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STAR-MENGER AND RELATED SPACES, II

Ljubiša D. Kočinac

Abstract. Some covering properties of topological spaces which are defined in terms of possibility to choose from a given sequence of covers (of some kind) a cover of the same or a different sort are considered. In particular we are interested in preservation of such properties in the preimage direction under several sorts of continuous mappings. The properties include the classical concepts: the Menger property, Rothberger's property and so on. For example, it is shown that the Menger property (in all finite powers) is an inverse invariant of closed irreducible finite-to-one mappings.

0. Introduction and definitions

In this paper we use the usual topological notation and terminology as in [1] and [4] and assume that all spaces are Hausdorff and all mappings are continuous surjections.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of a topological space X. Then the symbol $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(B_n : n \in \mathbb{N})$ such that for each n, $B_n \in A_n$ and $\{B_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$. The symbol $\mathsf{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, B_n is a finite subset of A_n and $\bigcup_{n\in\mathbb{N}} B_n$ is an element of $\mathcal{B}$ (see [8], [14]).

In [9] we introduced the following selection hypotheses:

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0.1. Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of families of subsets of a space X. Then:

(a) The symbol $\mathsf{S}_1^*(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each n, $U_n \in \mathcal{U}_n$ and $\{St(U_n,\mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$;

(b) The symbol $\mathsf{S}_{\mathbf{fin}}^*(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis: for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and $\bigcup_{n\in\mathbb{N}}\{St(V,\mathcal{U}_n) : V \in \mathcal{V}_n\} \in \mathcal{B}$;

(c) Let $\mathcal{K}$ be a family of subsets of X. Then we say that X belongs to the class $\mathbf{SS}_{\mathcal{K}}^*(\mathcal{A},\mathcal{B})$ if X satisfies the following selection hypothesis: for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(K_n : n \in \mathbb{N})$ of elements of $\mathcal{K}$ such that $\{St(K_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

When $\mathcal{K}$ is the collection of all one-point [resp., finite, compact] subspaces of X we write $\mathbf{SS}_1^*(\mathcal{A},\mathcal{B})$ [resp., $\mathbf{SS}_{\mathbf{fin}}^*(\mathcal{A},\mathcal{B})$, $\mathbf{SS}_{\mathbf{comp}}^*(\mathcal{A},\mathcal{B})$] instead of $\mathbf{SS}_{\mathcal{K}}^*(\mathcal{A},\mathcal{B})$. □

Here, as usual, for a subset A of a space X and a collection $\mathcal{P}$ of subsets of X, $St(A,\mathcal{P})$ denotes the star of A with respect to $\mathcal{P}$, that is the set $\bigcup\{P \in \mathcal{P} : A \cap P \neq \emptyset\}$; for $A = \{x\}$, $x \in X$, we write $St(x,\mathcal{P})$ instead of $St(\{x\},\mathcal{P})$.

Let us introduce now the following definition:

0.2. Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of a space X and let $\mathcal{K}$ be a family of subsets of X. Then we say that X belongs to the class $\mathbf{NSS}_{\mathcal{K}}^*(\mathcal{A},\mathcal{B})$ if X satisfies the following selection hypothesis: for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(K_n : n \in \mathbb{N})$ of elements of $\mathcal{K}$ such that for each neighborhood O_n of K_n, $n \in \mathbb{N}$, we have $\{St(O_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

When $\mathcal{K}$ is the collection of all one-point [resp., finite, compact] subspaces of X we write $\mathbf{NSS}_1^*(\mathcal{A},\mathcal{B})$ [resp., $\mathbf{NSS}_{\mathbf{fin}}^*(\mathcal{A},\mathcal{B})$, $\mathbf{NSS}_{\mathbf{comp}}^*(\mathcal{A},\mathcal{B})$] instead of $\mathbf{NSS}_{\mathcal{K}}^*(\mathcal{A},\mathcal{B})$. □

Note that all the selection hypotheses mentioned above have the games (of lenght ω) for two players, ONE and TWO, associated with them (see [14], [15], [9]). For example, the game $\mathsf{G}_{\mathbf{fin}}(\mathcal{A},\mathcal{B})$ corresponding to the selection hypothesis $\mathsf{S}_{\mathbf{fin}}(\mathcal{A},\mathcal{B})$ is played as follows: in the n-th round ONE chooses an element $A_n \in \mathcal{A}_n$ and TWO responds choosing a finite $B_n \subset A_n$. TWO wins the game if $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$; otherwise, ONE wins.

The game G associated with the selection hypothesis $\mathbf{NSS}_{\mathcal{K}}^*(\mathcal{A},\mathcal{B})$ is played in the following way: in the n-th round ONE chooses an element $A_n \in \mathcal{A}_n$ and TWO responds by choosing an element $K_n \in \mathcal{K}$. TWO wins if for each neighborhood O_n of K_n, $n \in \mathbb{N}$, $\{St(O_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

In this paper $\mathcal{A}$ and $\mathcal{B}$ will be the following collections of subsets of a space $(X, \mathcal{T})$:

$\mathcal{O}$ — the collection of open covers of X;

$\mathcal{C}$ — the collection of (not necessarily open) covers of X;

Ω — the collection of ω-covers of X. An open cover $\mathcal{U}$ of X is an *ω-cover* [5] if X does not belong to $\mathcal{U}$ and every finite subset of X is contained in a member of $\mathcal{U}$;

Γ — the collection of γ-covers of X. An open cover $\mathcal{U}$ of X is a *γ-cover* [5] if it is infinite and for every $x \in X$ the set $\{U \in \mathcal{U} : x \notin U\}$ is finite;

$\mathcal{D}$ — the collection of $\mathcal{U} \subset \mathcal{T}$ whose union is dense in X;

$\mathcal{F}$ — the collection of $\mathcal{U} \subset \mathcal{T}$ for which $X = \bigcup\{\overline{U} : U \in \mathcal{U}\}$;

$\mathcal{J}$ — the collection of $\mathcal{U} \subset \mathcal{P}(X)$ such that $X = \bigcup\{\mathrm{int}(U) : U \in \mathcal{U}\}$.

Recall that a space X is said to have the *Menger property* [11], [6], [7] (resp. the *Rothberger property* [13]) if the selection hypothesis $\mathsf{S}_{\mathbf{fin}}(\mathcal{O}, \mathcal{O})$ (resp. $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$) is true for X (see also [12], [8], [14]).

A space X is said to have: (1) the *star-Rothberger property* (SR), (2) the *star-Menger property* (SM), (3) the *strongly star-Rothberger property* (SSR), (4) the *strongly star-Menger property* (SSM), (5) *strongly star-K-Menger property* ($\mathrm{SS}_K\mathrm{M}$) if it satisfies the selection hypothesis: (1') $\mathsf{S}^*_1(\mathcal{O}, \mathcal{O})$, (2') $\mathsf{S}^*_{\mathbf{fin}}(\mathcal{O}, \mathcal{O})$, (3') $\mathbf{SS}^*_1(\mathcal{O}, \mathcal{O})$, (4') $\mathbf{SS}^*_{\mathbf{fin}}(\mathcal{O}, \mathcal{O})$, (5') $\mathbf{SS}^*_{\mathbf{comp}}(\mathcal{O}, \mathcal{O})$ (see [9]).

We shall say that X has: (i) the *nearly strongly star-Menger property* (NSSM), (ii) the *nearly strongly star-Rothberger property* (NSSR), (iii) the *nearly strongly star-K-Menger property* ($\mathrm{NSS}_K\mathrm{M}$) if it satisfies:

(i) $\mathbf{NSS}^*_{\mathbf{fin}}(\mathcal{O}, \mathcal{O})$, (ii) $\mathbf{NSS}^*_1(\mathcal{O}, \mathcal{O})$, (iii) $\mathbf{NSS}^*_{\mathbf{comp}}(\mathcal{O}, \mathcal{O})$.

A continuous mapping $f : X \to Y$ is said to be *irreducible* if the only closed subset F of X satisfying $f(F) = Y$ is $F = X$. f is *finite-to-one* if for each $y \in Y$ the set $f^{\leftarrow}(y)$ is finite in X. For $A \subset X$ we denote by $f^{\#}(A)$ the set $\{y \in Y : f^{\leftarrow}(y) \subset A\} = Y \setminus f(X \setminus U)$.

In this paper we shall need the following simple (but useful) lemma.

Lemma A. *If $f : X \to Y$ is a closed irreducible finite-to-one mapping and $\mathcal{U}$ is an ω-cover of X, then $f^{\#}(\mathcal{U}) := \{f^{\#}(U) : U \in \mathcal{U}\}$ is an ω-cover of Y.*

Proof. Because f is closed irreducible for each $U \in \mathcal{U}$, $f^{\#}(U) \neq Y$ and all elements of $f^{\#}(\mathcal{U})$ are nonempty open sets. These elements form an ω-cover of Y. Indeed, let F be a finite subset of Y. The set $f^{\leftarrow}(F)$ is finite in X;

take a set $U \in \mathcal{U}$ containing it. So, for each $y \in F$ we have $f^{\leftarrow}(y) \subset U$ which means $y \in f^{\#}(U)$ and consequently $F \subset f^{\#}(U)$. $\square$

We shall also use the following result from [1;VI.110]:

Lemma B. *If $f : X \to Y$ is a closed irreducible mapping and U is an open subset of X, then $f(\overline{U}) = \overline{f^{\#}(U)}$ (and $\overline{U} = \overline{f^{\leftarrow} f^{\#}(U)}$).* $\square$

1. Remarks

We give the following diagram which gives relationships between some here defined properties. Recall that a space X is said to be *strongly starcompact* [*starcompact*] (see [3], [10]) if for every open cover $\mathcal{U}$ of X there is a finite subset A of X [a finite $\mathcal{V} \subset \mathcal{U}$] such that $St(A,\mathcal{U}) = X$ [$St(\cup\mathcal{V},\mathcal{U}) = X$].

Let us note that each property from the diagram (and also NSSR) is an invariant of continuous mappings.

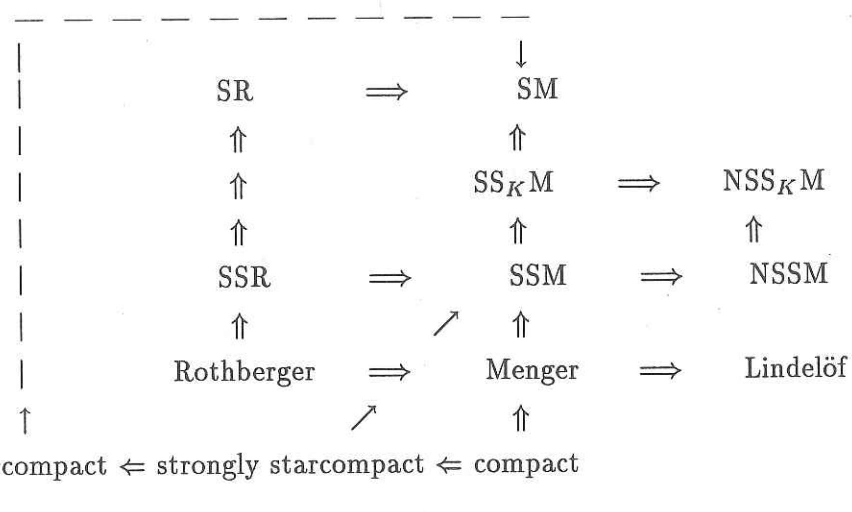

Diagram 1

Considering the definition of SSM spaces it might expect that the following natural definition would give a new concept: A space X is *extra strongly star-Menger* if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X there is a finite set F such that $X = \bigcup_{n\in\mathbb{N}} St(F,\mathcal{U}_n)$. (So, we require all finite sets in the definition of SSM spaces to be equal to the same set F.) But this gives nothing new because of:

1.1. Proposition. *A space X is extra* SSM *if and only if it is strongly starcompact.*

Proof. Let X be extra strongly star-Menger and let $\mathcal{U}$ be an open cover of X. Apply the assumption to the sequence $(\mathcal{U}_n \equiv \mathcal{U} : n \in \mathbb{N})$ and find a finite set $F \subset X$ with $St(F, \mathcal{U}) = X$.

Conversely, if X is strongly star compact and $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of open covers of X, then for a fixed $k \in \mathbb{N}$ there is a finite F such that $St(F, \mathcal{U}_k) = X$. Consequently, $X = \bigcup_{n \in \mathbb{N}} St(F, \mathcal{U}_n)$. □

Let us mention two results which show that in some classes of spaces some of the properties in the diagram coincide. The second result is also a new characterization of the Menger property for subsets of the real line. We omit the (simple) proof of the first assertion.

Recall that a space X is *mesocompact* if for each open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{V}$ which is compact-finite (in the sense that every compact subset of X intersects only finitely many members of $\mathcal{V}$).

1.2. Theorem. *A mesocompact space X is SSM if and only if it is SS_KM.*

1.3. Theorem. *For a paracompact (Hausdorff) space X the following are equivalent:*

(*a*) *X is a nearly strongly star-Menger space;*
(*b*) *X is a strongly star-Menger space;*
(*c*) *X is a strongly star-K-Menger space;*
(*d*) *X is a star-Menger space;*
(*e*) *X is a Menger space.*

Proof. We have to prove only that (a) implies (b), because $(b) \implies (a)$ is obvious and the other equivalences are shown in [9]. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of a paracompact nearly strongly star-Menger space X. For every $n \in \mathbb{N}$ let $\mathcal{V}_n$ be an open locally finite refinement of $\mathcal{U}_n$. Since X is nearly strongly star-Menger there exists a sequence $\{F_n : n \in \mathbb{N}\}$ of finite subsets of X witnessing for $(\mathcal{V}_n : n \in \mathbb{N})$ that fact. For each n there is a neighborhood O_n of F_n which meets only finitely many elements of $\mathcal{V}_n$. Pick a point from each of these intersections and denote by K_n the set of all such points. We get a finite subset of X satisfying $St(K_n, \mathcal{V}_n) = St(O_n, \mathcal{V}_n)$ so that we have $\bigcup_{n \in \mathbb{N}} St(K_n, \mathcal{V}_n) = \bigcup_{n \in \mathbb{N}} St(O_n, \mathcal{V}_n) = X$. For every $V \in \mathcal{V}_n$ let U_V be a member of $\mathcal{U}_n$ such that $V \subset U_V$. Then $\bigcup_{n \in \mathbb{N}} St(K_n, \mathcal{U}_n) = X$ which means that X is a strongly star-Menger space. □

So, for paracompact spaces we have

$$SSR \implies NSSR \implies NSSM \iff SSM \iff SS_KM \iff SM \iff M.$$

2. $S_{fin}(\mathcal{A},\mathcal{B})$

Very simple examples show that the Menger property is not an inverse invariant of continuous irreducible mappings. For instance, take any bijection $f : D(c) \to [0,1]$. But we have:

2.1. Theorem. *If $f : X \to Y$ is a closed irreducible finite-to-one mapping from a space X onto a Menger space Y, then X is also a Menger space.*

Proof. Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of open covers of X. Let $\mathbb{N} = N_1 \cup N_2 \cup \cdots \cup N_n \cup \ldots$ be a partition of $\mathbb{N}$ into countably many pairwise disjoint infinite subsets. For every $n \in \mathbb{N}$ let $\mathcal{V}_n$ be the set of elements of the form $U_{n_1} \cup \cdots \cup U_{n_k}$, $k \in \mathbb{N}$, $n_1 < n_2 < \cdots < n_k \in N_n$, $U_{n_i} \in \mathcal{U}_{n_i}$ for all $i = 1,2,\ldots,k$. The sequence $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of ω-covers of X. By Lemma A $(f^{\#}(\mathcal{V}_n))_{n\in\mathbb{N}}$ is a sequence of open (ω-) covers of Y. Using the fact that Y is in $S_{fin}(\mathcal{O},\mathcal{O})$ choose for each $n \in \mathbb{N}$ a finite set $\mathcal{W}_n \subset \mathcal{V}_n$ such that $\bigcup_{n\in\mathbb{N}}\{f^{\#}(W) : W \in \mathcal{W}_n\}$ is an open cover of Y. Then one can easily see that $X = \bigcup_{n\in\mathbb{N}} \bigcup f^{\leftarrow} f^{\#}(\mathcal{W}_n) \subset \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{W}_n$. But elements of $\mathcal{W}_n$'s may be augmented to a sequence of finite subsets of $\mathcal{U}_n$'s witnessing for $(\mathcal{U}_n)_{n\in\mathbb{N}}$ that X is in $S_{fin}(\mathcal{O},\mathcal{O})$. □

2.2. Example. The Menger property is not an inverse invariant of open finite-to-one mappings.

Let $X_1 = [0,\omega_1)$, $X_2 = [0,\omega_1]$, $X = X_1 \oplus X_2$, $Y = X_2$ and let $f : X \to Y$ be the mapping sending each point of X into the naturally corresponding point of Y. Then f is an open finite-to-one mapping, Y is a Menger space (because it is compact), but X is not Menger (because it is not Lindelöf). Let us mention that X is an SSM space which is not metacompact, because by Theorem 2.4 in [9] a metacompact SSM space is a Menger space. □

2.3. Example. The Menger property is not an inverse invariant of perfect mappings.

Let $X = [0,\omega_1) \times [0,\omega_1]$, $f : X \to [0,\omega_1]$. Then f is a perfect mapping onto a Menger space $[0,\omega_1]$, but X is not a Menger space. □

Using the reasoning similar to those from the proof of Theorem 2.1 one proves:

2.4. Theorem. *$S_{fin}(\Omega,\Omega)$ is an inverse invariant of closed irreducible finite-to-one mappings.* □

We shall mention now some consequences of this theorem and Theorem 2.1. For this we need some notation and terminology. For a Tychonoff space X $C_p(X)$ denotes the space of all continuous real-valued functions on X with the pointwise topology. Recall that a space X has *countable fan tightness* if for each sequence $(A_n; n \in \mathbb{N})$ of subsets of X and each point $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that for each n, B_n is a finite subset of A_n and $x \in \overline{\bigcup_{n\in\mathbb{N}} B_n}$ [2]. The symbol

$$\mathcal{A} \to [\mathcal{B}]^2_2$$

denotes the statement; for each member $A \in \mathcal{A}$ and for every coloring $f : [A]^2 \to \{1,2\}$ there are $i \in \{1,2\}$, a $B \subset A$ in $\mathcal{B}$ and a finite-to-one function $g : B \to \omega$ with: for all x and y in B, $g(x) \neq g(y)$ implies $f(\{x,y\}) = i$ (see [14]).

2.5. Corollary. *Let X and Y be Tychonoff spaces and let $f : X \to Y$ be a closed irreducible finite-to-one mapping. If Y has any of the properties $(a) - (d)$ below so does X:*

(a) Y has the Menger property in all finite powers;

(b) $C_p(Y)$ has countable fan tightness;

(c) ONE does not have a winning strategy in the game $\mathsf{G}_{\mathbf{fin}}(\mathcal{O},\mathcal{O})$ (played on Y);

(d) Y satisfies $\Omega \to [\Omega]^2_2$.

Proof. (a) By Theorem 3.9 in [8] Y^n has the Menger property for all $n \in \mathbb{N}$ if and only if Y is in $\mathsf{S}_{\mathbf{fin}}(\Omega,\Omega)$. By Theorem 2.4 X is also in $\mathsf{S}_{\mathbf{fin}}(\Omega,\Omega)$ and thus for all $n \in \mathbb{N}$ X^n has the Menger property.

(b) In [2], it was shown that $C_p(X)$ has countable fan tightness if and only if X^n has the Menger property for each positive integer n.

(c) This is a consequence of Theorem 2.4 and a result of Hurewicz (see Theorem 4 in [15]): Y has the Menger property if and only if ONE does not have a winning strategy in $\mathsf{G}_{\mathbf{fin}}(\mathcal{O},\mathcal{O})$.

(d) By Theorem 6.2 in [8] (d) is equivalent to the assertion Y is in $\mathsf{S}_{\mathbf{fin}}(\Omega,\Omega)$. □

Call a space X *almost Menger* [*almost Rothberger*] if it satisfies the selection hypothesis $\mathsf{S}_{\mathbf{fin}}(\mathcal{O},\mathcal{F})$ [$\mathsf{S}_1(\mathcal{O},\mathcal{F})$].

It is not difficult to prove (see the proof of Theorem 3.4):

2.6. Theorem. *The almost Menger property is an inverse invariant of closed irreducible finite-to-one mappings.* □

We also have:

2.7. Theorem. $\mathsf{S}_{\mathbf{fin}}(\mathcal{D},\mathcal{D})$ *is preserved in the preimage direction under closed irreducible mappings.* □

2.8. Remark. This class (and also the class $S_1(\mathcal{D},\mathcal{D})$) is not an inverse invariant of open mappings. The Souslin line S belongs to the class $S_1(\mathcal{D},\mathcal{D})$ and consequently to $S_{fin}(\mathcal{D},\mathcal{D})$. But it is well known that S^2 is not *ccc* and thus $S^2 \notin S_{fin}(\mathcal{D},\mathcal{D})$. □

3. $S_1(\mathcal{A},\mathcal{B})$

3.1. Theorem. *If $f: X \to Y$ is a closed irreducible mapping from a space X onto a space Y in $S_1(\mathcal{C},\mathcal{J})$, then X is in $S_1(\mathcal{O},\mathcal{F})$.*

Proof. Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of open covers of X. Then $(f(\mathcal{U}_n) : n \in \mathbb{N})$ is a sequence of covers of Y. Since Y satisfies $S_1(\mathcal{C},\mathcal{J})$, for each $n \in \mathbb{N}$ there is an element $U_n \in \mathcal{U}_n$ such that $Y = \bigcup_{n\in\mathbb{N}} \mathrm{int} f(U_n)$. We are going to prove $X = \bigcup_{n\in\mathbb{N}} \overline{U_n}$.

Let $x \in X$. Then $y = f(x) \in \mathrm{int} f(U_k)$ for some positive integer k. Let us prove $x \in \overline{U_k}$. Suppose $x \notin \overline{U_k}$. Let G be a neighborhood of x such that $G \cap U_k = \emptyset$. We have $f^{\#}(G) \cap f(U_k) = \emptyset$. On the other hand, by Lemma B, $y = f(x) \in \overline{f^{\#}(G)}$ and consequently $y \in \overline{f^{\#}(G)} \cap \mathrm{int} f(U_k)$; this implies $f^{\#}(G) \cap f(U_k) \neq \emptyset$ and we have a contradiction. Therefore, $x \in \overline{U_k}$, i.e. X is in $S_1(\mathcal{O},\mathcal{F})$. □

3.2. Theorem. *If $f: X \to Y$ is a closed irreducible mapping from a space X onto a space Y in $S_1(\mathcal{F},\mathcal{D})$, then X is also in $S_1(\mathcal{F},\mathcal{D})$.*

Proof. Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of elements from $\mathcal{F}$. Then, according to Lemma B, $(f^{\#}(\mathcal{U}_n) : n \in \mathbb{N})$ is a sequence of elements of $\mathcal{F}$ in Y. Applying the fact that Y satisfies $S_1(\mathcal{F},\mathcal{D})$, for each $n \in \mathbb{N}$ take an element $U_n \in \mathcal{U}_n$ such that $Y = \overline{\bigcup_{n\in\mathbb{N}} f^{\#}(U_n)}$. Then, again by Lemma B and using the fact that f is closed, it follows

$$X = f^{\leftarrow}(\overline{\bigcup_{n\in\mathbb{N}} f^{\#}(U_n)}) = \overline{\bigcup_{n\in\mathbb{N}} f^{\leftarrow} f^{\#}(U_n)} \subset \overline{\bigcup_{n\in\mathbb{N}} U_n}.$$

This witnesses membership of X to the class $S_1(\mathcal{F},\mathcal{D})$. □

The following two statements show that both the Rothberger property and almost Rothberger property are inverse invariants of closed irreducible finite-to-one mappings. Since the proofs are quite similar we show only the second assertion.

3.3. Theorem. *Let f be a closed irreducible finite-to-one mapping from a space X onto a space Y. Then X is in the same class as Y in the following cases:*

(1) *Y is a Rothberger space;*
(2) *Y is in $S_1(\Omega,\Omega)$.*
(3) *Y is in $S_1(\mathcal{O},\mathcal{D}) \equiv S_1(\Omega,\mathcal{D})$.* □

3.4. Theorem. *If f is a closed irreducible finite-to-one mapping from a space X onto an almost Rothberger space Y (i.e. $Y \in \mathsf{S}_1(\mathcal{O},\mathcal{F})$), then X is also almost Rothberger.*

Proof. Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of open covers of X. Working as in the proof of Theorem 2.1 construct the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of ω-covers of X. By Lemma A $(f^{\#}(\mathcal{V}_n))_{n\in\mathbb{N}}$ is a sequence of open (ω-) covers of Y. Using the fact that Y is in $\mathsf{S}_1(\mathcal{O},\mathcal{F})$ choose for each $n \in \mathbb{N}$ a set $V_n \in \mathcal{V}_n$ such that $Y = \bigcup_{n\in\mathbb{N}} \overline{f^{\#}(V_n)}$. Then, as can be easily verified, $X = \bigcup_{n\in\mathbb{N}} \overline{V_n}$. But each V_n is of the form $U_{n_1} \cup \cdots \cup U_{n_k}$, $k \in \mathbb{N}$, $n_1 < n_2 < \cdots < n_k \in N_n$, $U_{n_i} \in \mathcal{U}_{n_i}$ for all $i = 1, 2, \ldots, k$ (see the proof of Theorem 2.1). So, elements V_n will give a sequence of elements of $\mathcal{U}_n$'s (one from each) which guarantees that X is in $\mathsf{S}_1(\mathcal{O},\mathcal{F})$. □

Sets satisfying the selection hypothesis $\mathsf{S}_1(\Omega,\Gamma)$ are called *γ-sets*; every γ-set has Rothberger's property [5].

3.5. Theorem. *γ-sets are preserved in the preimage direction by closed irreducible finite-to-one mappings.*

Proof. Let $f : X \to Y$ be a closed irreducible mapping from a space X onto a γ-set Y and let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of ω-covers of X. Then $(f^{\#}(\mathcal{U}_n))_{n\in\mathbb{N}}$ is a sequence of ω-covers of Y so that there are $f^{\#}(U_n) \in f^{\#}(\mathcal{U}_n)$, $n \in \mathbb{N}$, such that $\{f^{\#}(U_n) : n \in \mathbb{N}\}$ is a γ-cover of Y. We prove that $\{U_n : n \in \mathbb{N}\}$ is a γ-cover of X. Let $x \in X$. Then $f(x)$ belongs to all but finitely many elements of $\{f^{\#}(U_n) : n \in \mathbb{N}\}$, say $f^{\#}(U_{n_1}), \ldots, f^{\#}(U_{n_k})$. Clearly, x belongs to all the sets U_n for $n \notin \{n_1, \ldots, n_k\}$. □

4. Star-covering properties

4.1. Example. The SSM property is not an inverse invariant of irreducible finite-to-one mappings.

Let $T = [0,\omega_1] \times [0,\omega] \setminus \{(\omega_1,\omega)\}$ be the Tychonoff plank. Let X denote the Dieudonné plank: the underlying set is T and open sets in X are all singletons of $[0,\omega_1)\times[0,\omega)$ and all sets of the form $U_\alpha(\beta) = \{(\beta,\gamma) : \alpha < \gamma \leq \omega\}$ and $V_\alpha(\beta) = \{(\gamma,\beta) : \alpha < \gamma \leq \omega_1\}$ (see [16]). It is known that X is metacompact non-Lindelöf space, hence X is not a Menger space. Since the Dieudonné topology is finer than the Tychonoff topology the identity mapping $i : X \to T$ is continuous, and obviously irreducible finite-to-one. It was shown in [9] that T is an SSM space. But X cannot be such a space because otherwise, by Theorem 2.4 in [9] which states that every metacompact SSM space is a Menger space, it would be a Menger space. □

4.2. Example. There is a consistent example of an SSM space X whose product with a compact space Y is not SSM [9]. So, the SSM property is not an inverse invariant of perfect mappings. □

4.3. Theorem. *Let $f : X \to Y$ be a closed irreducible mapping. If Y is in $\mathbf{SS}^*_{\mathbf{fin}}(\mathrm{int}\mathcal{C}, \Omega)$, then X is in $\mathbf{SS}^*_{\mathbf{fin}}(\overline{\mathcal{O}}, \Omega)$.*

Proof. Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of open covers of X. Then the sequence $(f(\mathcal{U}_n))_{n\in\mathbb{N}}$ is a sequence of covers (from $\mathcal{C}$) of Y. Apply the fact $Y \in \mathbf{SS}^*_{\mathbf{fin}}(\mathrm{int}\mathcal{C}, \Omega)$ and find a sequence $(B_n)_{n\in\mathbb{N}}$ of finite subset of Y such that $\{St(B_n, \mathrm{int}f(\mathcal{U}_n)) : n \in \mathbb{N}\}$ is an ω-cover of Y. We shall prove that $\{St(f^{\leftarrow}(B_n), \overline{\mathcal{U}_n}) : n \in \mathbb{N}\}$ is an ω-cover of X.

Let K be a finite subset of X.Then $f(K) \subset St(B_m, \mathrm{int}f(\mathcal{U}_m))$ for some $m \in \mathbb{N}$. Working similarly as in the proof of the corresponding part of Theorem 3.1 (see also the proof of Theorem 4.4) one proves $K \subset St(f^{\leftarrow}(B_m), \overline{\mathcal{U}_m})$ which is enough to finish the proof. □

4.4. Theorem. *Let $f : X \to Y$ be a closed irreducible finite-to-one mapping. If Y is in $\mathbf{SS}^*_{\mathbf{fin}}(\Omega, \mathcal{O})$ (resp. $\mathbf{SS}^*_{\mathbf{fin}}(\Omega, \Omega)$, $\mathbf{SS}^*_{\mathbf{1}}(\Omega, \Omega)$), then X is in the same class.*

Proof. We consider only the first class. Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of ω-covers of X. Then the sequence $(f^{\#}(\mathcal{U}_n))_{n\in\mathbb{N}}$ is a sequence of ω-covers of Y. Apply the fact $Y \in \mathbf{SS}^*_{\mathbf{fin}}(\Omega, \mathcal{O})$ and choose for each $n \in \mathbb{N}$ a finite set $B_n \subset Y$ such that $\bigcup_{n\in\mathbb{N}} St(B_n, f^{\#}(\mathcal{U}_n)) = Y$. It is enough now to prove $X = \bigcup_{n\in\mathbb{N}} St(f^{\leftarrow}(B_n), \mathcal{U}_n)$.

Let $x \in X$.Then $y = f(x) \in St(B_k, f^{\#}(\mathcal{U}_k)$ for some $k \in \mathbb{N}$. So, there is a $U \in \mathcal{U}_k$ such that $y \in f^{\#}(U))$ and $f^{\#}(U) \cap B_k \neq \emptyset$. Then $x \in f^{\leftarrow}(y) \subset f^{\leftarrow}f^{\#}(U) \subset U$ and $U \cap f^{\leftarrow}(B_k) \supset f^{\leftarrow}f^{\#}(U) \cap f^{\leftarrow}(B_k) = f^{\leftarrow}(f^{\#}(U) \cap B_k) \neq \emptyset$. Thus $x \in St(f^{\leftarrow}(B_k), \mathcal{U}_k)$. □

4.5. Theorem. *The classes $\mathbf{S}^*_{\mathbf{fin}}(\Omega, \mathcal{O})$, $\mathbf{S}^*_{\mathbf{1}}(\Omega, \mathcal{O})$ and $\mathbf{S}^*_{\mathbf{1}}(\Omega, \Omega)$ are inverse invariants of closed irreducible finite-to-one mappings.*

Proof. We consider only the first class. Let $Y \in \mathbf{S}^*_{\mathbf{fin}}(\Omega, \mathcal{O})$ and let $f : X \to Y$ be a closed irreducible mapping. Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of ω-covers of X. For the sequence $(f^{\#}(\mathcal{U}_n))_{n\in\mathbb{N}}$ of ω-covers of Y there is a sequence $(f^{\#}(\mathcal{V}_n))_{n\in\mathbb{N}}$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $Y = \bigcup_{n\in\mathbb{N}} St(\cup\mathcal{V}_n, \mathcal{U}_n)$. We want to prove $X = \bigcup_{n\in\mathbb{N}} St(\cup\mathcal{V}_n, \mathcal{U}_n)$.

Take a point $x \in X$. Then $y = f(x) \in St(\cup f^{\#}(\mathcal{V}_m), \mathcal{U}_m)$ for some $m \in \mathbb{N}$, i.e. y belongs to some $f^{\#}(U) \in f^{\#}(\mathcal{U}_m)$ such that $f^{\#}(U)$ meets a $f^{\#}(V)$ in $f^{\#}(\mathcal{V}_m)$. It follows that $U \cap V \neq \emptyset$ and since $x \in U$ we have $x \in St(\cup\mathcal{V}_m, \mathcal{U}_m)$. □

4.6. Theorem. *Let $f : X \to Y$ be a closed irreducible finite-to-one mapping from a space X onto a space Y in $\mathbf{NSS}^*_{\mathbf{comp}}(\Omega, \mathcal{O})$ (resp. $\mathbf{NSS}^*_{\mathbf{fin}}(\Omega, \mathcal{O})$). Then X is also in $\mathbf{NSS}^*_{\mathbf{comp}}(\Omega, \mathcal{O})$ (resp. $\mathbf{NSS}^*_{\mathbf{fin}}(\Omega, \mathcal{O})$).*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequnce of ω-covers of X. Then, by Lemma A, $(f^\#(\mathcal{U}_n) : n \in \mathbb{N})$ is a sequence of ω-covers of Y. So, there is a sequence $(K_n : n \in \mathbb{N})$ of compact subspaces of Y witnessing for $(f^\#(\mathcal{U}_n) : n \in \mathbb{N})$ that Y is in $\mathbf{NSS}^*_{\mathbf{comp}}(\Omega, \mathcal{O})$. Consider the sequence $(f^\leftarrow(K_n) : n \in \mathbb{N})$ of compact subspaces of X. We are going to prove that this sequence witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that X is in $\mathbf{NSS}^*_{\mathbf{comp}}(\Omega, \mathcal{O})$.

Let $(O_n : n \in \mathbb{N})$ be a sequence of neighborhoods of $f^\leftarrow(K_n)$, $n \in \mathbb{N}$. Since f is closed for every $n \in \mathbb{N}$ there is a neighborhood V_n of K_n such that $f^\leftarrow(V_n) \subset O_n$. Since $Y = \bigcup_{n \in \mathbb{N}} St(V_n, f^\#(\mathcal{U}_n))$ we have (as can be easily checked)

$$X = f^\leftarrow(Y) = \bigcup_{n \in \mathbb{N}} St(f^\leftarrow(V_n), f^\leftarrow f^\#(\mathcal{U}_n)) \subset \bigcup_{n \in \mathbb{N}} St(O_n, \mathcal{U}_n),$$

i.e. X belongs to the class $\mathbf{NSS}^*_{\mathbf{comp}}(\Omega, \mathcal{O})$.

The second assertion is shown quite similarly. □

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Department of Mathematics, Faculty of Sciences and Mathematics, University of Niš
Ćirila i Metodija 2, 18000 Niš, Yugoslavia
E-mail: lkocinac@ptt.yu