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LOG–MAXIMAL FINITE SEMIGROUPS DO NOT EXIST

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Abstract. In the present note we prove the nonexistence of finite semigroups with log-maximal free spectra (asymptotically behaving as k^{k^n} for k-element semigroups). Also, several examples of finite semigroups with log-exponential free spectra are presented, as well as some problems concerning the topic.

Let $\mathcal{V}$ be a variety of algebras and let $\mathbf{F}_n(\mathcal{V})$ denote the free algebra on n free generators in $\mathcal{V}$. The *free spectrum of the variety* $\mathcal{V}$ is the sequence of cardinal numbers $f_n(\mathcal{V}) = |\mathbf{F}_n(\mathcal{V})|$. If $\mathbf{A} = \langle A, F\rangle$ is a given algebra, we define $f_n(\mathbf{A})$, *the free spectrum of* $\mathbf{A}$, just as the free spectrum of the variety that $\mathbf{A}$ generates.

The free spectrum is an important invariant of algebras (resp. varieties). However, it is clear that the amount of information provided by the free spectrum becomes significant only in the case that its members are finite cardinals, that is, natural numbers. This is the case precisely with locally finite varieties and, in particular, with all finite algebras. Indeed, if $|A| = k$, it is easy to obtain the well known upper bound $f_n(\mathbf{A}) \leq k^{k^n}$, with equality holding for *primal algebras*.

Finite algebras with large free spectra (in the sense of being close to k^{k^n}) and the structure of their clones had been studied earlier by Murskiĭ [8] and Berman [1], to mention some. Their results are surveyed in detail in McKenzie, McNulty and Taylor [7]. Following Berman [1], we call a nontrivial finite algebra $\mathbf{A}$ *log-maximal* if

$$\lim_{n\to\infty} \frac{\log_{|A|} f_n(\mathbf{A})}{|A|^n} = 1.$$

In this short note, we are interested in the existence of log-maximal semigroups. To prove that such semigroups do not exist, we are going to use

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the following two results. Recall that an algebra is *functionally complete* if all finitary operations of its universe are algebraic (i.e. representable by a polynomial expression of the corresponding type).

Result 1. (Berman, [1]) *If* $\mathbf{A}$ *is a log maximal algebra, then* $\mathbf{A}$ *is functionally complete and has no nontrivial automorphisms.*

Result 2. (Werner, [10]) *Let* $\mathbf{A}$ *be a nontrivial algebra generating a congruence-permutable variety. Then* $\mathbf{A}$ *is functionally complete if and only if* $\mathbf{Con}\ \mathbf{A}^2 \cong \mathbf{2}^2$*, where* $\mathbf{2}$ *denotes the two-element lattice.*

Equiped with these two, we prove the following

Theorem. *Let* $\mathbf{S}$ *be a finite semigroup. Then* $\mathbf{S}$ *is not log-maximal.*

Proof. Suppose $\mathbf{S} = \langle S, \cdot \rangle$ is a log-maximal finite semigroup. Then by Result 1, $\mathbf{S}$ is functionally complete and thus simple in the sense that it has no nontrivial congruences (which is to be distinguished from the same expression used in semigroup theory for semigroups having no proper ideals). Now we consider three cases.

If $\mathbf{S}$ has no zero element and $|S| \geq 3$, then $\mathbf{S}$ is a simple group, by Theorem III.6.2 in [6]. Since (by Result 1) $\mathbf{S}$ has no nontrivial automorphisms, it follows that the inner groups automorphisms $\sigma_a(x) = a^{-1}xa$ must be identity mappings for all $a \in S$. Thus $ax = xa$ holds for all $a, x \in S$, i.e. $\mathbf{S}$ is commutative. Hence, $\mathbf{S}$ is a cyclic group of prime order p. An elementary calculation shows that $\mathbf{Z}_p \times \mathbf{Z}_p$ has five congruences, so Result 2 implies that $\mathbf{S}$ is not functionally complete, a contradiction. Of course, it makes no differnce if $\mathbf{S}$ is considered as a semigroup or as a group (having all group operations in its language) because x^{p-1} expresses the inverse operation and $x^p = 1$ holds in $\mathbf{S}$.

Now assume that $\mathbf{S}$ has zero element 0. But in this case $\mathbf{S}$ is immediately seen not to be functionally complete because for each unary polynomial operation f of $\mathbf{S}$ it has to be $f(0) = 0$. Therefore, in this case we have also reached a contradiction (of course, we assume the nontriviality of $\mathbf{S}$).

Finally, it remains to analyze the two-element semigroups. This is an easy task, since we have only five of them (up to an isomorphism): the two-element semilattice ($f_n = 2^n - 1$), the two-element left and right zero semigroups ($f_n = n$), the two-element null semigroup ($f_n = n + 1$) and $\mathbf{Z}_2$ ($f_n = 2^n$). The proof is now complete. □

From the above proof, it is not difficult to derive

Corollary. *The only finite functionally complete semigroups are the non-abelian simple groups and the trivial semigroup.*

On the other hand, there certainly exist semigroups having log-exponential free spectra, that is, with $\log f_n(\mathbf{S})$ bounded below by an exponential function c^n for some $c > 1$. Namely, the results of Neumann [9] and Higman [5] yield that a finite *group* $\mathbf{G}$ is nilpotent of index q if and only if $\log f_n(\mathbf{G})$ is bounded above by a polynomial in n of degree q and if $\mathbf{G}$ is not nilpotent, then $f_n(\mathbf{G}) \geq 2^{2^n}$, which means that $\mathbf{G}$ has log-exponential free spectrum. This gives a basis for the following example.

Example 1. Let $\mathbf{G}$ be a finite group which is not nilpotent (this can be taken to be, for example, the dihedral group $\mathbf{D}_3$ or any nonsolvable group such as $\mathbf{A}_5$) and put $\mathbf{S}$ to be the Rees matrix semigroup $\mathcal{M}[\mathbf{G}; I, \Lambda; P]$, where the index sets I, Λ are finite. Clearly, the semigroup reduct of $\mathbf{G}$ is embeddable into $\mathbf{S}$ and hence all semigroup reducts of members of the variety $\mathcal{W}$ generated by $\mathbf{G}$ belong to the variety $\mathcal{V}$ generated by $\mathbf{S}$. Of course, this is also true for the semigroup reduct of $\mathbf{F}_n(\mathcal{W})$. Obviously, it is $2n$-generated as a semigroup, thus it is a homomorphic image of $\mathbf{F}_{2n}(\mathcal{V})$. These considerations prove the inequality $f_{2n}(\mathbf{S}) \geq f_n(\mathbf{G})$, which together with the facts above imply that

$$f_n(\mathbf{S}) \geq 2^{2^{\left[\frac{n}{2}\right]}}.$$

Therefore, we obtain infinitely many nonisomorphic finite simple semigroups (most of which are *not* reducts of groups) with log-exponential free spectra.

The above example motivates the following question.

Problem 1. *Characterize all finite semigroups with log-exponential free spectra. Is there any lower bound on* $\log f_n(\mathbf{S})$ *of the form* c^n $(c > 1)$ *for such semigroups?*

An additional remark should be given at this final point of the paper. Here we recall that Crvenković and Ruškuc [3] fully described all varieties of semigroups having log-linear free spectra (varieties for which $\log f_n(\mathcal{V}) \leq cn$ for some constant $c > 0$). Their result implies that finite semigroups generating such varieties are exactly the finite nilpotent ideal extensions of medial semigroups. But, as the next example shows, there exist locally finite varieties of semigroups which are neither log-linear, nor log-exponential.

Example 2. Let $\mathcal{V}$ be the variety of semigroups determined by the identity $xyx = z^2$. An easy argument shows that $f_n(\mathcal{V})$ is just equal to the number of all sequences over $\{1, \ldots, n\}$ with all members different, that is

$$f_n(\mathcal{V}) = \sum_{k=0}^{n} \binom{n}{k} k! = [en!].$$

(Recall that the above numbers play a major role in Ramsey theory, because $[en!]+1$ vertices in a complete graph ensure the existence of a monochromatic triangle in any n-colouring of its edges, see [2], Exercise 7.2.3.) By Stirling's formula, we have that $\log f_n(\mathcal{V}) \sim n \log n$. Consequently, $f_n(\mathcal{V})$ is bounded above by a quadric (but *not* by a linear) polynomial in n. However, it is not hard to prove that $\mathcal{V}$ is not finitely generated, because one obtains the following formula for the finitely generated $\mathcal{V}$-free algebras:

$$f_n(\mathbf{F}_m(\mathcal{V})) = \sum_{k=0}^{m} \binom{n}{k} k!,$$

which is for $n > m$ strictly less then $f_n(\mathcal{V})$.

This discussion leads to some further problems.

Problem 2. *Does there exists a finite semigroup whose free spectrum is neither log-linear, nor log-exponential?*

Of course, the negative answer would yield a trivial solution to our first problem. But if the answer is positive, the above question could be investigated in more detail.

Problem 3. *If Problem 2 has positive solution, is there a log-polynomial finite semigroup which is not log-linear? If yes, characterize all log-polynomial finite semigroups.*

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