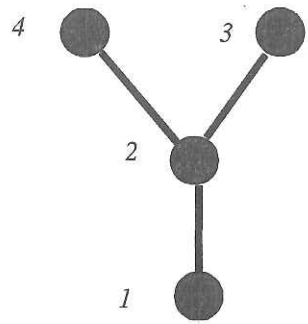
Theorem 2. *For every prime number $p > 2$ and every natural number $k < p$:*

$$!p \not\equiv 0 \pmod{p} \Leftrightarrow !(p-k)+k-2-\sum_{i=1}^{k-2} i(p-k+i-1)(p-k+i-1)! \not\equiv 0 \pmod{p}$$

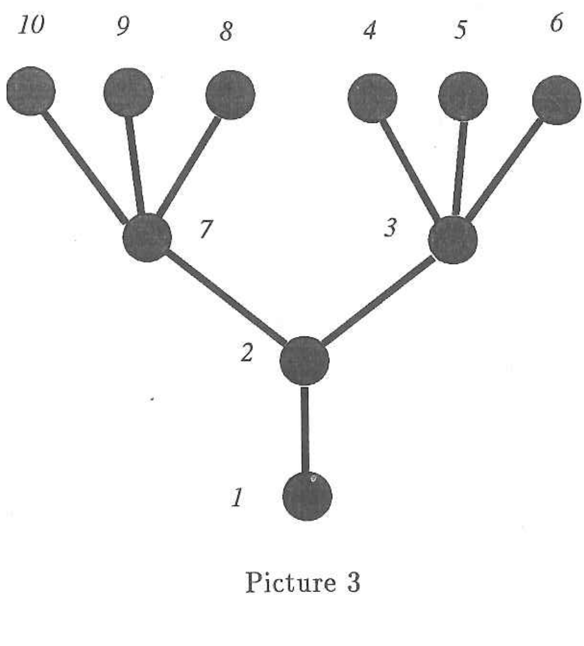
Corollary 4. *For every prime number $p > 2$:*

$$!p \equiv -\sum_{i=2}^{p-1} i^2\, i! \pmod{p}.$$

Proof. In Theorem 2 we choose $k = p - 1$ and use the identity for prime numbers $3 \equiv (p-2)^2(p-2)! + (p-1)^2(p-1)! \pmod{p}$.



Picture 2



Picture 3

Sequences $D_i(n)$ and B_i

If we generate a tree as in (2) for $n = 3$, we get the tree on Picture 2. The number of nodes of this tree is $!3$. If we subtract nodes denoted with 1, 2, 3 and put in nodes 1, 2, and subtract nodes 1, 2, 4 we get: $3! = !3 + 2$. Let $D_3(n) = 0$. Then for $n = 3$: $n! = !n + 2D_3(n) + 2$.

If we generate a tree as in (2) for $n = 4$, we get the tree on Picture 3. The number of nodes of this tree is $!4$. If we subtract nodes denoted with 1, 2, 3, 4, put in nodes 1, 2, 3, subtract nodes 1, 2, 3, 5, put in nodes 1, 2, 3, subtract nodes 1, 2, 3, 6, put in nodes 1, 2, and repeat whole procedure with nodes 1, 2, 7, 8, 9 we get $4! = !4 + 2 \cdot 6 + 2$. Let $D_4(n) = (n-1)(n-2)$. Then for $n = 4$:

$$n! = !n + 2D_4(n) + 2\,.$$

If we repeat the procedure for $n = 5$ and $D_5(n) = (n-1)(n-2) + ((n-2) + (n-1)(n-2))(n-3)$, we get:

$$n! = !n + 2D_5(n) + 2\,.$$

A generalization of this procedure gives the next definition of the sequence $D_i(n)$, for natural numbers $n > 2, i > 2$,

(4)

$$D_3(n) = 0$$

$$D_i(n) = (n-i+3)D_{i-1}(n) + (n-i+3)(n-i+2), \text{ for } n > 2,\ i > 3\,.$$

Theorem 3. *For every natural number $n > 2$:*

$$n! = !n + 2D_n(n) + 2$$

Using Theorem 3, we can prove the next equivalent KH:

Corollary 5. *For every natural number $n > 2$:*

$$!n \equiv 2D_n(n) + 2 \pmod{n}\,.$$

Corollary 6. *For every natural number $n > 2$:*

$$\sum_{i=2}^{n-1} (i-1)\, i! = 2D_n(n) + 2\,.$$

Proof. Using Theorems 1 and 2 .

On the basis of definition (4) of the sequence $D_i(n)$, we define the sequence B_i in a following way:

$$B_3 = 0 \tag{5}$$
$$B_i = -(i-3)B_{i-1} + (i-3)(i-2), \quad \text{for } i > 3\,.$$

Numbers of sequence B_i are: $B_3 = 0$, $B_4 = 2$, $B_5 = 2$, $B_6 = 6$, $B_7 = -4$, $B_8 = 50, \ldots$
Using (4), (5) and Theorem 3, the next equivalent of KH can be proven:

Corollary 7. *For every natural number $n > 2$:*

$$-!n \equiv 2B_n + 2 \pmod{n}\,.$$

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