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FUZZY B-OPEN SETS AND FUZZY B-SEPARATION AXIOMS

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Abstract. The concepts of fuzzy b-open sets, fuzzy b-continuous mappings and fuzzy weakly b-continuous mappings are introduced. Further fuzzy separation axioms have been introduced and investigated with the help of fuzzy b-open sets.

1. Introduction

Since semiopen sets were introduced by Levine [9] in 1963, many studies have been done on this topic. The preopen sets were introduced by Mashhour, Abd El-Monsef and El-Deep [10] in 1982. Mashhour [11] in 1983 defined the class of semipreopen sets.

As a generalization of these notions, in fuzzy topology Azad [2], Singal [13] and Park [12] have introduced the fuzzy semiopen sets, fuzzy preopen sets and fuzzy semipreopen sets, respectively.

The class of b-open sets was introduced and studied by Andrijevic [1] in 1996.

Here we introduce the class of fuzzy b-open sets and establish some of their properties. Also we discuss the relationship between this class and the classes above mentioned.

In the Section 3 we show that the concept of fuzzy b-open sets is weaker than any one of the concepts of fuzzy semiopen or fuzzy preopen sets. On the other hand, it is stronger than the concept of fuzzy semipreopen sets.

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In the Section 4 and the Section 5 new weaker forms of fuzzy continuity: fuzzy b-continuity and fuzzy weak b-continuity are introduced. In addition, following the concept used in [4, 5, 14] we discuss the relations between these new weaker forms and some other weaker forms of continuity defined earlier.

The separation axioms in fuzzy tpological spaces were studied by many authors, see e.g. [6, 7, 13]. Here, in the Section 6 we give an extension of fuzzy separation notions using the fuzzy b-open sets. By means of numerous examples, we point out the non-coincidence of different notions of b-separation.

2. Preliminaries

We now introduce some basic notions and results that are used in the sequel. In this work by (X, τ) or simply by X we will denote a fuzzy topological space (fts) due to Chang [3]. The interior, closure, and the complement of a fuzzy set A will be denoted by intA, clA and A^c, respectively.

Definition 2.1. *Let A be a fuzzy set of an fts X. Then A is called*

(1) a fuzzy semiopen set if and only if there exists a fuzzy open set U such that $U \leq A \leq clU$ [2];

(2) a fuzzy preopen set if and only if $A \leq int(clA)$ [13];

(3) a fuzzy semipreopen set if and only if there exists a fuzzy preopen set U such that $U \leq A \leq clU$ [12].

The family of all fuzzy semiopen sets, fuzzy preopen sets and fuzzy semipreopen sets of fts (X, τ) will be denoted by FSO(τ), FPO(τ) and FSPO(τ), respectively.

Lemma 2.1. *Let A be a fuzzy set of an fts X. Then A is*

(1) a fuzzy semiopen set if and only if $A \leq cl(intA)$[2];

(2) a fuzzy semipreopen set of X if and only if $A \leq cl(pintA)$[12]. ||

Definition 2.2. *Let A be a fuzzy set of an fts X. Then A is called*

(1) a fuzzy semiclosed set if and only if A^c *is a fuzzy semiopen set* [2];

(2) a fuzzy preclosed set if and only if A^c *is a fuzzy preopen set* [13];

(3) a fuzzy semipreclosed set if and only if A^c *is a fuzzy semipreopen set* [12].

The family of all semiclosed sets, fuzzy preclosed sets and fuzzy semipreclosed sets of (X, τ) will be denoted by FSC(τ), FPC(τ) and FSPC(τ), respectively.

Definition 2.3. *Let A be a fuzzy set of an fts X. Then,*

$sintA = \{B \mid B \leq A, B \in FSO(\tau)\}$, *is called the fuzzy semi-interior of A* [2];

$sclA = \{B \mid B \geq A, B \in FSC(\tau)\}$, *is called the fuzzy semiclosure of A* [2];

$pintA = \{B \mid B \leq A, B \in FPO(\tau)\}$, *is called the fuzzy preinterior of A* [13];

pclA$= \{B \mid B \geq A, B \in$*FPC*$(\tau)\}$, *is called the fuzzy preclosure of* A [13];

spintA$= \{B \mid B \leq A, B \in$*FSPO*$(\tau)\}$, *is called the fuzzy semipreinterior of* A [12];

spclA$= \{B \mid B \geq A, B \in$*FSPC*$(\tau)\}$, *is called the fuzzy semipreclosure of* A [12].

Lemma 2.2. [13] *Let* A *be a fuzzy set of an fts* X. *Then,*

(1) $pclA^c = (pintA)^c$;

(2) $pintA^c = (pclA)^c$. ||

Lemma 2. 3. *Let* A *be a fuzzy set of an fts* X. *Then,*

(1) $pclA \geq A \vee cl(intA)$;

(2) $pintA \leq A \wedge int(clA)$. ||

Definition 2.4. [15] *Let* f *be a mapping from a set* X *into a set* Y. *Let* A *and* B *be fuzzy sets of* X *and* Y *respectively. Then* $f(A)$ *is a fuzzy set of* Y *defined by*

$$f(A) = \begin{cases} \sup_{x\in f^{-1}(y)} A(x), & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}.$$

and $f^{-1}(B)$ *is a fuzzy set of* X, *defined by*

$$f^{-1}(B)(x) = B(f(x)), \text{ for each } x \in X.$$

Lemma 2.4. [2] *Let* $f : X \to Y$ *be a mapping. For any fuzzy sets* A *and* B *of* X *and* Y *receptively, the following statements hold:*

(1) $ff^{-1}(B) \leq B$;

(2) $f^{-1}f(A) \geq A$;

(3) $f(A^c) \geq f(A)^c$;

(4) $f^{-1}(B^c) = f^{-1}(B)^c$;

(5) *If* f *is injective, then* $f^{-1}f(A) = A$;

(6) *If* f *is surjective, then* $ff^{-1}(B) = B$;

(7) *If* f *is bijective, then* $f(A^c) = f(A)^c$. ||

Definition 2.5. *Let* $f : (X, \tau_1) \to (Y, \tau_2)$ *be a mapping from an fts* (X, τ_1) *into an fts* (Y, τ_2). *The mapping* f *is called*

(1) *fuzzy continuous if* $f^{-1}(B)$ *is a fuzzy open set of* X, *for each* $B \in \tau_2$[3];

(2) *fuzzy semicontinuous if* $f^{-1}(B)$ *is a fuzzy semiopen set of* X, *for each* $B \in \tau_2$[2];

(3) *fuzzy precontinuous if* $f^{-1}(B)$ *is a fuzzy preopen set of* X, *for each* $B \in \tau_2$[13];

(4) *fuzzy semiprecontinuous if* $f^{-1}(B)$ *is a fuzzy semipreopen set of* X, *for each* $B \in \tau_2$[12];

(5) *fuzzy weakly continuous if* $f^{-1}(B) \leq intf^{-1}(clB)$, *for each* $B \in \tau_2$[2].

Lemma 2.5. [2] *Let* $g : X \to X \times Y$ *be a graph of a mapping* $f : X \to Y$. *If* A *is a fuzzy set of* X *and* B *is a fuzzy set of* Y, *then* $g^{-1}(A \times B) = A \wedge f^{-1}(B)$. ||

Definition 2.6. [8] *A fuzzy set of an fts X is called a fuzzy singleton if it takes the zero value for all points x in X except one point. The point at which the fuzzy singleton takes the non-zero value is called a support and the corresponding element of (0,1], is called its value. A fuzzy singleton with value 1 is called a crisp fuzzy singleton.*

Definition 2.7. [7] *An fts X is called fuzzy* $T_0(FT_0)$ *if and only if for each pair of fuzzy singletons* p_1, p_2 *with different supports, there exists a fuzzy open set U such that* $p_1 \leq U \leq p_2^c$ *or* $p_2 \leq U \leq p_1^c$.

3. Fuzzy b-open sets and fuzzy b-closed sets

Definition 3.1. *Let A be a fuzzy set of an fts X. Then A is called*
(1) a fuzzy b-open set if and only if A ≤pcl(pintA).
(2) a fuzzy b-closed set if and only if A ≥pint(pclA).

Remark 3.1. Immediately from the Definition 3.1, it follows that a fuzzy set A is fuzzy b-open if and only if A^c is fuzzy b-closed.

The family of all fuzzy b-open and fuzzy b-closed sets of (X, τ) will be denoted by FBO(τ) and FBC(τ), respectively. The family FBO(τ) contains τ; it may not be a fuzzy topology on X, but it is closed under arbitrary unions.

Theorem 3.1. *Let* (X, τ) *be an fts. Then,*

$$FSO(\tau) \cup FPO(\tau) \subseteq FBO(\tau) \subseteq FSPO(\tau).$$

Proof. Let A be a fuzzy preopen set. Then $A = pintA$. From $A \leq pclA = pcl(pintA)$ it follows that A is a fuzzy b-open set. If A is a fuzzy semiopen set then $A \leq cl(intA) = pcl(intA) \leq pcl(pintA)$. The proof of the second inclusion follows from A≤pcl(pintA)≤ cl(pintA). ||

The next example shows that the inclusions can not be replaced by equalities.

Example 3.1. Let $X = \{a, b, c\}$ and A, B, C be fuzzy sets of X defined as follows:

$$\begin{array}{lll} A(a) = 0,5 & A(b) = 0,2 & A(c) = 0,6, \\ B(a) = 0,3 & B(b) = 0,4 & B(c) = 0,3, \\ C(a) = 0,5 & C(b) = 0,5 & C(c) = 0,5, \\ D(a) = 0,2 & D(b) = 0,6 & D(c) = 0,2. \end{array}$$

Let $\tau = \{0, A, B, A \vee B, A \wedge B, 1\}$. By easy computation it can be shown that D is fuzzy semipreopen, but not fuzzy b-open. The fuzzy set C is fuzzy b-open, but C is neither fuzzy semiopen nor fuzzy preopen.

Lemma 3.2. *Let* $\{A_\alpha | \alpha \in I\}$ *, be a family of fuzzy sets of an fts X. Then,* *(1)* $\vee_{\alpha\in I}$ $pcl(A_\alpha) \leq pcl(\vee_{\alpha\in I}A)$;

(2) $\vee_{\alpha\in I} pint(A_\alpha) \leq pint(\vee_{\alpha\in I}A)$.

Proof. We prove only the statement (1). From $\mathrm{pcl}(A_\alpha) \leq pcl(\vee_{\alpha\in I}A_\alpha)$, for every $\alpha \in I$ we obtain $\vee_{\alpha\in I}\mathrm{pcl}(A_\alpha) \leq \mathrm{pcl}(\vee_{\alpha\in I}A_\alpha)$. ||

Theorem 3.3. *Let X be an fts.*

(1) Any union of fuzzy b-open sets is a fuzzy b-open set.

(2) Any intersection of fuzzy b-closed sets is a fuzzy b-closed set.

Proof. We prove only the statement (1). Let $\{\mathrm{A}_\alpha\}_{\alpha\in I}$ be a family of fuzzy b-open sets. Then $\mathrm{A}_\alpha \leq \mathrm{pcl}(\mathrm{pintA}_\alpha)$, for each $\alpha \in$I. Hence $\vee_{\alpha\in I}$ $\mathrm{A}_\alpha \leq \vee_{\alpha\in I}\mathrm{pcl}(\mathrm{pintA}_\alpha) \leq \mathrm{pcl}(\mathrm{pint}(\vee_{\alpha\in I}\mathrm{A}_\alpha))$. ||

Definition 3.2. *Let A be a fuzzy set of an fts X.*

(1) The union of all fuzzy b-open sets contained in A is called the fuzzy b-interior of A, denoted by bintA.

(2) The intersection of all fuzzy b-closed sets containing A is called the fuzzy b-closure of A, denoted by bclA.

Theorem 3.4. *Let A and B be fuzzy sets of an fts (X,τ). Then,*

(1) $A \in FBO(\tau) \Leftrightarrow A=bintA, \quad A \in FBC(\tau) \Leftrightarrow A=bclA$;

2) $A \leq B \Leftrightarrow bintA \leq bintB$, $\quad A \leq B \Leftrightarrow bclA \leq bclB$.

Proof. It follows from the Definition 3.2 and the Theorem 3.3. ||

The next statement gives the relationship between the operators fuzzy b-interior and fuzzy b-closure.

Theorem 3.5. *Let A be a fuzzy set of an fts X. Then:*

(1) $bclA^c = (bintA)^c$;

(2) $bintA^c = (bclA)^c$.

Proof. (1) $(\mathrm{bintA})^c=(\vee\{d|d\leq A,\ d\in FBO(\tau)\})^c$

$$= \wedge\{d^c\ |d\leq A,\ d\in FBO(\tau)\} =$$

$$=\wedge\{c|c\geq A^c,\ c\in FBC(\tau)\}=bclA^c.$$

(2) $(\mathrm{bclA})^c = (\mathrm{bcl}(A^c)^c)^c = ((\mathrm{bintA}^c)^c)^c=\mathrm{bintA}^c$. ||

Theorem 3.6. *Let A be a fuzzy set of an fts X. Then,*

(1) $intA\leq pintA\leq bintA\leq spintA\leq A \leq splA\leq bclA\leq pclA\leq clA$;

(2) $intA\leq sintA\leq bintA\leq spintA\leq A \leq spclA\leq bclA\leq sclA\leq clA$.

Proof. It follows from the definitions of the corresponding operators. ||

4. Fuzzy b-continuity

Definition 4.1. *A mapping $f:(X,\tau_1)\rightarrow(Y,\tau_2)$ from an fts X into an fts Y is called fuzzy b-continuous if $f^{-1}(B) \in FBO(\tau_1)$, for each $B \in \tau_2$.*

Remark 4.1. The following statements for the mapping $f : X \rightarrow Y$, where X and Y are fts, are valid:

(1) If f is fuzzy semicontinuous or f is fuzzy precontinuous, then f is a b-continuous mapping.

(2) If f is fuzzy b-continuous, then f is a fuzzy semiprecontinuous mapping.

The following example shows that the reverse statement may be not true.

Example 4.1. We consider the Example 3.1. If we put $\tau_1 = \{0, C, 1\}$ and f=id:$(X,\tau) \to (X,\tau_1)$ we conclude that f is fuzzy b-continuous but f is neither fuzzy semicontinuous nor fuzzy precontinuous. If we put $\tau_2 = \{0, D, 1\}$, then f=id:$(X,\tau) \to (X,\tau_2)$ is fuzzy semiprecontinuous, but not fuzzy b-continuous.

Theorem 4.1. *Let $f : (X,\tau_1) \to (Y,\tau_2)$ be a mapping from an fts (X,τ_1) into an fts (Y,τ_2). Then the following statements are equivalent:*

(i) f is a fuzzy b-continuous mapping.

(ii) *$f^{-1}(B)$ is a fuzzy b-closed set of X for each fuzzy closed set B of Y.*

(iii) $f(bclA) \leq clf(A)$, for each fuzzy set A of X.

(iv) $bclf^{-1}(B) \leq f^{-1}(clB)$, for each fuzzy set B of Y.

(v) *$f^{-1}(intB) \leq bintf^{-1}(B)$, for each fuzzy set B of Y.*

(vi) There exists a base β for τ_2 such that $f^{-1}(B)$ is fuzzy b-open set of X for each $B \in \beta$.

(vii) There exists a base β for τ_2 such that $f^{-1}(B)$ is fuzzy b-closed set of X for each $B^c \in \beta$.

Proof. (i)⇔(ii) Let B be a fuzzy closed set of Y. Then B^c is fuzzy open set of Y. According to the assumption $f^{-1}(B^c)$ is a fuzzy b-open set of X. From the Lemma 2.4 we conclude that $f^{-1}(B)$ is a fuzzy b-closed set of X.

(ii)⇔(iii) Let A be a fuzzy set of X. Then clf(A) is a fuzzy closed set of Y, and from (ii), we obtain $f^{-1}(\text{clf(A)})$ is a fuzzy b-closed set of X. Thus, $\text{bclA} \leq \text{bclf}^{-1} f(\text{A}) \leq \text{bclf}^{-1}(\text{clf(A)}) = f^{-1}(\text{clf(A)})$. Hence $f(\text{bclA}) \leq \text{clf(A)}$.

(iii)⇔(iv) Let B be a fuzzy set of Y. According to the assumption we have $f(\text{bcl}f^{-1}(\text{B})) \leq \text{cl}(ff^{-1}(\text{B})) \leq \leq \text{clB}$. Thus $\text{bcl}f^{-1}(\text{B}) \leq f^{-1}f(\text{bcl}f^{-1}(\text{B})) \leq f^{-1}(\text{clB})$.

(iv)⇔ (v) Let B be a fuzzy set of Y. From (iv) we obtain $f^{-1}(\text{clB}^c) \geq \text{bclf}^{-1}(B^c) = \text{bclf}^{-1}(\text{B})^c$, hence $f^{-1}(\text{intB}) = f^{-1}(\text{clB}^c)^c \leq (\text{bcl} f^{-1}(\text{B})^c)^c = \text{bint} f^{-1}(\text{B})$.

(v)⇔(i) Let B be a fuzzy open set of Y. Then B=intB. From (v) we obtain $f^{-1}(\text{B}) = f^{-1}(\text{intB}) \leq \leq \text{bint} f^{-1}(\text{B}) \leq f^{-1}(\text{B})$. Thus $f^{-1}(\text{B}) = \text{bint} f^{-1}(\text{B})$. Hence f is a fuzzy b-continuous mapping.

(i)⇔(vi) Obvious.

(vi)⇔(i) Let C be a fuzzy open set of Y. Then there exists a subfamily β_1 of β such that $\text{C} = \vee_{B\in\beta_1}\text{B}$, and $f^{-1}(\text{C}) = \vee_{B\in\beta_1} f^{-1}(\text{B})$. According to the assumption $f^{-1}(\text{B})$ is a b-open set. Thus $f^{-1}(\text{C})$ is a b-open set as a union of fuzzy b-open sets, hence f is a fuzzy b-continuous mapping.

(vi)⇒(vii) Can be easily proved. ||

Theorem 4.2. *Let $f : X \to Y$ be a mapping from an fts X into an fts Y. Then the following statements are equivalent:*

(i) f is fuzzy b-continuous.

(ii) pint(pcl$f^{-1}(B)) \leq f^{-1}(clB)$, for each fuzzy set B of Y.

(iii) f(pint(pcl$A$)) $\leq$clf(A), for each fuzzy set A of X.

Proof. (i)⇔(ii) Let B be a fuzzy set of Y. Then f^{-1}(clB) is a fuzzy b-closed set, hence f^{-1}(clB)$\geq$ $\geq$pint(pclf^{-1}(clB))$\geq$pint(pclf^{-1}(B)).

(ii)⇔(iii) Let A be a fuzzy set of X. Let us put B=f(A), then A$\leq f^{-1}$(B). According to the assumption we obtain pint(pclA)$\leq$pint(pcl f^{-1} (B)) $\leq f^{-1}$ (clB). Hence, f(pint(pclA)) $\leq$ clB=clf(A).

(iii)⇔ (*i*) Let D be a fuzzy closed set of Y. According to the assumption, f(pint(pclf^{-1}(D))$\leq$clff^{-1}(D)$\leq$clD=D.

Thus pint(pclf^{-1}(D))$\leq f^{-1}f$(pint(pclf^{-1}(D))$\leq f^{-1}$(D).

Hence f^{-1}(D) is a fuzzy b-closed set. ||

Theorem 4.3. *Let $f : X \to Y$ be* a *bijective mapping from an fts X into an fts Y. The mapping f is fuzzy b-continuous if and only if intf(A)$\leq$ f(bint$A$), for each fuzzy set A of X.*

Proof. Let f be fuzzy b-continuous and let A be a fuzzy set of X. Then f^{-1}(intf(A)) is a fuzzy b-open set of X. From Theorem 4.1, since f is injective we have f^{-1}(intf(A)) $\leq$bintf^{-1}(intf(A)) $\leq$bintf^{-1}f(A)=bintA. Again, since f is surjective, we obtain, intf(A) = ff^{-1}(intf(A)) $\leq f$ (bintA).

Conversely, let B be a fuzzy open set of Y. Then intB=B. According to the assumption, f(bintf^{-1}(B))$\geq$ $\geq$intff^{-1}(B)=intB=B. This implies that $f^{-1}f$(bintf$^{-1}(B)) \geq f^{-1}(B)$. Since f is injective we obtain bint f^{-1} (B)= $f^{-1}f$ (bint f^{-1} (B) $\geq f^{-1}$ (B). Thus bint f^{-1} (B)= f^{-1} (B). Hence f is fuzzy b-continuous. ||

Theorem 4.4. *Let $f : X \to Y$ and $g : Y \to Z$ be mappings, where X, Y and Z are fts's. If f is fuzzy b-continuous and g is fuzzy continuous, then gf is fuzzy b-continuous.*

Proof. It follows from the relation (gf)$^{-1}$(B)=$f^{-1}(g^{-1}$(B)), for each fuzzy set B of Z. ||

Corollary 4.5. *Let X, X_1 and X_2 be fts's and let $p_i : X_1 \times X_2 \to X_i$ $(i = 1,2)$ be the projections of $X_1 \times X_2$ onto X_i. If $f : X \to X_1 \times X_2$ is fuzzy b-continuous, then $p_i f$ is also fuzzy b-continuous.*

Proof. It follows from the fact that p_i $(i = 1,2)$ are fuzzy continuous mappings. ||

Theorem 4.6. *Let $f : X \to Y$ be a mapping from an fts X into an fts Y. If the graph $g : X \to X \times Y$* of *f is fuzzy b-continuous, then f is also fuzzy b-continuous.*

Proof. From the Lemma 2.5, for each fuzzy open set B of Y, f^{-1}(B)=1 $\wedge f^{-1}$(B)=g^{-1}(1×B). Since g is fuzzy b-continuous and 1×B is a fuzzy open set of X×Y, f^{-1}(B) is a fuzzy b-open set of X and hence f is fuzzy b-continuous. ||

5. Fuzzy weak b-continuity

Definition 5.1. *A mapping $f:(X,\tau_1)\to(Y,\tau_2)$ from an fts X into an fts Y is called fuzzy weakly b-continuous if $f^{-1}(B)\leq bint f^{-1}(bclB)$, for each $B\in\tau_2$.*

Remark 5.1. It is obvious that fuzzy b-continuity implies fuzzy weak b-continuity. From the next example we conclude that the implication is not reversible.

Example 5.1. Let X=$\{a,b,c\}$ and A,B,D be fuzzy sets of X defined as follows:

$$A(a)=0,4 \qquad A(b)=0,2 \qquad A(c)=0,1;$$

$$B(a)=0,5 \qquad B(b)=0,5 \qquad B(c)=0,5;$$

$$D(a)=0,3 \qquad D(b)=0,2 \qquad D(c)=0,6.$$

If we put $\tau_1=\{0, A, B, 1\}$ and $\tau_2=\{0, D, 1\}$, then f=id:$(X,\tau_1)\to(X,\tau_2)$ is fuzzy weakly b-continuous, but not fuzzy b-continuous.

Example 5.2. Let X=$\{a,b,c\}$ and A, B be fuzzy sets of X defined as follows:

$$A(a)=0,3 \qquad A(b)=0,1 \qquad A(c)=0,4;$$

$$B(a)=0,6 \qquad B(b)=0,7 \qquad B(c)=0,5.$$

If we put $\tau_1=\{0, B, 1\}$ and $\tau_2=\{0, A, 1\}$, then f=id:$(X,\tau_1)\to(X,\tau_2)$ is fuzzy weakly continuous, but not fuzzy weakly b-continuous.

Example 5.3. Let X=$\{a,b,c\}$ and A, B, C be fuzzy sets of X defined as follows:

$$A(a)=0,4 \qquad A(b)=0,2 \qquad A(c)=0,1;$$

$$B(a)=0,5 \qquad B(b)=0,5 \qquad B(c)=0,5;$$

$$C(a)=0,3 \qquad C(b)=0,2 \qquad C(c)=0.$$

If we put $\tau_1=\{0, C, 1\}$ and $\tau_2=\{0,A,B,1\}$, then f=id:$(X,\tau_1)\to(X,\tau_2)$ is fuzzy weakly b-continuous, but fuzzy weakly continuous.

The Examples 5.2 and 5.3 establish the following:

Theorem 5.1. *Fuzzy weak continuity and fuzzy weak b-continuity are independent notions.* ||

In the following theorems we give some characterizations of the fuzzy weakly b-continuous mappings.

Theorem 5.2. *A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ from an fts X into an fts Y is fuzzy weakly b-continuous if for each $B \in \tau_2, f^{-1}(bclB)$ is a fuzzy b-open set of X.*

Proof. It follows immediately from the Definition 5.1. ||

Theorem 5.3. *A mapping $f : X \to Y$ from an fts X into an fts Y is fuzzy weakly b-continuous if and only if for any fuzzy singleton x_α of X and any fuzzy open set B of Y containing $f(x_\alpha)$, there exists a fuzzy b-open set A of X containing x_α such that $f(A) \leq bclB$.*

Proof. Let f be fuzzy weakly b-continuous, x_α be a fuzzy singleton of X and B be a fuzzy open set of Y such that $f(x_\alpha) \leq$B. Then $x_\alpha \leq f^{-1}(\mathrm{B}) \leq \mathrm{bint} f^{-1}(\mathrm{bclB})$. Let A=bint$f^{-1}$(bclB). Then A is a fuzzy b-open set, and $f(\mathrm{A})=f(\mathrm{bint} f^{-1}(\mathrm{bclB})) \leq f f^{-1}(\mathrm{bclB}) \leq \mathrm{bclB}$.

Conversely, let B be a fuzzy open set of Y and let x_α be a fuzzy singleton of X such that $x_\alpha \leq f^{-1}(\mathrm{B})$. According to the assumption there exists a fuzzy b-open set A of X such that $x_\alpha \leq$A and $f(\mathrm{A}) \leq$bclB. Hence $x_\alpha \leq \mathrm{A} \leq f^{-1}f(\mathrm{A}) \leq f^{-1}(\mathrm{bclB})$ and $x_\alpha \leq \mathrm{A}=\mathrm{bintA} \leq \mathrm{bint} f^{-1}(\mathrm{bclB})$. Since x_α is arbitrary and $f^{-1}(\mathrm{B})$ is the union of all fuzzy singletons of $f^{-1}(\mathrm{B})$, $f^{-1}(\mathrm{B}) \leq \mathrm{bint} f^{-1}(\mathrm{bclB})$. Thus f is fuzzy weakly b-continuous. ||

Theorem 5.4. *Let $f : X \to Y$ be a mapping from an fts X into an fts Y. Then the following statements are equivalent:*

(i) f is fuzzy weakly b-continuous;

(ii) $bclf^{-1}(bintB) \leq f^{-1}(B)$ for each fuzzy closed set B of Y;

(iii) $f^{-1}(intB) \leq bintf^{-1}(bclB)$, for each fuzzy set B of Y;

(iv) $bclf^{-1}(bintB) \leq f^{-1}(clB)$, for each fuzzy set B of Y.

Proof. (i)⇔(ii) Let B be a fuzzy closed set of Y. Then B^c is a fuzzy open set of Y. According to the assumption, $f^{-1}(\mathrm{B}^c) \leq \mathrm{bint} f^{-1}(\mathrm{bclB}^c)$. From the Lemma 2.4 we have $f^{-1}(\mathrm{B})^c = f^{-1}(\mathrm{B}^c) \leq \mathrm{bint} f^{-1}(\mathrm{bclB}^c) = (\mathrm{bcl} f^{-1}(\mathrm{bintB}))^c$. Thus $\mathrm{bcl} f^{-1}(\mathrm{bintB}) \leq f^{-1}(\mathrm{B})$.

(ii)⇔(iii) Let B be a fuzzy set of Y. From the assumption we obtain

$$f^{-1}(\mathrm{clB}^c) \geq \mathrm{bcl} f^{-1}(\mathrm{bint}(\mathrm{clB}^c)).$$

Hence $f^{-1}(\mathrm{intB}) \leq \mathrm{bint} f^{-1}(\mathrm{bcl}(\mathrm{intB})) \leq \mathrm{bint} f^{-1}(\mathrm{bclB}))$.

(iii)⇔(iv) Can be proved by using the complement.

(iv)⇔(i) Let B be any fuzzy open set of Y. Then B^c is a fuzzy closed set of Y. According to the assumption $\mathrm{bcl} f^{-1}(\mathrm{bintB}^c) \leq f^{-1}(\mathrm{clB}^c) = f^{-1}(\mathrm{B}^c)$. Hence $\mathrm{bint} f^{-1}(\mathrm{bclB}) \geq f^{-1}(\mathrm{B})$. ||

Corollary 5.5. *Let $f : X \to Y$ be a mapping from an fts X into an fts Y. If f is fuzzy weakly b-continuous, then $bclf^{-1}(B) \leq f^{-1}(clB)$, for each* *fuzzy open set B of Y.* ||

Theorem 5.6. *Let $f : X \to Y$ be a mapping from an fts X into an fts Y. If f is fuzzy open and fuzzy weakly b-continuous then $f(bclA) \leq clf(A)$, for each fuzzy open set A of X.*

Proof. Let A be a fuzzy open set of X and let f(A)=B. Since f is fuzzy open, we conclude that B is a fuzzy open set of Y. Since A $\leq f^{-1}f$ (A)= f^{-1} (B) and f is fuzzy weakly b-continuous, from the Theorem 5.4 we obtain bcl f^{-1} (B) $\leq f^{-1}$ (clB). Thus bclA $\leq f^{-1}$ (clB). Hence f (bclA) $\leq$ clB=clf(A). ||

The following example shows that the composition of two fuzzy weakly b-continuous mappings may be not fuzzy weakly b-continuous.

Example 5.4. Let X=$\{a,b,c\}$ and A, B, D_1, D_2 be fuzzy sets of X defined as follows:

$$A(a) = 0,4 \qquad A(b) = 0,2 \qquad A(c) = 0,1;$$

$$B(a) = 0,5 \qquad B(b) = 0,5 \qquad B(c) = 0,5;$$

$$D_1(a) = 0,2 \qquad D_1(b) = 0,2 \qquad D_1(c) = 0,6;$$

$$D_2(a) = 0,3 \qquad D_2(b) = 0,2 \qquad D_2(c) = 0,4;$$

If we put $\tau_1 = \{0, D_1, 1\}, \tau_2 = \{0, A, B, 1\}, \tau_3 = \{0, D_2, 1\}$, then the mappings f=id:$(X,\tau_1) \to (X,\tau_2)$ and g=id:$(X,\tau_2) \to (X,\tau_3)$ are fuzzy weakly b-continuous, but gf is not fuzzy weakly b-continuous.

Definition 5.2. *A mapping $f : X \to Y$ from an fts X into an fts Y is called fuzzy b-irresolute continuous if $f^{-1}(B)$ is a fuzzy b-open set of X, for each fuzzy b-open set B of Y.*

Theorem 5.7. *If $f : X \to Y$ is fuzzy b-irresolute continuous and $g : Y \to Z$ is fuzzy weakly b-continuous, then gf is fuzzy weakly b-continuous.*

Proof. Let B be a fuzzy open set of Z. Then

$$(gf)^{-1}(\mathrm{B})=f^{-1}(g^{-1}(\mathrm{B}))\leq f^{-1}(\mathrm{bint}(g^{-1}(\mathrm{bclB}))=\mathrm{bint}f^{-1}(\mathrm{bint}(g^{-1}(\mathrm{bclB}))\leq$$
$$\leq \mathrm{bint}f^{-1}(g^{-1}(\mathrm{bclB}))=\mathrm{bint}(gf)^{-1}(\mathrm{bclB}).$$

Thus gf is fuzzy weakly b-continuous. ||

6. Fuzzy b-separation axioms

Definition 6.1. *An fts X is called fuzzy $b-T_0(FBT_0)$ if and only if for each pair of fuzzy singletons p_1, p_2 with different supports, there exists a fuzzy b-open set U such that $p_1 \leq U \leq p_2^c$ or $p_2 \leq U \leq p_1^c$.*

It is not difficult to conclude that FT_0 space is FBT_0. The following example shows that the converse may be not true.

Example 6.1. Let X={a,b} and A, B be fuzzy sets of X defined as follows:

$$A(a) = 1/2, \qquad A(b) = 1;$$

$$B(a) = 1, \qquad B(b) = 0.$$

Let $\tau = \{0, A, 1\}$. Then B is a fuzzy b-open set but not a fuzzy open set. By easy computation it can be shown that (X, τ) is FBT_0 but not FT_0.

Theorem 6.1. *If an fts X is FBT_0 then fuzzy b-closures of any two crisp fuzzy singletons, with different supports, are distinct.*

Proof. Let X be FBT_0 and p_1, p_2 be two crisp fuzzy singletons with different supports. According to the assumption there exists a fuzzy b-open set U such that $p_1 \leq U \leq p_2^c$. Then $p_2 \leq \text{bcl}p_2 \leq U^c$. Since p_1 is a crisp fuzzy singleton we have $p_1 \not\leq U^c$, hence $p_1 \not\leq bclp_2$. Therefore $\text{bcl}p_1 \neq \text{bcl}p_2$. ||

Definition 6.2. *An fts X is said to be fuzzy $b - T_1(FBT_1)$ if and only if for each pair of fuzzy singletons p_1, p_2 with different supports, there exist fuzzy b-open sets U and V such that $p_1 \leq U \leq p_2^c$ and $p_2 \leq V \leq p_1^c$.*

The following theorem gives a nice characterization for FBT_1 spaces.

Theorem 6.2. *An fts X is FBT_1 if and only if every crisp fuzzy singleton is b-closed.*

Proof. We consider a crisp fuzzy singleton p_1 in X with support x_1. For any fuzzy singleton p_2 with support x_2 different from x_1, there exist fuzzy b-open sets U_1 and U_2 such that $p_1 \leq U_1 \leq p_2^c$ and $p_2 \leq U_2 \leq p_1^c$. Since each fuzzy set can be considered as a union of the fuzzy singletons which it contains, we have $p_1^c = \vee_{p \leq p_1^c} p$, hence $p_1^c = \vee_{p \leq p_1^c} U_2$. Thus p_1^c is a fuzzy b-open set. Hence p_1 is fuzzy b-closed.

Conversely, let p_1 and p_2 be any pair of fuzzy singletons with different supports. We choose crisp fuzzy singletons q_1 and q_2 such that $\text{supp}(q_1)=\text{supp}(p_1)$ and $\text{supp}(q_2)=\text{supp}(p_2)$. The fuzzy sets q_1^c and q_2^c are b-open and satisfy the conditions $p_1 \leq q_2^c \leq p_2^c$ and $p_2 \leq q_1^c \leq p_1^c$. ||

Remark 6.1. Each FBT_1 space is FBT_0 but the converse may be not true. The fts in the next example is FBT_0 but it is not FBT_1.

Example 6.2. Let $X = \{a, b\}$ and A, B be fuzzy sets of X defined as follows:

$$A(a) = 0, \qquad A(b) = 1;$$

$$B(a) = 1/3, \qquad B(b) = 0.$$

Let $\tau = \{0, A, B, A \vee B, 1\}$. By easy computation it can be shown that (X, τ) is FBT_0 but not FBT_1.

Definition 6.3. *An fts X is called fuzzy b-strong $T_1(FBT_s)$ if and only if each fuzzy singleton is a fuzzy b-closed set.*

It is obvious that each FBT_s space is an FBT_1 space but the converse may be not true as it is shown by the following example.

Example 6.3. Let X={a,b} and A, B be fuzzy sets of X defined as follows:

$$A(a) = 1/2, \quad A(b) = 0;$$

$$B(a) = 1/4, \quad B(b) = 3/4;$$

Let $\tau = \{0, A, B, A \vee B, A \wedge B, 1\}$. By easy computation it can be shown that (X, τ) is FBT_1, but not FBT_s.

Definition 6.3. *An fts X is called fuzzy b-Hausdorff (FBT_2) if for each pair of fuzzy singletons p_1, p_2 with different supports, there exist two fuzzy b-open sets U and V such that $p_1 \leq U \leq p_2^c, p_2 \leq V \leq p_1^c$ and $U \leq V^c$.*

Theorem 6.3. *An fts X is FBT_2 if and only if for each pair of fuzzy singletons, with different supports, there exists a fuzzy b-open set U such that $p_1 \leq U \leq bclU \leq p_2^c$.*

Proof. Obvious and omitted. ||

Example 6.4. Consider the fuzzy topology τ on X defined by $\tau = \{U|$ suppU^c is finite$\}$. Obviously, every fuzzy singleton of X is fuzzy b-closed. Consequently (X, τ) is FBT_s, but it is not FBT_2.

Remark 6.2. The fts (X, τ) in the Example 6.3 is FBT_2, but not FBT_s. Hence, we can conclude that the classes of FBT_s spaces and FBT_2 spaces are independent.

Definition 6.4. *An fts X is called fuzzy b-Uryshon ($FBT_{2\frac{1}{2}}$) if and only if for each pair of fuzzy singletons p_1, p_2 with different supports, there exist two fuzzy b-open sets U and V such that $p_1 \leq U \leq p_2^c, p_2 \leq V \leq p_1^c$ and $bclU \leq (bclV)^c$.*

Clearly, an $FBT_{2\frac{1}{2}}$ space is an FBT_2 space but the converse may be not true.

Example 6.5. Let X=I and A and B be fuzzy sets of X defined as follows:

$$A = \{U | U(1/2) \leq 1/2\}$$

$B = \{U | U^c \text{ is finite}\}$.

Then $A \cup B$ is a fuzzy topology on X. The corresponding space is FBT_2 but not $FBT_{2\frac{1}{2}}$.

Definition 6.5. *An fts X is called fuzzy b-regular (FBR) if and only if for each fuzzy singleton p and each closed fuzzy set F, such that $p \leq F^c$, there exist two fuzzy b-open sets U and V such that $p \leq U, F \leq V$ and $U \leq V^c$.*

Remark 6.3. The fts in the Example 6.3 is $FBT_{2\frac{1}{2}}$ but not FBR.

Theorem 6.4. *An fts X is FBR if and only if for each fuzzy singleton p and each fuzzy open set U such that $p \leq U$, there exists a fuzzy b-open set W such that $p \leq W \leq bclW \leq U$.*

Proof. Obvious and omitted. ||

Definition 6.6. *An fts X is called fuzzy $b - T_3(FBT_3)$ if and only if it is FBR as well as FBT_s.*

Theorem 6.5. *For any closed fuzzy set F in an FBR space X and each fuzzy singleton p such that $p \leq F^c$, there exists a fuzzy b-open sets U and V such that $p \leq U, F \leq V$ and $bclU \leq (bclV)^c$.*

Proof. Since X is FBR, there exists a fuzzy b-open set U such that $p \leq U \leq bclU \leq F^c$. Let $V=(bclU)^c$, then V is a fuzzy b-open set, $F \leq V$ and $bclU \leq (bclV)^c$. ||

Corollary 6.6. *Each FBT_3 space is $FBT_{2\frac{1}{2}}$.*

Next, we obtain the following interesting result.

Theorem 6.7. *Let X be a FBR space which is also FT_0. Then X is FBT_2.*

Proof. Consider two fuzzy singletons p and q with different supports in X. Since the space is FT_0, we have $p \leq U \leq q^c$, where U is a fuzzy open set of X. Let $F = U^c$ which implies $p \leq F^c$. Now by the Theorem 6.5 there exist two fuzzy b-open sets G and H such that $p \leq G, F \leq H$ and $bclG \leq (bclH)^c$. Since $q \leq F$, it follows that $q \leq F \leq H$ and $bclG \leq (bclH)^c$. Hence X is $FBT_{2\frac{1}{2}}$. ||

Definition 6.7. *An fts X is called fuzzy b-normal (FBN) if and only if for each pair of closed fuzzy sets F_1 and F_2 such that $F_1 \leq F_2^c$ there exist two fuzzy b-open sets U and V such that $F_1 \leq U, F_2 \leq V$ and $U \leq V^c$.*

An FBN space which is also FBT_s, is called FBT_4.

Theorem 6.8. *An fts X is FBN if and only if for each closed fuzzy set F and each fuzzy open set U with $F \leq U$, there exists a fuzzy b-open set W such that $F \leq W \leq bclW \leq U$.*

Proof. Straightforward. ||

Definition 6.7. *An fts X is called fuzzy weakly b-normal (FWBN) if and only if for each pair of closed fuzzy sets F_1 and F_2 such that $F_1 \wedge F_2 = \emptyset$, there exist two fuzzy b-open sets U and V such that $F_1 \leq U, F_2 \leq V$ and $U \leq V^c$.*

Obviously, the notions of b-normality and weak b-normality coincide for ordinary topological spaces. For fuzzy topological spaces, the following property may be proved.

Theorem 6.9. *Each b-normal fts is weakly b-normal.*

Proof. The property is an immediate consequence of the implicatin $F_1 \wedge F_2 = \emptyset \Leftrightarrow F_1 \leq F_2^c$, which holds for every fuzzy sets F_1,F_2 of X. ||

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