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INEQUALITIES FOR LINEAR COMBINATIONS OF RANDOM VARIABLES WITH LAPLACE DISTRIBUTION

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Abstract. We present some known results of Schur–convexity for distribution functions of linear combination of independent random variables. In addition, we present some new particular results for the Laplace distribution.

1. Introduction

Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbf{R}^n$, and let $x_{[i]}, y_{[i]}$ denote the i-th largest component of $\mathbf{x}, \mathbf{y}$ respectively. Then we say that $\mathbf{x} \prec \mathbf{y}$ ($\mathbf{x}$ is majorized by $\mathbf{y}$) if (see [6] for details)

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}\ , k = 1, 2, \ldots, n-1, \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}\ .$$

A function f of n arguments is Schur–convex on a set $A \subset \mathbf{R}^n$ if, for all $\mathbf{x}, \mathbf{y} \in A$,

$$\mathbf{x} \prec \mathbf{y} \Rightarrow f(\mathbf{x}) \leq f(\mathbf{y})\ .$$

A function f is Schur–concave if

$$\mathbf{x} \prec \mathbf{y} \Rightarrow f(\mathbf{x}) \geq f(\mathbf{y})\ .$$

A function f is Schur–concave if $-f$ is Schur–convex.

Let f be a Schur–convex function on a convex set A. Then

$$f(x_1, x_2, \ldots, x_n) \geq f(m, m, \ldots, m),\ m = \frac{x_1 + \cdots + x_n}{n}. \tag{1}$$

We can apply Schur–convexity to produce useful inequalities that are sometimes very hard to prove using other methods.

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Consider $X_1, \dots, X_n$ independent random variables with the same distribution. Let

$$F(c_1, \dots, c_n; t) = P(c_1X_1 + \cdots + c_nX_n \leq t)\ c_i \geq 0,\ i = 1, 2, \dots, n$$

We investigate Schur–convexity of this function with respect to $(c_1, \dots, c_n)$, because it leads to interesting conclusions about the behaviour of tail probabilities for the linear combination of random variables (see [1] or [2]).

For symetric absolutely continuous distributions a general result is known. Let f be the common density of $X_1, \dots, X_n$. Proschan [8] showed that if f is symetric about zero and $\log f$ is concave , then the function

$$F(c_1, \dots, c_n; t) = P(c_1X_1 + \cdots + c_nX_n \leq t) \tag{2}$$

is Schur–concave in $(c_1, \dots, c_n)$ for every $t \geq 0$.

For positive random variables with nonsymetric distributions very little is known. Bock, Diaconis and Perlman [1] proved some results regarding Gamma and Weibull distribution. Their results may be summarized as follows.

Let X_i have a Gamma(α, β) density

$$f(x) = e^{-\beta x}x^{\alpha-1}\frac{\beta^\alpha}{\Gamma(\alpha)},\ (x \geq 0), \quad f(x) = 0,\ (x < 0).$$

By a result in [1], for $n = 2$, function F defined in (4) is Schur–convex in $\mathbf{c}$ if $t \leq \alpha(c_1 + c_2)/\beta$ and is Schur–concave for $t \geq (\alpha + \frac{1}{2})(c_1 + c_2)/\beta$.

For a general n and the same Gamma distribution, it is proved in [1] that F is Schur–convex in $\mathbf{c}$ in region

$$\left\{c : \min_{1 \leq i \leq n} c_i \geq \frac{t\beta}{n\alpha + 1}\right\},$$

and this function is Schur–concave in $\mathbf{c}$ for

$$t \geq \frac{(n\alpha + 1)(c_1 + \cdots + c_n)}{\beta}.$$

If X_1, X_2 are independent random variables with a Weibull density

$$f(x) = \gamma\beta x^{\beta-1}e^{\gamma x^\beta},\ (x > 0) \quad f(x) = 0,\ (x \leq 0),$$

then (according to [1]) the function F (for $n = 2$) is Schur–concave in $\mathbf{c}$ if

$$t \geq C(\beta)(c_1 + c_2)\left(\frac{1}{\gamma}\left(1 + \frac{1}{2\beta}\right)\right)^{1/\beta},$$

where $C(\beta) = 2^{(1/\beta)} - 1$ for $0 < \beta < 1$ and $C(\beta) = 1$for $\beta \geq 1$.

2. Laplace's random variables

In this part we give two different proofs for result related to Laplace distributions. The first proof is simpler then the other one, but the second gives the method for proving the results of this type (see [1] and [7]).

Theorem 1. Let X and Y be independent random variables with Laplace distribution, i.e. both variables has the same density

$$g(x) = \frac{1}{2\lambda} e^{-\frac{|x-\mu|}{\lambda}} .$$

Then the function

$$F(c_1, c_2, z) = P(c_1 X + c_2 Y \leq z) , \quad c_1 + c_2 = 1, \ c_1, c_2 > 0 ,$$

is Schur–convex in (c_1, c_2) if $z < \mu$ and it is Schur–concave in (c_1, c_2) if $z \geq \mu$.

In the proof of this theorem we require the following simple result.

Let $\varphi(t) = 1 + e^t - \frac{4}{2-t}$. Then $\varphi(t) > 0$ for $t < 0$.

Indeed, it is clear that $\varphi(0) = 0$. We shall prove that function $\varphi(t)$ have no zeros on interval $(-\infty, 0)$. The equation $\varphi(t) = 0$ is equivalent to

$$e^t = \frac{2+t}{2-t}. \tag{3}$$

The right side in (3) is positive for $-2 < t < 2$. Therefore, (3) can have real solutions only on interval $(-2, 2)$. In this case (3) is equivalent to

$$\psi(t) = \log(2+t) - \log(2-t) - t = 0 , \quad -2 < t < 2. \tag{4}$$

It is clear that $\psi(0) = 0$. Suppose now that the equation $\varphi(t) = 0$ has at least one solution $\alpha \in (-2, 0]$. Then the function ψ defined by (4) would also have root α, and we would have

$$\psi(0) = \psi(\alpha) = 0 .$$

By Rolle's theorem this would imply that the derivative ψ' has one root on interval $(-2, 0]$. However, this is not possible because root of ψ' is 0 only. This means that equation $\psi(0) = 0$ does not have solutions on interval $(-2, 0]$.

For $t \leq -2$, we have $\frac{4}{2-t} \leq 1$, so that $\varphi(t) > 0$ for $t \leq -2$. It follows that $\varphi(t) > 0$ for $t < 0$ because equation $\varphi(t) = 0$ have no solutions on $(-2, 0]$.

Proof of Theorem 1. It is easy to show that, for $-\infty < \mu < \infty$, $\lambda > 0$, we have for $c_1 \neq c_2$

$$(5) \quad F(c_1, c_2, z) = \begin{cases} \dfrac{1}{2}\dfrac{c_1^2 e^{(z-\mu)/c_1\lambda} - c_2^2 e^{(z-\mu)/c_2\lambda}}{c_1 - c_2}, & z < \mu, \\ 1 - \dfrac{c_1^2 e^{-(z-\mu)/c_1\lambda} - c_2^2 e^{-(z-\mu)/c_2\lambda}}{2(c_1 - c_2)}, & z \geq \mu. \end{cases}$$

and

$$(6) \qquad F\left(\frac{1}{2}, \frac{1}{2}, z\right) = \begin{cases} \dfrac{1}{2}\left(1 - \dfrac{z-\mu}{\lambda}\right) e^{2(z-\mu)/\lambda}, & z < \mu, \\ 1 - \dfrac{1}{2}\left(\dfrac{z-\mu}{\lambda}\right) e^{-2(z-\mu)/\lambda}, & z \geq \mu. \end{cases}$$

First, we can prove the Theorem 1 by using a result of Proschan [7]. Indeed, from the density function of X: $g(x) = \frac{1}{2\lambda} e^{-\frac{|x-\mu|}{\lambda}}$ for $t = x - \mu$, we get

$$(7) \qquad g(t) = \frac{1}{2\lambda} e^{-\frac{|t|}{\lambda}}, \; -\infty < t < \infty.$$

Let X^* and Y^* be independent random variables with the same density function (7). Since the common density $g(t,s) = \frac{1}{4\lambda^2} e^{(-|t|-|s|)/\lambda}$ of X^* and Y^* is simmetric about zero and $\log f$ is concave, by Proschran's theorem it follows that the function

$$\begin{aligned} (8) \quad F^*(c_1, c_2, u) &= P\{c_1 X^* + c_2 Y^* \leq u\} \\ &= \begin{cases} \dfrac{1}{2}\dfrac{c_1^2 e^{u/c_1\lambda} - c_2^2 e^{u/c_2\lambda}}{c_1 - c_2}, & c_1 \neq c_2, \quad u < 0, \\ 1 - \dfrac{c_1^2 e^{-u/c_1\lambda} - c_2^2 e^{-u/c_2\lambda}}{2(c_1 - c_2)}, & c_1 \neq c_2, \quad u \geq 0. \end{cases} \end{aligned}$$

and

$$(9) \qquad F^*\left(\frac{1}{2}, \frac{1}{2}, u\right) = \begin{cases} \frac{1}{2}\left(1 - \frac{u}{\lambda}\right) e^{2u/\lambda}, & u < 0, \\ 1 - \frac{1}{2}\left(1 - \frac{u}{\lambda}\right) e^{-2u/\lambda}, & u \geq 0. \end{cases}$$

is Schur–concave in $(c_1, \ldots, c_n)$ for $u \geq 0$ and Schur–convex for $u < 0$. If in (5) we put $z - \mu = u$, we get the function $F^*(c_1, c_2, u)$. Now, it follows that function F is Schur–concave for $z - \mu \geq 0$, i.e. for $z \geq \mu$ and Schur–convex for $z < \mu$.

Now we present our second proof of Theorem 1.

Since $\lim_{c_1 \to 1/2} F(c_1, 1-c_1, z) = F(1/2, 1/2, z)$, it suffices to show Schur–convexity (Schur–concavity) of F with respect to (c_1, c_2) in the domain $c_1 + c_2 = 1, c_1 \neq c_2, c_1, c_2 > 0$.

Let $z < \mu$. If we assume that $c_1 > c_2$ then by $c_1 + c_2 = 1$ we have $c_1 > 1/2$ and $c_2 = 1 - c_1$. For $c_1 = c$, from (5), we get

$$(10) \qquad f(c,z) = \frac{1}{2} \frac{c^2 e^{(z-\mu)/c\lambda} - (1-c)^2 e^{(z-\mu)/(1-c)\lambda}}{2c-1} .$$

To show that function F from (5) is Schur–convex it is enough to show that f from (10) is nondecreasing in $c \in (1/2, 1]$.

The derivative of f with respect to c is

$$\begin{aligned} \frac{\partial f(c,z)}{\partial c} &= \frac{1}{2} \frac{1}{(2c-1)^2} \left[e^{-(z-\mu)/c\lambda} \left(2c^2 - 2c - 2c\frac{z-\mu}{\lambda} + \frac{z-\mu}{\lambda} \right) \right. \\ &+ \left. e^{(z-\mu)/(1-c)\lambda} \left(2c - 2c^2 - 2c\frac{z-\mu}{\lambda} + \frac{z-\mu}{\lambda} \right) \right] . \end{aligned}$$

For $1/2 < c < 1$ and $z > 0$, the sign of the latter expression is the same as the sign of

$$(11) \quad A(z,c) = 1 + e^{[(z-\mu)/(1-c)\lambda - (z-\mu)/c\lambda]} + \frac{4c(c-1)}{2c(1-c) + (z-\mu)(1-2c)/\lambda} .$$

Let

$$(12) \qquad t = \frac{z-\mu}{(1-c)\lambda} - \frac{z-\mu}{c\lambda} = \frac{(z-\mu)(2c-1)}{\lambda(1-c)c}, \ \lambda > 0$$

By substitution in (11), we get

$$A(z,c) = 1 + e^t - \frac{4}{2-t} = \varphi(t) .$$

Now, by (12), we have that $A(z,c) > 0$ for $z < \mu$, and the function f is nondecreasing on c, which implies Schur–convexity of F for $z < \mu$.

Let $z \geq \mu$. We assume, again, that $c_1 > c_2$, then by $c_1 + c_2 = 1$ we have $c_1 > 1/2$ and $c_2 = 1 - c_1$. For $c_1 = c$ from (5), we get

$$(13) \quad F(c_1, c_2, z) = F(c_1, 1-c_1, z) = F(c, 1-c, z) = 1 - h(c,z), \ z \geq 0,$$

where

$$h(c,z) = \frac{c^2 e^{-(z-\mu)/c\lambda} - (1-c)^2 e^{-(z-\mu)/(1-c)\lambda}}{2c-1} .$$

The function F in (13) is Schur–convex (Schur–concave) if and only if the function h is nonincreasing (nondecreasing) in $c \in (1/2, 1]$.

The derivative of the function h with respect to c reads:

$$\begin{aligned}\frac{\partial h(c,z)}{\partial c} &= \frac{1}{(2c-1)^2}\left[e^{-(z-\mu)/c\lambda}\left(2c^2 - 2c + 2c\frac{z-\mu}{\lambda} - \frac{z-\mu}{\lambda}\right)\right. \\ &+ \left. e^{-(z-\mu)/(1-c)\lambda}\left(2c - 2c^2 + 2c\frac{z-\mu}{\lambda} - \frac{z-\mu}{\lambda}\right)\right] .\end{aligned}$$

For $1/2 < c < 1$ and $z > 0$, the sign of the latter expression is the same as the sign of

$$(14)\quad B(z,c) = 1 + e^{-[(z-\mu)/(1-c)\lambda - (z-\mu)/c\lambda]} + \frac{4c(c-1)}{2c(1-c) + (z-\mu)(2c-1)/\lambda} .$$

Let

$$(15)\qquad t = \frac{z-\mu}{(1-c)\lambda} - \frac{z-\mu}{c\lambda}$$

By substitution in (14), we get

$$B(z,c) = 1 + e^{-t} - \frac{4}{2+t} = \varphi(-t) .$$

Now we have that $B(z,c) > 0$ for $z > \mu$, and the function h is nondecreasing on c, which implies Schur–concavity of F for $z > \mu$. The proof is completed.

Corollary 1. Let X and Y be independent random variables with double Exponential distribution, i.e. both variables have the same density $g(x) = \frac{1}{2}e^{-|x|}$, $-\infty < x < \infty$. Then the function

$$F(c_1, c_2, z) = P(c_1X + c_2Y \le z) , \quad c_1 + c_2 = 1,\ c_1, c_2 > 0 ,$$

is Schur–convex on (c_1, c_2) if $z < 0$ and it is Schur–concave on (c_1, c_2) if $z \ge 0$.

It follows from Theorem 1, for $\mu = 0$ and $\lambda = 1$.

Corollary 2. For X, Y, c_1, c_2 as in the Theorem 1, the following inequalities hold:

$$\begin{aligned} P(c_1X + c_2Y \le z) &\ge P\left(\frac{X+Y}{2} \le z\right) \quad \text{if } z < \mu \\ P(c_1X + c_2Y \le z) &\le P\left(\frac{X_1+X_2}{2} \le z\right) \quad \text{if } z \ge \mu .\end{aligned}$$

Proof. Immediate by Theorem 1 and the inequality (3).

Comment. By theorem of Proschan it follows that the function

$$F(c_1, \dots, c_n; y) = P(c_1X_1 + \dots + c_nX_n \le z),\ c_i > 0, i = 1, \dots, n, \sum_{i=1}^{n} c_i = 1$$

is Schur–convex in $(c_1, \dots, c_n)$ for every $z < \mu$ and it is is Schur–concave in $(c_1, \dots, c_n)$ for every $t \ge \mu$.

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