

REMARKS ON FUZZY TOPOLOGICAL SUBALGEBRAS IN HILBERT ALGEBRAS

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Abstract. The purpose of this paper is to introduce the concept of fuzzy topological subalgebras and deductive systems in Hilbert algebras, and to obtain its some results.

1. Introduction

The concept of Hilbert algebra was introduced in early 50-ties by L.Henkin and T.Skolem for some investigations of implication in intuicionistic and other non-classical logics. In 60-ties, these algebras were studied especially by A.Horn and A.Diego from algebraic point of view. A.Diego proved (cf. [3]) that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by D.Busneag (cf. [1], [2]) and Y.B.Jun (cf. [6]) and some of their filters forming deductive systems were recognized. W.A.Dudek (cf. [4]) considered the fuzzification of subalgebras and deductive systems in Hilbert algebras. The concept of a fuzzy set, which was introduced in [9], provides a natural framework for generalizing many of the concepts of general topology to what might be called fuzzy topological spaces. D.H.Foster (cf. [5]) combined the structure of a fuzzy topological spaces with that of a fuzzy group, introduced by A.Rosenfeld (cf. [8]), to formulate the elements of a theory of fuzzy topological groups. In the present paper, we introduce the concept of fuzzy topological subalgebras of Hilbert algebras and apply some of Foster's results on homomorphic images and inverse images to fuzzy topological subalgebras.

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2. Preliminaries

Since there exist various versions of the definition of Hilbert algebra, we use that of [1] with the modified symbol of an operation. Namely, we use \star instead of \rightarrow .

Definition 2.1. A nonempty set X with a constant 1 and a binary operation \star is called a *Hilbert algebra* if for all $x, y, z \in X$ the following axioms are satisfied:

- (I) $x \star (y \star x) = 1$,
- (II) $(x \star (y \star z)) \star ((x \star y) \star (x \star z)) = 1$,
- (III) $x \star y = 1$ and $y \star x = 1$ imply $x = y$.

In a Hilbert algebra, the following hold (cf. for example [3], [7]):

- (1) $x \star x = 1$,
- (2) $1 \star x = x$,
- (3) $x \star 1 = 1$.

It is easily checked that in a Hilbert algebra X the relation \leq defined by

$$x \leq y \iff x \star y = 1$$

is a partial order on X with 1 as the largest element.

We now review some fuzzy concepts from [5] and [9].

Let X be a set. A *fuzzy set* A in X is characterized by a membership function $\mu_A : X \rightarrow [0, 1]$. Let α be a mapping from the set X to the set Y . Let B be a fuzzy set in Y with membership function μ_B . The *inverse image* of B , denoted $\alpha^{-1}(B)$, is the fuzzy set in X with membership function $\mu_{\alpha^{-1}(B)}$ defined by $\mu_{\alpha^{-1}(B)}(x) = \mu_B(\alpha(x))$ for all $x \in X$. Conversely, let A be a fuzzy set in X with membership function μ_A . Then the *image* of A , denoted by $\alpha(A)$, is the fuzzy set in Y such that

$$\mu_{\alpha(A)}(y) = \begin{cases} \sup_{z \in \alpha^{-1}(y)} \mu_A(z) & \text{if } \alpha^{-1}(y) = \{x : \alpha(x) = y\} \text{ is non-empty,} \\ 0 & \text{otherwise.} \end{cases}$$

A *fuzzy topology* on a set X is a family \mathcal{T} of fuzzy sets in X which satisfies the following conditions:

- (i) For all $c \in [0, 1]$, $k_c \in \mathcal{T}$, where k_c have constant membership functions,
- (ii) If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$,
- (iii) If $A_j \in \mathcal{T}$ for all $j \in J$, then $\bigcup_{j \in J} A_j \in \mathcal{T}$.

The pair (X, \mathcal{T}) is called a *fuzzy topological space* and members of \mathcal{T} are *open fuzzy sets*.

Let A be a fuzzy set in X and \mathcal{T} a fuzzy topology on X . Then the *induced fuzzy topology* on A is the family of fuzzy subsets of A which are the intersection with A of \mathcal{T} -open fuzzy sets in X . The induced fuzzy topology is denoted by \mathcal{T}_A , and the pair (A, \mathcal{T}_A) is called a *fuzzy subspace* of (X, \mathcal{T}) .

Let (X, \mathcal{T}) and (Y, \mathcal{U}) be two fuzzy topological spaces. A mapping α of (X, \mathcal{T}) into (Y, \mathcal{U}) is *fuzzy continuous* if for each open fuzzy set U in \mathcal{U} the inverse image $\alpha^{-1}(U)$ is in \mathcal{T} . Conversely, α is *fuzzy open* if for each open fuzzy set V in \mathcal{T} , the image $\alpha(V)$ is in \mathcal{U} .

Let (A, \mathcal{T}_A) and (B, \mathcal{U}_B) be fuzzy subspaces of fuzzy topological spaces (X, \mathcal{T}) and (Y, \mathcal{U}) respectively, and let α be a mapping from (X, \mathcal{T}) to (Y, \mathcal{U}) . Then α is a mapping of (A, \mathcal{T}_A) into (B, \mathcal{U}_B) if $\alpha(A) \subset B$. Furthermore α is *relatively fuzzy continuous* if for each open fuzzy set V' in \mathcal{U}_B , the intersection $\alpha^{-1}(V') \cap A$ is in \mathcal{T}_A . Conversely, α is *relatively fuzzy open* if for each open fuzzy set U' in \mathcal{T}_A , the image $\alpha(U')$ is in \mathcal{U}_B .

Lemma 2.1 [5]. *Let (A, \mathcal{T}_A) , (B, \mathcal{U}_B) be fuzzy subspaces of fuzzy topological spaces (X, \mathcal{T}) , (Y, \mathcal{U}) respectively, and let α be a fuzzy continuous mapping of (X, \mathcal{T}) into (Y, \mathcal{U}) such that $\alpha(A) \subset B$. Then α is a relatively fuzzy continuous mapping of (A, \mathcal{T}_A) into (B, \mathcal{U}_B) .*

3. Fuzzy topological subalgebras

Definition 3.1 [4]. A fuzzy set F in a Hilbert algebra X with membership function μ_F is called a *fuzzy subalgebra* of X if

$$\mu_F(x \star y) \geq \min\{\mu_F(x), \mu_F(y)\}, \quad \forall x, y \in X.$$

Example 3.2 [4]. Let $X = \{a, b, c, d, 1\}$ be a set with Cayley table (Table 1) and Hasse diagram (Diagram 1) as follows:

\star	a	b	c	d	1
a	1	1	1	1	1
b	a	1	c	1	1
c	a	b	1	1	1
d	a	b	c	1	1
1	a	b	c	d	1

Table 1

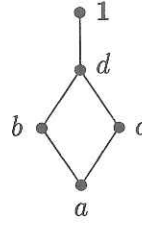


Diagram 1

By routine calculations we know that $(X, \star, 1)$ is a Hilbert algebra and $\mu : X \rightarrow [0, 1]$ defined by $\mu(1) = 0.8$ and $\mu(x) = 0.4$ for all $x \neq 1$ is a fuzzy set which is a fuzzy subalgebra of X . \square

Proposition 3.3. *Let α be a homomorphism of a Hilbert algebra X into a Hilbert algebra Y and G a fuzzy subalgebra of Y with membership function μ_G . Then the inverse image $\alpha^{-1}(G)$ of G is a fuzzy subalgebra of X .*

Proof. Let $x, y \in X$. Then

$$\begin{aligned}
 \mu_{\alpha^{-1}(G)}(x \star y) &= \mu_G(\alpha(x \star y)) = \mu_G(\alpha(x) \star \alpha(y)) \\
 &\geq \min\{\mu_G(\alpha(x)), \mu_G(\alpha(y))\} \\
 &= \min\{\mu_{\alpha^{-1}(G)}(x), \mu_{\alpha^{-1}(G)}(y)\}.
 \end{aligned}$$

This completes the proof. \square

For images, we need the following definition [8].

Definition 3.4. A fuzzy set F in a Hilbert algebra X with membership function μ_F is said to have the *sup property* if, for any subset $T \subset X$, there exists $t_0 \in T$ such that

$$\mu_F(t_0) = \sup_{t \in T} \mu_F(t).$$

Proposition 3.5. *Let α be a homomorphism of a Hilbert algebra X onto a Hilbert algebra Y and let F be a fuzzy subalgebra of X that has the sup property. Then the image $\alpha(F)$ of F is a fuzzy subalgebra of Y .*

Proof. For $u, v \in Y$, let $x_0 \in \alpha^{-1}(u)$, $y_0 \in \alpha^{-1}(v)$ such that

$$\mu_F(x_0) = \sup_{t \in \alpha^{-1}(u)} \mu_F(t), \quad \mu_F(y_0) = \sup_{t \in \alpha^{-1}(v)} \mu_F(t).$$

Then, by the definition of $\mu_{\alpha(F)}$, we have

$$\begin{aligned}
 \mu_{\alpha(F)}(u \star v) &= \sup_{t \in \alpha^{-1}(u \star v)} \mu_F(t) \\
 &\geq \mu_F(x_0 \star y_0) \\
 &\geq \min\{\mu_F(x_0), \mu_F(y_0)\} \\
 &= \min\left\{\sup_{t \in \alpha^{-1}(u)} \mu_F(t), \sup_{t \in \alpha^{-1}(v)} \mu_F(t)\right\} \\
 &= \min\{\mu_{\alpha(F)}(u), \mu_{\alpha(F)}(v)\},
 \end{aligned}$$

ending the proof. \square

For any Hilbert algebra X and any element $a \in X$ we use L_a denote the left translation of X defined by $L_a(x) = a \star x$ for all $x \in X$. It is clear that $L_a(1) = 1$ for all $a \in X$. Obviously L_1 is the identity mapping of X .

Definition 3.6. Let X be a Hilbert algebra and \mathcal{T} a fuzzy topology on X . Let F be a fuzzy subalgebra of X with induced topology \mathcal{T}_F . Then F is called a *fuzzy topological subalgebra* of X if for each $a \in X$ the mapping L_a of $(F, \mathcal{T}_F) \rightarrow (F, \mathcal{T}_F)$ is relatively fuzzy continuous.

Theorem 3.7. Given Hilbert algebras X, Y and a homomorphism $\alpha : X \rightarrow Y$, let \mathcal{U} and \mathcal{T} be the fuzzy topologies on Y and X respectively, such that $\mathcal{T} = \alpha^{-1}(\mathcal{U})$. Let G be a fuzzy topological subalgebra of Y with membership function μ_G . Then $\alpha^{-1}(G)$ is a fuzzy topological subalgebra of X with membership function $\mu_{\alpha^{-1}(G)}$.

Proof. We have to show that, for each $a \in X$, the mapping

$$L_a : (\alpha^{-1}(G), \mathcal{T}_{\alpha^{-1}(G)}) \longrightarrow (\alpha^{-1}(G), \mathcal{T}_{\alpha^{-1}(G)})$$

is relatively fuzzy continuous. Let U be an open fuzzy set in $\mathcal{T}_{\alpha^{-1}(G)}$ on $\alpha^{-1}(G)$. Since α is a fuzzy continuous mapping of (X, \mathcal{T}) into (Y, \mathcal{U}) , it follows from Lemma 2.1 that α is a relatively fuzzy continuous mapping of $(\alpha^{-1}(G), \mathcal{T}_{\alpha^{-1}(G)})$ into (G, \mathcal{U}_G) . Note that there exists an open fuzzy set $V \in \mathcal{U}_G$ such that $\alpha^{-1}(V) = U$. The membership function of $L_a^{-1}(U)$ is given by

$$\begin{aligned}
 \mu_{L_a^{-1}(U)}(x) &= \mu_U(L_a(x)) = \mu_U(a \star x) = \mu_{\alpha^{-1}(V)}(a \star x) \\
 &= \mu_V(\alpha(a \star x)) = \mu_V(\alpha(a) \star \alpha(x)).
 \end{aligned}$$

As G is a fuzzy topological subalgebra of Y , the mapping

$$L_b : (G, \mathcal{U}_G) \rightarrow (G, \mathcal{U}_G)$$

is relatively fuzzy continuous for each $b \in Y$. Hence

$$\begin{aligned}\mu_{L_a^{-1}(U)}(x) &= \mu_V(\alpha(a) \star \alpha(x)) = \mu_V(L_{\alpha(a)}(\alpha(x))) \\ &= \mu_{L_{\alpha(a)}^{-1}(V)}(\alpha(x)) = \mu_{\alpha^{-1}(L_{\alpha(a)}^{-1}(V))}(x),\end{aligned}$$

which implies that $L_a^{-1}(U) = \alpha^{-1}(L_{\alpha(a)}^{-1}(V))$ so that

$$L_a^{-1}(U) \cap \alpha^{-1}(G) = \alpha^{-1}(L_{\alpha(a)}^{-1}(V)) \cap \alpha^{-1}(G)$$

is open in the induced fuzzy topology on $\alpha^{-1}(G)$. This completes the proof. \square

We say that the membership function μ_G of a fuzzy subalgebra G of a Hilbert algebra X is α -invariant [8] if, for all $x, y \in X$, $\alpha(x) = \alpha(y)$ implies $\mu_G(x) = \mu_G(y)$.

Theorem 3.8. *Given Hilbert algebras X, Y and a homomorphism α of X onto Y , let \mathcal{T} be the fuzzy topology on X and \mathcal{U} be the fuzzy topology on Y such that $\alpha(\mathcal{T}) = \mathcal{U}$. Let F be a fuzzy topological subalgebra of X . If the membership function μ_F of F is α -invariant, then $\alpha(F)$ is a fuzzy topological subalgebra of Y .*

Proof. It is sufficient to show that the mapping

$$L_b : (\alpha(F), \mathcal{U}_{\alpha(F)}) \longrightarrow (\alpha(F), \mathcal{U}_{\alpha(F)})$$

is relatively fuzzy continuous for each $b \in Y$. Note that α is relatively fuzzy open; for if $U' \in \mathcal{T}_F$, there exists $U \in \mathcal{T}$ such that $U' = U \cap F$ and by the α -invariance of μ_F ,

$$\alpha(U') = \alpha(U) \cap \alpha(F) \in \mathcal{U}_{\alpha(F)}.$$

Let V' be an open fuzzy set in $\mathcal{U}_{\alpha(F)}$. Since α is onto, for each $b \in Y$ there exists $a \in X$ such that $b = \alpha(a)$. Hence

$$\begin{aligned}\mu_{\alpha^{-1}(L_b^{-1}(V'))}(x) &= \mu_{\alpha^{-1}(L_{\alpha(a)}^{-1}(V'))}(x) = \mu_{L_{\alpha(a)}^{-1}(V')}(\alpha(x)) \\ &= \mu_{V'}(L_{\alpha(a)}(\alpha(x))) = \mu_{V'}(\alpha(a) \star \alpha(x)) \\ &= \mu_{V'}(\alpha(a \star x)) = \mu_{\alpha^{-1}(V')}(a \star x) \\ &= \mu_{\alpha^{-1}(V')}(L_a(x)) = \mu_{L_a^{-1}(\alpha^{-1}(V'))}(x),\end{aligned}$$

which implies that $\alpha^{-1}(L_b^{-1}(V')) = L_a^{-1}(\alpha^{-1}(V'))$.

By hypothesis, L_a is a relatively fuzzy continuous mapping from (F, \mathcal{T}_F) to (F, \mathcal{T}_F) and α is a relatively fuzzy continuous mapping from (F, \mathcal{T}_F) to $(\alpha(F), \mathcal{U}_{\alpha(F)})$. Hence

$$\alpha^{-1}(L_b^{-1}(V')) \cap G = L_a^{-1}(\alpha^{-1}(V')) \cap F$$

is open in \mathcal{T}_F . Since α is relatively fuzzy open,

$$\alpha(\alpha^{-1}(L_b^{-1}(V')) \cap F) = L_b^{-1}(V') \cap \alpha(F)$$

is open in $\mathcal{U}_{\alpha(F)}$. This completes the proof. \square

4. Fuzzy topological deductive systems

First we briefly review the concepts of fuzzy deductive systems of Hilbert algebras (cf. [4]).

Definition 4.1. A fuzzy set D in X with membership function μ_D is called a *fuzzy deductive system* of X if

- (a) $\mu_D(1) \geq \mu_D(x), \quad \forall x \in X,$
- (b) $\mu_D(y) \geq \min\{\mu_D(x \star y), \mu_D(x)\}, \quad \forall x, y \in X.$

Proposition 4.2. Let α be a homomorphism of a Hilbert algebra X into a Hilbert algebra Y and B a fuzzy deductive system of Y with membership function μ_B . Then the inverse image $\alpha^{-1}(B)$ of B is a fuzzy deductive system of X .

Proof. Since α is a homomorphism of X into Y , then $\alpha(1) = 1 \in Y$ and, by the assumption, $\mu_B(\alpha(1)) = \mu_B(1) \geq \mu_B(y)$ for every $y \in Y$. In particular, $\mu_B(\alpha(1)) \geq \mu_B(\alpha(x))$ for $x \in X$. Thus $\mu_{\alpha^{-1}(B)}(1) \geq \mu_{\alpha^{-1}(B)}(x)$, which proves (a).

Now, let $x, y \in X$. Then, by the assumption on μ_B , we have

$$\begin{aligned} \mu_{\alpha^{-1}(B)}(y) &= \mu_B(\alpha(y)) \\ &\geq \min\{\mu_B(v \star \alpha(y)), \mu_B(v)\} \end{aligned}$$

for all $\alpha(y), v \in Y$. In particular, for $v = \alpha(x)$, we have

$$\begin{aligned} \mu_{\alpha^{-1}(B)}(y) &\geq \min\{\mu_B(\alpha(x) \star \alpha(y)), \mu_B(\alpha(x))\} \\ &= \min\{\mu_B(\alpha(x \star y)), \mu_B(\alpha(x))\} \\ &= \min\{\mu_{\alpha^{-1}(B)}(x \star y), \mu_{\alpha^{-1}(B)}(x)\}, \end{aligned}$$

which proves (b). The proof is complete. \square

Since any fuzzy deductive system is a fuzzy subalgebra (cf. [4]), then as a consequence of the above results and Theorem 3.7, we obtain

Corollary 4.3. *Given Hilbert algebras X, Y and a homomorphism $\alpha : X \rightarrow Y$, let \mathcal{U} and \mathcal{T} be the fuzzy topologies on Y and X respectively, such that $\mathcal{T} = \alpha^{-1}(\mathcal{U})$. Let G be a fuzzy topological deductive system of Y with membership function μ_G . Then $\alpha^{-1}(G)$ is a fuzzy topological deductive system of X with membership function $\mu_{\alpha^{-1}(G)}$.* \square

It is not difficult to see that if the membership function μ_G of a fuzzy deductive system G of a Hilbert algebra X is α -invariant, then a homomorphic image $\alpha(G)$ of G is a fuzzy deductive system. Thus from Theorem 3.8 follows

Corollary 4.4. *Given Hilbert algebras X, Y and a homomorphism α of X onto Y , let \mathcal{T} be the fuzzy topology on X and \mathcal{U} be the fuzzy topology on Y such that $\alpha(\mathcal{T}) = \mathcal{U}$. Let F be a fuzzy topological deductive system of X . If the membership function μ_F of F is α -invariant, then $\alpha(F)$ is a fuzzy topological deductive system of Y .* \square

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