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Some Asymptotic Results of the Ruin Probabilities in a Bidimensional Renewal Risk Model with Brownian Perturbation

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Abstract. In this paper, a bidimensional renewal risk model with constant force of interest and Brownian perturbation is considered. Assuming that the claim-size distribution function is from the subexponential class, three types of the finite-time ruin probabilities under this model are discussed. We obtain the asymptotic formulas for the three types, which hold uniformly for any finite-time horizon.

1. Introduction

In this paper, we consider the bidimensional surplus process $\vec{U}(x,t) = (U_1(x_1,t), U_2(x_2,t))^T, t \geq 0$, described by the following renewal risk model with constant force of interest and Brownian perturbation:

$$\begin{pmatrix} U_1(x_1,t) \\ U_2(x_2,t) \end{pmatrix} = \begin{pmatrix} x_1 e^{rt} \\ x_2 e^{rt} \end{pmatrix} + \begin{pmatrix} p_1 \int_0^t e^{r(t-s)} ds \\ p_2 \int_0^t e^{r(t-s)} ds \end{pmatrix} - \begin{pmatrix} \sum_{j=1}^{N_1(t)} X_{1j} e^{r(t-\tau_j^{(1)})} + \sum_{i=1}^{N_3(t)} X_{2i} e^{r(t-\tau_i^{(3)})} \\ \sum_{k=1}^{N_2(t)} Y_{1k} e^{r(t-\tau_k^{(2)})} + \sum_{i=1}^{N_3(t)} Y_{2i} e^{r(t-\tau_i^{(3)})} \end{pmatrix} + \begin{pmatrix} \delta_1 \int_0^t e^{r(t-s)} dB_1(s) \\ \delta_2 \int_0^t e^{r(t-s)} dB_2(s) \end{pmatrix}, \quad (1)$$

where $x_1, x_2 \geq 0$ denote the initial surplus, $r \geq 0$ the force of interest, $p_1, p_2 \geq 0$ the premium rate, $\delta_1, \delta_2 \geq 0$ the volatility factor, $\{B_1(t), B_2(t); t \geq 0\}$ the diffusion perturbation which are independent standard Brownian motions, $\{X_{1j}, j \geq 1; X_{2i}, i \geq 1\}$ the sequence of claim sizes which are independent and identically distributed(i.i.d), $\{Y_{1k}, k \geq 1; Y_{2i}, i \geq 1\}$ the sequence of claim sizes which are independent and identically distributed(i.i.d).

We denote by $\tau_k^{(i)}, k = 1, 2, ...,$ the arrival times of the renewal counting process $N_i(t), i = 1, 2, 3$. And $N_1(t), N_2(t), N_3(t)$ are three independent renewal processes. In reality, the common shock $N_3(t)$ can depict the effect of a natural disaster that causes various kinds of insurance claims.

Throughout this paper, we assume that $\{X_{1j}, j \geq 1; X_{2i}, i \geq 1\}$, $\{Y_{1k}, k \geq 1; Y_{2i}, i \geq 1\}$, $\{B_1(t); t \geq 0\}$, $\{B_2(t); t \geq 0\}$ and $\{N_i(t); t \geq 0, i = 1, 2, 3\}$ are mutually independent.

Define the ruin times of the two marginal processes as:

$$T(x_i) = \inf\{t : U_i(t) < 0 | U_i(0) = x_i\}, i = 1, 2.$$

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For the bidimensional model, motivated by Lu and Zhang (2016), we investigate four sorts of ruin time here,

(1)$T_{max}(\vec{x}) = \inf\left\{s \geq 0 : \max\{U_1(s), U_2(s)\} < 0 \,|\, \vec{U}(0) = \vec{x}\right\}$;

(2)$T_{min}(\vec{x}) = \inf\left\{s \geq 0 : \min\{U_1(s), U_2(s)\} < 0 \,|\, \vec{U}(0) = \vec{x}\right\}$;

(3)$T_{\vee}(\vec{x}) = \max\{T(x_1), T(x_2)\} = T(x_1) \vee T(x_2)$;

(4)$T_{\wedge}(\vec{x}) = \min\{T(x_1), T(x_2)\} = T(x_1) \wedge T(x_2)$.

It is obvious to obtain the following inequalities:

$$T_{max}(\vec{x}) \geq T_{\vee}(\vec{x}) \geq T_{\wedge}(\vec{x}) = T_{min}(\vec{x}).$$

Therefore, we just need to consider three types of ruin probability.

$$\Psi_{\vee}(\vec{x}, t) = \mathbb{P}\left(T_{\vee}(\vec{x}) \leq t\right) = \mathbb{P}\left(\inf_{0\leq s\leq t} U_1(s) < 0, \inf_{0\leq s\leq t} U_2(s) < 0\right),$$

represents both of the surplus processes go below 0 in the finite time.

$$\Psi_{max}(\vec{x}, t) = \mathbb{P}\left(T_{max}(\vec{x}) \leq t\right) = \mathbb{P}\left(\inf_{0\leq s\leq t}\{U_1(s) \vee U_2(s)\} < 0\right),$$

represents the maximum of the two surplus processes goes blow 0 in the finite time.

$$\Psi_{min}(\vec{x}, t) = \mathbb{P}\left(T_{min}(\vec{x}) \leq t\right) = \mathbb{P}\left(\{\inf_{0\leq s\leq t} U_1(s) < 0\} \bigcup \{\inf_{0\leq s\leq t} U_2(s) < 0\}\right),$$

represents at least one of the surplus processes goes below 0 in the finite time.

It is obvious to obtain the following inequalities:

$$\Psi_{min}(\vec{x}, t) \geq \Psi_{\vee}(\vec{x}, t) \geq \Psi_{max}(\vec{x}, t). \tag{2}$$

It is well-known that there are increasing researchers having studied the asymptotic behavior of finite-time ruin probabilities for renewal risk models with heavy-tailed claims and Brownian perturbation, for example, see Li et al. (2007), Chen et al. (2013), Gao and Yang (2014), Cheng et al. (2016), Cheng and Wang (2016), Li (2017), and the references therein.

In Li (2017), he considered a renewal risk model with constant force of interest and Brownian perturbation, and derived for the finite-time ruin probability a precise asymptotic expansion. In Yang and Li (2014), they studied a bidimensional renewal risk model with constant interest force and dependent subexponential claims. Under the assumption that the claim size vectors form a sequence of i.i.d. random vectors following a common bivariate F-G-M distribution, they derived for the finite-time ruin probability an explicit asymptotic formula. In this paper, on the one hand, comparing with Li (2017), we extend the model to a bidimensional renewal risk model; on the other hand, comparing with Yang and Li (2014), we extend the model to a perturbed risk model.

The rest of paper is organized as follows. In Section 2, we present some useful lemmas. In Section 3, we introduce the main results. In Section 4, we give a useful lemma which will be used in the whole proofs, and the proofs of our results.

2. Preliminaries

Throughout this article, for the sake of clarity, let $\overline{V}(x) = 1 - V(x)$ be the proper distribution function V's tail. In risk theory, heavy-tailed distributions are very important. Now we recall two important subclasses of heavy-tailed distributions.

A distribution function V on $[0,\infty)$ is said to belong to the subexponential class, written as $V \in \mathcal{S}$, if $\overline{V} > 0$ for all $x \geq 0$ and the relation

$$\lim_{x\to\infty} \frac{\overline{V^{n*}}(x)}{\overline{V(x)}} = n$$

holds for all (or, equivalently, for some) $n \geq 2$, where $V^{n*}(x)$ is the $n-$fold convolution of V with itself. It is known that if $V \in \mathcal{S}$ then $V \in \mathcal{L}$, which stands for the class of long-tailed distributions characterized by $\overline{V}(x) > 0$ for all $x \geq 0$ and the relation

$$\lim_{x\to\infty} \frac{\overline{V}(x+y)}{\overline{V(x)}} = 1,\ y \in (-\infty,\infty).$$

To obtain the main results in this paper, we need the following series of lemmas which play a key role in the proofs. Our first lemma is a restatement of Lemma 1.3.5 (b) of Embrechts et al. (1997).

Lemma 2.1. *If $\overline{V} \in \mathcal{S}$, then, for every $\varepsilon > 0$, it holds that*

$$e^{-\varepsilon x} = o\left(\overline{V}(x)\right).$$

The following two lemmas are from Lemma 3.2 and Lemma 3.4 of Li (2017).

Lemma 2.2. *Let $\{Z_i; i \geq 1\}$ be a sequence of independent real-valued random variables with distribution functions $V_1, V_2, \ldots$, respectively. Assume that there is a distribution function $V \in \mathcal{S}$ such that $\overline{V}_i(x) \sim l_i\overline{V}(x)$ with some positive constant l_i for each $i \geq 1$. And let ξ be a real-valued random variable independent of $\{Z_i; i \geq 1\}$ such that $\mathbb{P}(\xi > x) = o\left(\overline{V}(x/a)\right)$ for some $a > 0$. Then, for each $n \geq 1$ and every $b \geq a$, it holds uniformly for $(c_1,\ldots,c_n) \in [a,b]^n$ that*

$$\mathbb{P}\left(\sum_{i=1}^{n} c_i Z_i + \xi > x\right) \sim \sum_{i=1}^{n} l_i\overline{V}(x/c_i).$$

Lemma 2.3. *Let $\{Z_i; i \geq 1\}$ be a sequence of independent real-valued random variables with common distribution function $V \in \mathcal{S}$. Let ξ be a real-valued random variable independent of $\{Z_i; i \geq 1\}$ such that $\mathbb{P}(\xi > x) = O\left(\overline{V}(x/a)\right)$ for some $a > 0$. Then, for every $\varepsilon > 0$, and some constant $K > 0$ such that the relation*

$$\mathbb{P}\left(c\sum_{i=1}^{n} Z_i + \xi > x\right) \leq K(1+\varepsilon)^n\overline{V}(x/c)$$

holds for all $x \in (-\infty,\infty)$, $c \geq a$, and $n \geq 1$.

Hereafter, all limit relationships are for $x \to \infty$ unless stated otherwise. For two positive functions $f(x)$ and $g(x)$, we write $f(x) \lesssim g(x)$ if $\limsup f(x)/g(x) \leq 1$, write $f(x) \gtrsim g(x)$ if $\liminf f(x)/g(x) \geq 1$, write $f(x) \sim g(x)$ if $\lim f(x)/g(x) = 1$. Furthermore, for two positive bivariate functions $f(x,t)$ and $g(x,t)$, we say that the asymptotic relation $f(x,t) \sim g(x,t)$ holds uniformly for t in a nonempty set Δ if

$$\lim_{x\to\infty} \sup_{t\in\Delta} \left|\frac{f(x,t)}{g(x,t)} - 1\right| = 0.$$

Clearly, the asymptotic relation $f(x,t) \sim g(x,t)$ holds uniformly for $t \in \Delta$ if and only if

$$\limsup_{x\to\infty} \sup_{t\in\Delta} \frac{f(x,t)}{g(x,t)} \leq 1 \text{ and } \liminf_{x\to\infty} \inf_{t\in\Delta} \frac{f(x,t)}{g(x,t)} \geq 1,$$

which mean that the relations $f(x,t) \lesssim g(x,t)$ and $f(x,t) \gtrsim g(x,t)$, respectively, hold uniformly for $t \in \Delta$.

For convenience, denote by $\lambda_i(t) = EN_i(t) = \sum_{j=1}^{\infty} \mathbb{P}\left(\tau_j^{(i)} \le t\right), t \ge 0$, the renewal function of $\{N_i(t); t \ge 0\}$, and it is natural to restrict the region of the variable t to the set $\Lambda_i = \{t : 0 < \lambda_i(t) \le \infty\}, i = 1, 2, 3$. Moreover, we write $\Lambda_T = \Lambda_1 \cap \Lambda_2 \cap \Lambda_3 \cap [0, T]$, where T is an arbitrarily positive number. Let

$$M_i(t) = \delta_i \int_0^t e^{-rs} dB_i(s)$$

with

$$\underline{M_i}(t) = \inf_{0\le s\le t} M_i(s) \le 0 \text{ and } \overline{M_i}(t) = \sup_{0\le s\le t} M_i(s) \ge 0. \tag{3}$$

It holds for every $T > 0$ that

$$\mathbb{P}\left(\underline{M_i}(T) < -x_i\right) = \mathbb{P}\left(\overline{M_i}(T) > x_i\right) \sim 2\overline{\Phi}\left(\frac{\sqrt{2r}x_i}{\delta_i\sqrt{1-e^{-2rT}}}\right). \tag{4}$$

The second relation of (4) indicates that, for every $T > 0$, $\overline{M}_1(T)$ and $\overline{M}_2(T)$ have ultimate tails of Gaussian type, which are negligible compared to any subexponential tail in view of Lemma 2.1. Hence, $\overline{M}_1(T)$ and $\overline{M}_2(T)$ satisfy all the requirements imposed on ξ in Lemma 2.2 and Lemma 2.3. Besides, it is well-known that the moment generating function of $N_1(T)$, $N_2(T)$ and $N_3(T)$ are analytic in a neighborhood of 0; see, e.g., Stein (1946).

3. Main results

The main results of this paper are given below. We get the asymptotic formulas for $\Psi_\vee(\vec{x}, t)$ and $\Psi_{max}(\vec{x}, t)$ in the following two theorems.

Theorem 3.1. *Let $\{X_{1j}, j \ge 1; X_{2i}, i \ge 1\}$ be the sequence of claim sizes which are i.i.d. with distribution function $F_1 \in \mathcal{S}$, and $\{Y_{1k}, k \ge 1; Y_{2i}, i \ge 1\}$ be the sequence of claim sizes which are i.i.d. with distribution function $F_2 \in \mathcal{S}$. Then, for the bidimensional perturbed renewal risk model (1), it holds uniformly for $t \in \Lambda_T$ that*

$$\begin{aligned}\Psi_\vee(\vec{x}, t) \sim &\int_{0-}^t \overline{F}_1(x_1e^{rs_1})d\lambda_1(s_1)\int_{0-}^t \overline{F}_2(x_2e^{rs_2})d\lambda_2(s_2) + \int_{0-}^t \overline{F}_1(x_1e^{rs_1})d\lambda_1(s_1)\int_{0-}^t \overline{F}_2(x_2e^{rs_3})d\lambda_3(s_3)\\ &+ \int_{0-}^t \overline{F}_1(x_1e^{rs_3})d\lambda_3(s_3)\int_{0-}^t \overline{F}_2(x_2e^{rs_2})d\lambda_2(s_2) + \int_{0-}^t \overline{F}_1(x_1e^{rs_3})\overline{F}_2(x_2e^{rs_3})d\lambda_3(s_3)\\ &+ \iint_{\substack{s_3,w\ge 0\\ s_3+w\le t}} \left(\overline{F}_1(x_1e^{r(s_3+w)})\overline{F}_2(x_2e^{rs_3}) + \overline{F}_1(x_1e^{rs_3})\overline{F}_2(x_2e^{r(s_3+w)})\right) d\lambda_3(s_3)d\lambda_3(w).\end{aligned} \tag{5}$$

Theorem 3.2. *Let $\{X_{1j}, j \ge 1; X_{2i}, i \ge 1\}$ be the sequence of claim sizes which are i.i.d. with distribution function $F_1 \in \mathcal{S}$, and $\{Y_{1k}, k \ge 1; Y_{2i}, i \ge 1\}$ be the sequence of claim sizes which are i.i.d. with distribution function $F_2 \in \mathcal{S}$. Then, for the bidimensional perturbed renewal risk model (1), it holds uniformly for $t \in \Lambda_T$ that*

$$\begin{aligned}\Psi_{max}(\vec{x}, t) \sim &\int_{0-}^t \overline{F}_1(x_1e^{rs_1})d\lambda_1(s_1)\int_{0-}^t \overline{F}_2(x_2e^{rs_2})d\lambda_2(s_2) + \int_{0-}^t \overline{F}_1(x_1e^{rs_1})d\lambda_1(s_1)\int_{0-}^t \overline{F}_2(x_2e^{rs_3})d\lambda_3(s_3)\\ &+ \int_{0-}^t \overline{F}_1(x_1e^{rs_3})d\lambda_3(s_3)\int_{0-}^t \overline{F}_2(x_2e^{rs_2})d\lambda_2(s_2) + \int_{0-}^t \overline{F}_1(x_1e^{rs_3})\overline{F}_2(x_2e^{rs_3})d\lambda_3(s_3)\\ &+ \iint_{\substack{s_3,w\ge 0\\ s_3+w\le t}} \left(\overline{F}_1(x_1e^{r(s_3+w)})\overline{F}_2(x_2e^{rs_3}) + \overline{F}_1(x_1e^{rs_3})\overline{F}_2(x_2e^{r(s_3+w)})\right) d\lambda_3(s_3)d\lambda_3(w).\end{aligned} \tag{6}$$

The theorem below is obtained by using Theorem 3.2.

Theorem 3.3. *Let $\{X_{1j}, j \geq 1; X_{2i}, i \geq 1\}$ be the sequence of claim sizes which are i.i.d. with distribution function $F_1 \in \mathcal{S}$, and $\{Y_{1k}, k \geq 1; Y_{2i}, i \geq 1\}$ be the sequence of claim sizes which are i.i.d. with distribution function $F_2 \in \mathcal{S}$. Then, for the bidimensional perturbed renewal risk model (1), it holds uniformly for $t \in \Lambda_T$ that*

$$\Psi_{min}(\vec{x}, t) \sim \int_{0-}^{t} \overline{F}_1(x_1 e^{rs_1}) d\lambda_1(s_1) + \int_{0-}^{t} \overline{F}_2(x_2 e^{rs_2}) d\lambda_2(s_2) + \int_{0-}^{t} \left(\overline{F}_1(x_1 e^{rs_3}) + \overline{F}_2(x_2 e^{rs_3})\right) d\lambda_3(s_3). \quad (7)$$

Not surprisingly, all the obtained results confirm the widely-agreed intuition that the Brownian perturbation with a Gaussian tail will not affect the asymptotic ruin probabilities of risk model with heavy-tailed claims.

4. Proof of main results

4.1. Useful Lemma

First, we give a useful lemma, which is the main ingredient of the proof of the asymptotic formulas for the three types. Then, we give the proofs of our main results .

Lemma 4.1. *Let $\{X_{1j}, j \geq 1; X_{2i}, i \geq 1\}$ be the sequence of claim sizes which are i.i.d. with distribution function $F_1 \in \mathcal{S}$, and $\{Y_{1k}, k \geq 1; Y_{2i}, i \geq 1\}$ be the sequence of claim sizes which are i.i.d. with distribution function $F_2 \in \mathcal{S}$. Then , under the conditions of the bidimensional perturbed renewal risk model (1), it holds uniformly for $t \in \Lambda_T$ that*

$$\mathbb{P}\left(\sum_{j=1}^{N_1(t)} X_{1j} e^{-r\tau_j^{(1)}} + \sum_{i=1}^{N_3(t)} X_{2i} e^{-r\tau_i^{(3)}} + \overline{M}_1(T) > x_1, \sum_{k=1}^{N_2(t)} Y_{1k} e^{-r\tau_k^{(2)}} + \sum_{i=1}^{N_3(t)} Y_{2i} e^{-r\tau_i^{(3)}} + \overline{M}_2(T) > x_2\right) \sim \phi(x_1, x_2) \quad (8)$$

and

$$\mathbb{P}\left(\sum_{j=1}^{N_1(t)} X_{1j} e^{-r\tau_j^{(1)}} + \sum_{i=1}^{N_3(t)} X_{2i} e^{-r\tau_i^{(3)}} - \overline{M}_1(T) > x_1, \sum_{k=1}^{N_2(t)} Y_{1k} e^{-r\tau_k^{(2)}} + \sum_{i=1}^{N_3(t)} Y_{2i} e^{-r\tau_i^{(3)}} - \overline{M}_2(T) > x_2\right) \sim \phi(x_1, x_2), \quad (9)$$

where $\overline{M}_i(T)$ is the random variable defined by (3), i=1,2. And

$$\begin{aligned}
\phi(x_1, x_2) \;=\; & \int_{0-}^{t} \overline{F}_1(x_1 e^{rs_1}) d\lambda_1(s_1) \int_{0-}^{t} \overline{F}_2(x_2 e^{rs_2}) d\lambda_2(s_2) + \int_{0-}^{t} \overline{F}_1(x_1 e^{rs_3}) d\lambda_3(s_3) \int_{0-}^{t} \overline{F}_2(x_2 e^{rs_2}) d\lambda_2(s_2) \\
& + \int_{0-}^{t} \overline{F}_1(x_1 e^{rs_1}) d\lambda_1(s_1) \int_{0-}^{t} \overline{F}_2(x_2 e^{rs_3}) d\lambda_3(s_3) + \int_{0-}^{t} \overline{F}_1(x_1 e^{rs_3}) \overline{F}_2(x_2 e^{rs_3}) d\lambda_3(s_3) \\
& + \iint_{\substack{s_3, w \geq 0 \\ s_3 + w \leq t}} \left(\overline{F}_1(x_1 e^{r(s_3+w)}) \overline{F}_2(x_2 e^{rs_3}) + \overline{F}_1(x_1 e^{rs_3}) \overline{F}_2(x_2 e^{r(s_3+w)})\right) d\lambda_3(s_3) d\lambda_3(w).
\end{aligned}$$

Proof. Now we deal with the relation (8). Before proving, we give the idea of the proof. Firstly, the proof is divided into $I_1, ..., I_8$, and $I_i = o(I_1), i = 2, ..., 8$. However, we only need to prove $I_2 = o(I_1)$ because of the similarity in the proof. Next, I_1 is divided into $I_{11}, ..., I_{18}$, and $I_{1i} = o(I_{11}), i = 2, ..., 8$. Similarly, we only need to prove $I_{12} = o(I_{11})$ and $I_{15} = o(I_{11})$. Hence, the emphasis of proof are the proofs of I_2, I_{11}, I_{12} and I_{15}. First, we know that

$$\begin{aligned}
& \mathbb{P}\left(\sum_{j=1}^{N_1(t)} X_{1j} e^{-r\tau_j^{(1)}} + \sum_{i=1}^{N_3(t)} X_{2i} e^{-r\tau_i^{(3)}} + \overline{M}_1(T) > x_1, \sum_{k=1}^{N_2(t)} Y_{1k} e^{-r\tau_k^{(2)}} + \sum_{i=1}^{N_3(t)} Y_{2i} e^{-r\tau_i^{(3)}} + \overline{M}_2(T) > x_2\right) \\
= \; & \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \mathbb{P}\left(\sum_{j=1}^{n} X_{1j} e^{-r\tau_j^{(1)}} + \sum_{i=1}^{m} X_{2i} e^{-r\tau_i^{(3)}} + \overline{M}_1(T) > x_1, \right. \\
& \left. \sum_{k=1}^{p} Y_{1k} e^{-r\tau_k^{(2)}} + \sum_{i=1}^{m} Y_{2i} e^{-r\tau_i^{(3)}} + \overline{M}_2(T) > x_2, N_1(t) = n, N_2(t) = p, N_3(t) = m\right). \quad (10)
\end{aligned}$$

Then, for some positive integer M, N, P, we split the right side of (10) into eight parts as

$$\begin{aligned}
&\Bigg(\sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{p=1}^{P}+\sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{p=P+1}^{\infty}+\sum_{m=1}^{M}\sum_{n=N+1}^{\infty}\sum_{p=1}^{P}+\sum_{m=1}^{M}\sum_{n=N+1}^{\infty}\sum_{p=P+1}^{\infty}+\sum_{m=M+1}^{\infty}\sum_{n=1}^{N}\sum_{p=1}^{P}+\sum_{m=M+1}^{\infty}\sum_{n=1}^{N}\sum_{p=P+1}^{\infty}\\
&+\sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty}\sum_{p=1}^{P}+\sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty}\sum_{p=P+1}^{\infty}\Bigg)\mathbb{P}\Bigg(\sum_{j=1}^{n}X_{1j}e^{-r\tau_j^{(1)}}+\sum_{i=1}^{m}X_{2i}e^{-r\tau_i^{(3)}}+\overline{M}_1(T)>x_1,\\
&\sum_{k=1}^{p}Y_{1k}e^{-r\tau_k^{(2)}}+\sum_{i=1}^{m}Y_{2i}e^{-r\tau_i^{(3)}}+\overline{M}_2(T)>x_2,N_1(t)=n,N_2(t)=p,N_3(t)=m\Bigg)\\
=:\ & I_1+I_2+\cdots+I_8. \qquad (11)
\end{aligned}$$

For I_1, by Lemma 2.2, we have

$$\begin{aligned}
I_1=&\sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{p=1}^{P}\int\cdots\int_{0<t_1^{(1)}<\ldots<t_n^{(1)}\le t<t_{n+1}^{(1)}}\int\cdots\int_{0<t_1^{(2)}<\ldots<t_p^{(2)}\le t<t_{p+1}^{(2)}}\int\cdots\int_{0<t_1^{(3)}<\ldots<t_m^{(3)}\le t<t_{m+1}^{(3)}}\\
&\mathbb{P}\left(\sum_{j=1}^{n}X_{1j}e^{-rt_j^{(1)}}+\sum_{i=1}^{m}X_{2i}e^{-rt_i^{(3)}}+\overline{M}_1(T)>x_1\right)\mathbb{P}\left(\sum_{k=1}^{p}Y_{1k}e^{-rt_k^{(2)}}+\sum_{i=1}^{m}Y_{2i}e^{-rt_i^{(3)}}+\overline{M}_2(T)>x_2\right)\\
&\mathbb{P}\left(\tau_1^{(1)}\in dt_1^{(1)},\ldots,\tau_{n+1}^{(1)}\in dt_{n+1}^{(1)}\right)\mathbb{P}\left(\tau_1^{(2)}\in dt_1^{(2)},\ldots,\tau_{p+1}^{(2)}\in dt_{p+1}^{(2)}\right)\mathbb{P}\left(\tau_1^{(3)}\in dt_1^{(3)},\ldots,\tau_{m+1}^{(3)}\in dt_{m+1}^{(3)}\right)\\
\sim&\sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{p=1}^{P}\int\cdots\int_{0<t_1^{(1)}<\ldots<t_n^{(1)}\le t<t_{n+1}^{(1)}}\int\cdots\int_{0<t_1^{(2)}<\ldots<t_p^{(2)}\le t<t_{p+1}^{(2)}}\int\cdots\int_{0<t_1^{(3)}<\ldots<t_m^{(3)}\le t<t_{m+1}^{(3)}}\\
&\left(\sum_{j=1}^{n}\mathbb{P}\left(X_{1j}e^{-rt_j^{(1)}}>x_1\right)+\sum_{i=1}^{m}\mathbb{P}\left(X_{2i}e^{-rt_i^{(3)}}>x_1\right)\right)\left(\sum_{k=1}^{p}\mathbb{P}\left(Y_{1k}e^{-rt_k^{(2)}}>x_2\right)+\sum_{i=1}^{m}\mathbb{P}\left(Y_{2i}e^{-rt_i^{(3)}}>x_2\right)\right)\\
&\mathbb{P}\left(\tau_1^{(1)}\in dt_1^{(1)},\ldots,\tau_{n+1}^{(1)}\in dt_{n+1}^{(1)}\right)\mathbb{P}\left(\tau_1^{(2)}\in dt_1^{(2)},\ldots,\tau_{p+1}^{(2)}\in dt_{p+1}^{(2)}\right)\mathbb{P}\left(\tau_1^{(3)}\in dt_1^{(3)},\ldots,\tau_{m+1}^{(3)}\in dt_{m+1}^{(3)}\right)\\
=&\Bigg(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=P+1}^{\infty}-\sum_{m=1}^{\infty}\sum_{n=N+1}^{\infty}\sum_{p=1}^{\infty}+\sum_{m=1}^{\infty}\sum_{n=N+1}^{\infty}\sum_{p=P+1}^{\infty}-\sum_{m=M+1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}+\sum_{m=M+1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=P+1}^{\infty}\\
&+\sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty}\sum_{p=1}^{\infty}-\sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty}\sum_{p=P+1}^{\infty}\Bigg)\\
&\Bigg(\sum_{j=1}^{n}\sum_{k=1}^{p}\mathbb{P}\left(X_{1j}e^{-r\tau_j^{(1)}}>x_1,N_1(t)=n\right)\mathbb{P}\left(Y_{1k}e^{-r\tau_k^{(2)}}>x_2,N_2(t)=p\right)\mathbb{P}\left(N_3(t)=m\right)\\
&+\sum_{i=1}^{m}\sum_{k=1}^{p}\mathbb{P}\left(X_{2i}e^{-r\tau_i^{(3)}}>x_1,N_3(t)=m\right)\mathbb{P}\left(Y_{1k}e^{-r\tau_k^{(2)}}>x_2,N_2(t)=p\right)\mathbb{P}\left(N_1(t)=n\right)\\
&+\sum_{j=1}^{n}\sum_{i=1}^{m}\mathbb{P}\left(X_{1j}e^{-r\tau_j^{(1)}}>x_1,N_1(t)=n\right)\mathbb{P}\left(Y_{2i}e^{-r\tau_i^{(3)}}>x_2,N_3(t)=m\right)\mathbb{P}\left(N_2(t)=p\right)\\
&+\sum_{i=1}^{m}\sum_{l=1}^{m}\mathbb{P}\left(X_{2i}e^{-r\tau_i^{(3)}}>x_1,Y_{2l}e^{-r\tau_l^{(3)}}>x_2,N_3(t)=m\right)\mathbb{P}\left(N_1(t)=n\right)\mathbb{P}\left(N_2(t)=p\right)\Bigg)\\
=:&I_{11}-I_{12}-I_{13}+I_{14}-I_{15}+I_{16}+I_{17}-I_{18}.
\end{aligned}$$

For I_{11}, it holds that

$$\begin{aligned}
I_{11} =& \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\Bigg(\sum_{j=1}^{n}\sum_{k=1}^{p}\mathbb{P}\left(X_{1j}e^{-r\tau_j^{(1)}} > x_1, N_1(t) = n\right)\mathbb{P}\left(Y_{1k}e^{-r\tau_k^{(2)}} > x_2, N_2(t) = p\right)\mathbb{P}\left(N_3(t) = m\right) \\
&+ \sum_{i=1}^{m}\sum_{k=1}^{p}\mathbb{P}\left(X_{2i}e^{-r\tau_i^{(3)}} > x_1, N_3(t) = m\right)\mathbb{P}\left(Y_{1k}e^{-r\tau_k^{(2)}} > x_2, N_2(t) = p\right)\mathbb{P}\left(N_1(t) = n\right) \\
&+ \sum_{j=1}^{n}\sum_{i=1}^{m}\mathbb{P}\left(X_{1j}e^{-r\tau_j^{(1)}} > x_1, N_1(t) = n\right)\mathbb{P}\left(Y_{2i}e^{-r\tau_i^{(3)}} > x_2, N_3(t) = m\right)\mathbb{P}\left(N_2(t) = p\right) \\
&+ \sum_{i=1}^{m}\sum_{l=1}^{m}\mathbb{P}\left(X_{2i}e^{-r\tau_i^{(3)}} > x_1, Y_{2l}e^{-r\tau_l^{(3)}} > x_2, N_3(t) = m\right)\mathbb{P}\left(N_1(t) = n\right)\mathbb{P}\left(N_2(t) = p\right)\Bigg) \\
=:& L_1 + L_2 + L_3 + L_4.
\end{aligned} \tag{12}$$

For L_1, interchanging the order of the sums leads to

$$\begin{aligned}
L_1 =& \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\mathbb{P}\left(X_{1j}e^{-r\tau_j^{(1)}} > x_1, \tau_j^{(1)} \le t\right)\mathbb{P}\left(Y_{1k}e^{-r\tau_k^{(2)}} > x_2, \tau_k^{(2)} \le t\right)\sum_{m=1}^{\infty}\mathbb{P}\left(N_3(t) = m\right) \\
=& \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\int_{0-}^{t}\mathbb{P}\left(X_{1j}e^{-rs_1} > x_1 \middle| \tau_j^{(1)} = s_1\right)\mathbb{P}\left(\tau_j^{(1)} \in ds_1\right)\int_{0-}^{t}\mathbb{P}\left(Y_{1k}e^{-rs_2} > x_2 \middle| \tau_k^{(2)} = s_2\right)\mathbb{P}\left(\tau_k^{(2)} \in ds_2\right) \\
=& \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\int_{0-}^{t}\mathbb{P}\left(X_{1j}e^{-rs_1} > x_1\right)\mathbb{P}\left(\tau_j^{(1)} \in ds_1\right)\int_{0-}^{t}\mathbb{P}\left(Y_{1k}e^{-rs_2} > x_2\right)\mathbb{P}\left(\tau_k^{(2)} \in ds_2\right) \\
=& \int_{0-}^{t}\overline{F}_1(x_1e^{rs_1})d\lambda_1(s_1)\int_{0-}^{t}\overline{F}_2(x_2e^{rs_2})d\lambda_2(s_2).
\end{aligned} \tag{13}$$

Similarly to (13), we can obtain that

$$\begin{aligned}
L_2 &= \int_{0-}^{t}\overline{F}_1(x_1e^{rs_3})d\lambda_3(s_3)\int_{0-}^{t}\overline{F}_2(x_2e^{rs_2})d\lambda_2(s_2), \\
L_3 &= \int_{0-}^{t}\overline{F}_1(x_1e^{rs_1})d\lambda_1(s_1)\int_{0-}^{t}\overline{F}_2(x_2e^{rs_3})d\lambda_3(s_3).
\end{aligned} \tag{14}$$

For L_4, interchanging the order of the sums, we have

$$\begin{aligned}
L_4 =& \sum_{i=1}^{\infty}\sum_{m=i}^{\infty}\sum_{l=1}^{m}\mathbb{P}\left(X_{2i}e^{-r\tau_i^{(3)}} > x_1, Y_{2l}e^{-r\tau_l^{(3)}} > x_2, N_3(t) = m\right)\sum_{n=1}^{\infty}\mathbb{P}\left(N_1(t) = n\right)\sum_{p=1}^{\infty}\mathbb{P}\left(N_2(t) = p\right) \\
=& \sum_{i=1}^{\infty}\sum_{m=i}^{\infty}\left(\sum_{l=1}^{i-1} + \sum_{l=i} + \sum_{l=i+1}^{m}\right)\mathbb{P}\left(X_{2i}e^{-r\tau_i^{(3)}} > x_1, Y_{2l}e^{-r\tau_l^{(3)}} > x_2, N_3(t) = m\right) \\
=& \Bigg(\sum_{l=1}^{\infty}\sum_{i=l+1}^{\infty}\mathbb{P}\left(X_{2i}e^{-r\tau_i^{(3)}} > x_1, Y_{2l}e^{-r\tau_l^{(3)}} > x_2, \tau_i^{(3)} \le t\right) + \sum_{i=1}^{\infty}\mathbb{P}\left(X_{2i}e^{-r\tau_i^{(3)}} > x_1, Y_{2i}e^{-r\tau_i^{(3)}} > x_2, \tau_i^{(3)} \le t\right) \\
&+ \sum_{i=1}^{\infty}\sum_{l=i+1}^{\infty}\mathbb{P}\left(X_{2i}e^{-r\tau_i^{(3)}} > x_1, Y_{2l}e^{-r\tau_l^{(3)}} > x_2, \tau_l^{(3)} \le t\right)\Bigg) \\
=:& L_{41} + L_{42} + L_{43}.
\end{aligned} \tag{15}$$

For L_{41}, since $\{N_3(t); t \geq 0\}$ is a renewal process, $\tau_i^{(3)} - \tau_l^{(3)}$ is independent of $\tau_l^{(3)}$ and has the same distribution as $\tau_{i-l}^{(3)}$. Hence, conditioning on the values of $\tau_l^{(3)}$ and $\tau_i^{(3)} - \tau_l^{(3)}$ yields

$$
\begin{aligned}
L_{41} &= \sum_{l=1}^{\infty} \sum_{i=l+1}^{\infty} \mathbb{P}\left(X_{2i}e^{-r\tau_i^{(3)}} > x_1, Y_{2l}e^{-r\tau_l^{(3)}} > x_2, \tau_i^{(3)} \leq t\right) \\
&= \sum_{l=1}^{\infty} \sum_{i=l+1}^{\infty} \mathbb{P}\left(X_{2i}e^{-r(\tau_l^{(3)}+(\tau_i^{(3)}-\tau_l^{(3)}))} > x_1, Y_{2l}e^{-r\tau_l^{(3)}} > x_2, \tau_l^{(3)} + (\tau_i^{(3)} - \tau_l^{(3)}) \leq t\right) \\
&= \sum_{l=1}^{\infty} \sum_{i=l+1}^{\infty} \iint_{\substack{s_3,w\geq 0\\ s_3+w\leq t}} \mathbb{P}\left(X_{2i}e^{-r(s_3+w)} > x_1, Y_{2l}e^{-rs_3} > x_2 \middle| \tau_i^{(3)} - \tau_l^{(3)} = w, \tau_l^{(3)} = s_3\right) \\
&\quad \mathbb{P}\left(\tau_l^{(3)} \in ds_3\right)\mathbb{P}\left(\tau_i^{(3)} - \tau_l^{(3)} \in dw\right) \\
&= \sum_{l=1}^{\infty} \sum_{i=l+1}^{\infty} \iint_{\substack{s_3,w\geq 0\\ s_3+w\leq t}} \mathbb{P}\left(X_{2i}e^{-r(s_3+w)} > x_1\right)\mathbb{P}\left(Y_{2l}e^{-rs_3} > x_2\right)\mathbb{P}\left(\tau_l^{(3)} \in ds_3\right)\mathbb{P}\left(\tau_i^{(3)} - \tau_l^{(3)} \in dw\right) \\
&= \iint_{\substack{s_3,w\geq 0\\ s_3+w\leq t}} \overline{F}_1(x_1e^{r(s_3+w)})\overline{F}_2(x_2e^{rs_3})d\lambda_3(s_3)d\lambda_3(w).
\end{aligned}
\tag{16}
$$

Similarly, we have

$$
L_{42} = \int_{0-}^{t} \overline{F}_1(x_1e^{rs_3})\overline{F}_2(x_2e^{rs_3})d\lambda_3(s_3), \tag{17}
$$

and

$$
L_{43} = \iint_{\substack{s_3,w\geq 0\\ s_3+w\leq t}} \overline{F}_1(x_1e^{rs_3})\overline{F}_2(x_2e^{r(s_3+w)})d\lambda_3(s_3)d\lambda_3(w). \tag{18}
$$

Plugging (16), (17) and (18) into (15), we can obtain that

$$
\begin{aligned}
L_4 &= \int_{0-}^{t} \overline{F}_1(x_1e^{rs_3})\overline{F}_2(x_2e^{rs_3})d\lambda_3(s_3) \\
&\quad + \iint_{\substack{s_3,w\geq 0\\ s_3+w\leq t}} \left(\overline{F}_1(x_1e^{r(s_3+w)})\overline{F}_2(x_2e^{rs_3}) + \overline{F}_1(x_1e^{rs_3})\overline{F}_2(x_2e^{r(s_3+w)})\right)d\lambda_3(s_3)d\lambda_3(w).
\end{aligned}
\tag{19}
$$

Plugging (13), (14) and (19) into (12) yields that it holds uniformly for $t \in \Lambda_T$ that

$$
\begin{aligned}
I_{11} &= \int_{0-}^{t} \overline{F}_1(x_1e^{rs_1})d\lambda_1(s_1)\int_{0-}^{t} \overline{F}_2(x_2e^{rs_2})d\lambda_2(s_2) + \int_{0-}^{t} \overline{F}_1(x_1e^{rs_3})d\lambda_3(s_3)\int_{0-}^{t} \overline{F}_2(x_2e^{rs_2})d\lambda_2(s_2) \\
&\quad + \int_{0-}^{t} \overline{F}_1(x_1e^{rs_1})d\lambda_1(s_1)\int_{0-}^{t} \overline{F}_2(x_2e^{rs_3})d\lambda_3(s_3) + \int_{0-}^{t} \overline{F}_1(x_1e^{rs_3})\overline{F}_2(x_2e^{rs_3})d\lambda_3(s_3) \\
&\quad + \iint_{\substack{s_3,w\geq 0\\ s_3+w\leq t}} \left(\overline{F}_1(x_1e^{r(s_3+w)})\overline{F}_2(x_2e^{rs_3}) + \overline{F}_1(x_1e^{rs_3})\overline{F}_2(x_2e^{r(s_3+w)})\right)d\lambda_3(s_3)d\lambda_3(w).
\end{aligned}
$$

For I_{12}, it holds that

$$\begin{aligned} I_{12} =& \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=P+1}^{\infty}\Bigg(\sum_{j=1}^{n}\sum_{k=1}^{p}\mathbb{P}\left(X_{1j}e^{-r\tau_j^{(1)}} > x_1, N_1(t)=n\right)\mathbb{P}\left(Y_{1k}e^{-r\tau_k^{(2)}} > x_2, N_2(t)=p\right)\mathbb{P}\left(N_3(t)=m\right) \\ &+ \sum_{i=1}^{m}\sum_{k=1}^{p}\mathbb{P}\left(X_{2i}e^{-r\tau_i^{(3)}} > x_1, N_3(t)=m\right)\mathbb{P}\left(Y_{1k}e^{-r\tau_k^{(2)}} > x_2, N_2(t)=p\right)\mathbb{P}\left(N_1(t)=n\right) \\ &+ \sum_{j=1}^{n}\sum_{i=1}^{m}\mathbb{P}\left(X_{1j}e^{-r\tau_j^{(1)}} > x_1, N_1(t)=n\right)\mathbb{P}\left(Y_{2i}e^{-r\tau_i^{(3)}} > x_2, N_3(t)=m\right)\mathbb{P}\left(N_2(t)=p\right) \\ &+ \sum_{i=1}^{m}\sum_{l=1}^{m}\mathbb{P}\left(X_{2i}e^{-r\tau_i^{(3)}} > x_1, Y_{2l}e^{-r\tau_l^{(3)}} > x_2, N_3(t)=m\right)\mathbb{P}\left(N_1(t)=n\right)\mathbb{P}\left(N_2(t)=p\right)\Bigg) \\ =&:E_1+E_2+E_3+E_4. \end{aligned} \tag{20}$$

For E_1, we interchange the order of the sums and choose P large enough, it holds uniformly for $t \in \Lambda_T$ that

$$\begin{aligned} E_1 =& \sum_{j=1}^{\infty}\sum_{p=P+1}^{\infty}\sum_{k=1}^{p}\mathbb{P}\left(X_{1j}e^{-r\tau_j^{(1)}} > x_1, N_1(t)=n\right)\mathbb{P}\left(Y_{1k}e^{-r\tau_k^{(2)}} > x_2, N_2(t)=p\right)\sum_{m=1}^{\infty}\mathbb{P}\left(N_3(t)=m\right) \\ \le& \sum_{j=1}^{\infty}\int_{0-}^{t}\mathbb{P}\left(X_{1j}e^{-rs_1} > x_1 \middle| \tau_j^{(1)} = s_1\right)\mathbb{P}\left(\tau_j^{(1)} \in ds_1\right) \\ & \sum_{p=P+1}^{\infty}\sum_{k=1}^{p}\int_{0-}^{t}\mathbb{P}\left(Y_{1k}e^{-rs_2} > x_2, \tau_p^{(2)} - \tau_1^{(2)} \le t - s_2 < \tau_{p+1}^{(2)} - \tau_1^{(2)} \middle| \tau_1^{(2)} = s_2\right)\mathbb{P}\left(\tau_1^{(2)} \in ds_2\right) \\ =& \int_{0-}^{t}\mathbb{P}\left(X_{1j}e^{-rs_1} > x_1\right)d\lambda_1(s_1)\sum_{p=P+1}^{\infty}\int_{0-}^{t}\sum_{k=1}^{p}\mathbb{P}\left(Y_{1k}e^{-rs_2} > x_2\right)\mathbb{P}\left(N_2(t-s_2) = p-1\right)\mathbb{P}\left(\tau_1^{(2)} \in ds_2\right) \\ \le& \int_{0-}^{t}\overline{F}_1(x_1e^{rs_1})d\lambda_1(s_1)\int_{0-}^{t}\overline{F}_2(x_2e^{rs_2})d\lambda_2(s_2)\sum_{p=P}^{\infty}(p+1)\,\mathbb{P}\left(N_2(T) \ge p\right) \\ =& o\left(L_1\right). \end{aligned} \tag{21}$$

Similarly to (21), we can obtain that

$$E_2 = o\left(L_2\right), E_3 = o\left(L_3\right). \tag{22}$$

For E_4, choose P large enough, and interchanging the order of the sums leads to

$$\begin{aligned} E_4 =& \sum_{m=1}^{\infty}\sum_{i=1}^{m}\sum_{l=1}^{m}\mathbb{P}\left(X_{2i}e^{-r\tau_i^{(3)}} > x_1, Y_{2l}e^{-r\tau_l^{(3)}} > x_2, N_3(t)=m\right)\sum_{n=1}^{\infty}\mathbb{P}\left(N_1(t)=n\right)\sum_{p=P+1}^{\infty}\mathbb{P}\left(N_2(t)=p\right) \\ =& L_4 \cdot \sum_{p=P+1}^{\infty}\mathbb{P}\left(N_2(t)=p\right) \\ =& o(L_4). \end{aligned} \tag{23}$$

Plugging (21), (22) and (23) into (20) yields that it holds uniformly for $t \in \Lambda_T$ that

$$I_{12} = o(I_{11}). \tag{24}$$

For $I_{1i}, i = 3, 4$, we can obtain the same conclusions as (24).

For I_{15}, it holds that

$$\begin{aligned} I_{15} = &\sum_{m=M+1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \left(\sum_{j=1}^{n} \sum_{k=1}^{p} \mathbb{P}\left(X_{1j} e^{-r\tau_j^{(1)}} > x_1, N_1(t) = n\right) \mathbb{P}\left(Y_{1k} e^{-r\tau_k^{(2)}} > x_2, N_2(t) = p\right) \mathbb{P}\left(N_3(t) = m\right) \right. \\ &+ \sum_{i=1}^{m} \sum_{k=1}^{p} \mathbb{P}\left(X_{2i} e^{-r\tau_i^{(3)}} > x_1, N_3(t) = m\right) \mathbb{P}\left(Y_{1k} e^{-r\tau_k^{(2)}} > x_2, N_2(t) = p\right) \mathbb{P}\left(N_1(t) = n\right) \\ &+ \sum_{j=1}^{n} \sum_{i=1}^{m} \mathbb{P}\left(X_{1j} e^{-r\tau_j^{(1)}} > x_1, N_1(t) = n\right) \mathbb{P}\left(Y_{2i} e^{-r\tau_i^{(3)}} > x_2, N_3(t) = m\right) \mathbb{P}\left(N_2(t) = p\right) \\ &\left. + \sum_{i=1}^{m} \sum_{l=1}^{m} \mathbb{P}\left(X_{2i} e^{-r\tau_i^{(3)}} > x_1, Y_{2l} e^{-r\tau_l^{(3)}} > x_2, N_3(t) = m\right) \mathbb{P}\left(N_1(t) = n\right) \mathbb{P}\left(N_2(t) = p\right) \right) \\ =: &F_1 + F_2 + F_3 + F_4. \end{aligned} \tag{25}$$

For F_1, F_2 and F_3, similarly to (21), we can obtain that

$$F_1 = o(L_1), F_2 = o(L_2), F_3 = o(L_3). \tag{26}$$

For F_4, choose M large enough, we have

$$\begin{aligned} F_4 = &\sum_{m=M+1}^{\infty} \sum_{1 \le i,l \le m} \mathbb{P}\left(X_{2i} e^{-r\tau_i^{(3)}} > x_1, Y_{2l} e^{-r\tau_l^{(3)}} > x_2, N_3(t) = m\right) \sum_{n=1}^{\infty} \mathbb{P}\left(N_1(t) = n\right) \sum_{p=1}^{\infty} \mathbb{P}\left(N_2(t) = p\right) \\ = &\sum_{m=M+1}^{\infty} \left(\sum_{1 \le i \ne l \le m} + \sum_{1 \le i = l \le m} \right) \mathbb{P}\left(X_{2i} e^{-r\tau_i^{(3)}} > x_1, Y_{2l} e^{-r\tau_l^{(3)}} > x_2, N_3(t) = m\right) \\ \le &\int_{0-}^{t} \mathbb{P}\left(X_{2i} e^{-rs_3} > x_1\right) \mathbb{P}\left(Y_{2l} e^{-rs_3} > x_2\right) \mathbb{P}\left(\tau_1^{(3)} \in ds_3\right) \sum_{m=M+1}^{\infty} m^2 \cdot \mathbb{P}\left(N_3(T) \ge m-1\right) \\ \le &\int_{0-}^{t} \overline{F}_1(x_1 e^{rs_3}) \overline{F}_2(x_2 e^{rs_3}) d\lambda_3(s_3) \sum_{m=M+1}^{\infty} m^2 \cdot \mathbb{P}\left(N_3(T) \ge m-1\right) \\ = &o(L_{42}). \end{aligned} \tag{27}$$

Plugging (26) and (27) into (25) yields that it holds uniformly for $t \in \Lambda_T$ that

$$I_{15} = o(I_{11}). \tag{28}$$

For $I_{1i}, i = 6, 7, 8$, we can obtain the same conclusions as (28).

For I_2, we have

$$\begin{aligned} I_2 = &\sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{p=P+1}^{\infty} \mathbb{P}\left(\sum_{j=1}^{n} X_{1j} e^{-r\tau_j^{(1)}} + \sum_{i=1}^{m} X_{2i} e^{-r\tau_i^{(3)}} + \overline{M}_1(T) > x_1, \right. \\ &\left. \sum_{k=1}^{p} Y_{1k} e^{-r\tau_k^{(2)}} + \sum_{i=1}^{m} Y_{2i} e^{-r\tau_i^{(3)}} + \overline{M}_2(T) > x_2, N_1(t) = n, N_2(t) = p, N_3(t) = m \right) \\ = &\sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{p=P+1}^{\infty} \int \cdots \int_{0 < t_1^{(1)} < \ldots < t_n^{(1)} \le t < t_{n+1}^{(1)}} \int \cdots \int_{0 < t_1^{(2)} < \ldots < t_p^{(2)} \le t < t_{p+1}^{(2)}} \int \cdots \int_{0 < t_1^{(3)} < \ldots < t_m^{(3)} \le t < t_{m+1}^{(3)}} \\ &\mathbb{P}\left(\sum_{j=1}^{n} X_{1j} e^{-rt_j^{(1)}} + \sum_{i=1}^{m} X_{2i} e^{-rt_i^{(3)}} + \overline{M}_1(T) > x_1 \right) \mathbb{P}\left(\sum_{k=1}^{p} Y_{1k} e^{-rt_k^{(2)}} + \sum_{i=1}^{m} Y_{2i} e^{-rt_i^{(3)}} + \overline{M}_2(T) > x_2 \right) \\ &\mathbb{P}\left(\tau_1^{(1)} \in dt_1^{(1)}, \ldots, \tau_{n+1}^{(1)} \in dt_{n+1}^{(1)}\right) \mathbb{P}\left(\tau_1^{(2)} \in dt_1^{(2)}, \ldots, \tau_{p+1}^{(2)} \in dt_{p+1}^{(2)}\right) \mathbb{P}\left(\tau_1^{(3)} \in dt_1^{(3)}, \ldots, \tau_{m+1}^{(3)} \in dt_{m+1}^{(3)}\right). \end{aligned}$$

For any $\varepsilon > 0$, by Lemma 2.3, there exists a constant K such that,

$$\begin{aligned}
I_2 \leq & K\sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{p=P+1}^{\infty}\int\cdots\int_{0<t_1^{(1)}<\ldots<t_n^{(1)}\leq t<t_{n+1}^{(1)}}\int\cdots\int_{0<t_1^{(2)}<\ldots<t_p^{(2)}\leq t<t_{p+1}^{(2)}}\int\cdots\int_{0<t_1^{(3)}<\ldots<t_m^{(3)}\leq t<t_{m+1}^{(3)}} \\
& (1+\varepsilon)^{(2m+n+p)}\left(\mathbb{P}\left(X_{11}e^{-rt_1^{(1)}}>x_1\right)+\mathbb{P}\left(X_{21}e^{-rt_1^{(3)}}>x_1\right)\right)\left(\mathbb{P}\left(Y_{11}e^{-rt_1^{(2)}}>x_2\right)+\mathbb{P}\left(Y_{21}e^{-rt_1^{(3)}}>x_2\right)\right) \\
& \mathbb{P}\left(\tau_1^{(1)}\in dt_1^{(1)},\ldots,\tau_{n+1}^{(1)}\in dt_{n+1}^{(1)}\right)\mathbb{P}\left(\tau_1^{(2)}\in dt_1^{(2)},\ldots,\tau_{p+1}^{(2)}\in dt_{p+1}^{(2)}\right)\mathbb{P}\left(\tau_1^{(3)}\in dt_1^{(3)},\ldots,\tau_{m+1}^{(3)}\in dt_{m+1}^{(3)}\right) \\
= & K\sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{p=P+1}^{\infty}(1+\varepsilon)^{(2m+n+p)}\left(\mathbb{P}\left(X_{11}e^{-r\tau_1^{(1)}}>x_1,N_1(t)=n\right)\mathbb{P}\left(Y_{11}e^{-r\tau_1^{(2)}}>x_2,N_2(t)=p\right)\mathbb{P}\left(N_3(t)=m\right)\right. \\
& +\mathbb{P}\left(X_{21}e^{-r\tau_1^{(3)}}>x_1,N_3(t)=m\right)\mathbb{P}\left(Y_{11}e^{-r\tau_1^{(2)}}>x_2,N_2(t)=p\right)\mathbb{P}\left(N_1(t)=n\right) \\
& +\mathbb{P}\left(X_{11}e^{-r\tau_1^{(1)}}>x_1,N_1(t)=n\right)\mathbb{P}\left(Y_{21}e^{-r\tau_1^{(3)}}>x_2,N_3(t)=m\right)\mathbb{P}\left(N_2(t)=p\right) \\
& \left.+\mathbb{P}\left(X_{21}e^{-r\tau_1^{(3)}}>x_1,Y_{21}e^{-r\tau_1^{(3)}}>x_2,N_3(t)=m\right)\mathbb{P}\left(N_1(t)=n\right)\mathbb{P}\left(N_2(t)=p\right)\right) \\
=: & I_{21}+I_{22}+I_{23}+I_{24}.
\end{aligned} \tag{29}$$

For I_{21}, choose M, N, P large enough, it holds that

$$\begin{aligned}
I_{21} = & K\sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{p=P+1}^{\infty}(1+\varepsilon)^{(2m+n+p)}\mathbb{P}\left(X_{11}e^{-r\tau_1^{(1)}}>x_1,N_1(t)=n\right)\mathbb{P}\left(Y_{11}e^{-r\tau_1^{(2)}}>x_2,N_2(t)=p\right)\mathbb{P}\left(N_3(t)=m\right) \\
= & K\sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{p=P+1}^{\infty}(1+\varepsilon)^{(2m+n+p)}\int_{0-}^{t}\mathbb{P}\left(X_{11}e^{-rs_1}>x_1,\tau_n^{(1)}-\tau_1^{(1)}\leq t-s_1<\tau_{n+1}^{(1)}-\tau_1^{(1)}\middle|\tau_1^{(1)}=s_1\right)\mathbb{P}\left(\tau_1^{(1)}\in ds_1\right) \\
& \int_{0-}^{t}\mathbb{P}\left(Y_{11}e^{-rs_2}>x_2,\tau_p^{(2)}-\tau_1^{(2)}\leq t-s_2<\tau_{p+1}^{(2)}-\tau_1^{(2)}\middle|\tau_1^{(2)}=s_2\right)\mathbb{P}\left(\tau_1^{(2)}\in ds_2\right)\mathbb{P}\left(N_3(t)=m\right) \\
\leq & K\int_{0-}^{t}\overline{F}_1(x_1e^{rs_1})d\lambda_1(s_1)\int_{0-}^{t}\overline{F}_2(x_2e^{rs_2})d\lambda_2(s_2) \\
& \sum_{m=1}^{M}(1+\varepsilon)^{2m}\mathbb{P}\left(N_3(t)=m\right)\sum_{n=1}^{N}(1+\varepsilon)^{n}\mathbb{P}\left(N_1(T)\geq n-1\right)\sum_{p=P+1}^{\infty}(1+\varepsilon)^{p}\mathbb{P}\left(N_2(T)\geq p-1\right),
\end{aligned}$$

thus, we can choose $\varepsilon > 0$ sufficiently small, it holds uniformly for $t \in \Lambda_T$ that

$$I_{21} = o\left(\int_{0-}^{t}\overline{F}_1(x_1e^{rs_1})d\lambda_1(s_1)\int_{0-}^{t}\overline{F}_2(x_2e^{rs_2})d\lambda_2(s_2)\right). \tag{30}$$

Similarly, we can obtain that

$$\begin{aligned}
I_{22} = & o\left(\int_{0-}^{t}\overline{F}_1(x_1e^{rs_3})d\lambda_3(s_3)\int_{0-}^{t}\overline{F}_2(x_2e^{rs_2})d\lambda_2(s_2)\right), \\
I_{23} = & o\left(\int_{0-}^{t}\overline{F}_1(x_1e^{rs_1})d\lambda_1(s_1)\int_{0-}^{t}\overline{F}_2(x_2e^{rs_3})d\lambda_3(s_3)\right), \\
I_{24} = & o\left(\int_{0-}^{t}\overline{F}_1(x_1e^{rs_3})\overline{F}_2(x_2e^{rs_3})d\lambda_3(s_3)\right).
\end{aligned} \tag{31}$$

Plugging (30) and (31) into (29) yields that it holds uniformly for $t \in \Lambda_T$ that

$$I_2 = o\left(I_1\right). \tag{32}$$

For $I_i, i = 3, ..., 8$, we can obtain the same conclusions as (32). Combining the discussion of $I_i, i = 1, 2, 3, ..., 8$ with (11), we conclude the asymptotic relation (8).

Combining Lemma 3.1, Lemma 3.3 of Li (2017) and following the proof of the relation (8) with slight modifications, we turn to the relation (9), then, we can obtain the same conclusion. □

4.2. Proof of Theorem 3.1

For the upper bound, we have

$$\begin{aligned}\Psi_\vee(\vec{x},t) &\le \mathbb{P}\left(\sum_{j=1}^{N_1(t)} X_{1j}e^{-r\tau_j^{(1)}} + \sum_{i=1}^{N_3(t)} X_{2i}e^{-r\tau_i^{(3)}} - \underline{M_1}(T) > x_1, \sum_{k=1}^{N_2(t)} Y_{1k}e^{-r\tau_k^{(2)}} + \sum_{i=1}^{N_3(t)} Y_{2i}e^{-r\tau_i^{(3)}} - \underline{M_2}(T) > x_2\right)\\ &= \mathbb{P}\left(\sum_{j=1}^{N_1(t)} X_{1j}e^{-r\tau_j^{(1)}} + \sum_{i=1}^{N_3(t)} X_{2i}e^{-r\tau_i^{(3)}} + \overline{M}_1(T) > x_1, \sum_{k=1}^{N_2(t)} Y_{1k}e^{-r\tau_k^{(2)}} + \sum_{i=1}^{N_3(t)} Y_{2i}e^{-r\tau_i^{(3)}} + \overline{M}_2(T) > x_2\right).\end{aligned}$$

By Lemma 4.1, we can easily get

$$\begin{aligned}\Psi_\vee(\vec{x},t) \lesssim &\int_{0-}^t \overline{F}_1(x_1e^{rs_1})d\lambda_1(s_1)\int_{0-}^t \overline{F}_2(x_2e^{rs_2})d\lambda_2(s_2) + \int_{0-}^t \overline{F}_1(x_1e^{rs_1})d\lambda_1(s_1)\int_{0-}^t \overline{F}_2(x_2e^{rs_3})d\lambda_3(s_3)\\ &+ \int_{0-}^t \overline{F}_1(x_1e^{rs_3})d\lambda_3(s_3)\int_{0-}^t \overline{F}_2(x_2e^{rs_2})d\lambda_2(s_2) + \int_{0-}^t \overline{F}_1(x_1e^{rs_3})\overline{F}_2(x_2e^{rs_3})d\lambda_3(s_3)\\ &+ \iint_{\substack{s_3,w\ge 0\\ s_3+w\le t}} \left(\overline{F}_1(x_1e^{r(s_3+w)})\overline{F}_2(x_2e^{rs_3}) + \overline{F}_1(x_1e^{rs_3})\overline{F}_2(x_2e^{r(s_3+w)})\right)d\lambda_3(s_3)d\lambda_3(w).\end{aligned} \tag{33}$$

For the lower bound, using Lemma 4.1 again, we have

$$\begin{aligned}\Psi_\vee(\vec{x},t) \ge&\mathbb{P}\left(\sum_{j=1}^{N_1(t)} X_{1j}e^{-r\tau_j^{(1)}} + \sum_{i=1}^{N_3(t)} X_{2i}e^{-r\tau_i^{(3)}} - \overline{M}_1(T) > x_1 + \frac{p_1}{r}, \sum_{k=1}^{N_2(t)} Y_{1k}e^{-r\tau_k^{(2)}} + \sum_{i=1}^{N_3(t)} Y_{2i}e^{-r\tau_i^{(3)}} - \overline{M}_2(T) > x_2 + \frac{p_2}{r}\right)\\ \sim &\int_{0-}^t \overline{F}_1(x_1e^{rs_1} + \frac{p_1}{r}e^{rs_1})d\lambda_1(s_1)\int_{0-}^t \overline{F}_2(x_2e^{rs_2} + \frac{p_2}{r}e^{rs_2})d\lambda_2(s_2)\\ &+ \int_{0-}^t \overline{F}_1(x_1e^{rs_1} + \frac{p_1}{r}e^{rs_1})d\lambda_1(s_1)\int_{0-}^t \overline{F}_2(x_2e^{rs_3} + \frac{p_2}{r}e^{rs_3})d\lambda_3(s_3)\\ &+ \int_{0-}^t \overline{F}_1(x_1e^{rs_3} + \frac{p_1}{r}e^{rs_3})d\lambda_3(s_3)\int_{0-}^t \overline{F}_2(x_2e^{rs_2} + \frac{p_2}{r}e^{rs_2})d\lambda_2(s_2)\\ &+ \int_{0-}^t \overline{F}_1(x_1e^{rs_3} + \frac{p_1}{r}e^{rs_3})\overline{F}_2(x_2e^{rs_3} + \frac{p_2}{r}e^{rs_3})d\lambda_3(s_3)\\ &+ \iint_{\substack{s_3,w\ge 0\\ s_3+w\le t}} \Big(\overline{F}_1(x_1e^{r(s_3+w)} + \frac{p_1}{r}e^{r(s_3+w)})\overline{F}_2(x_2e^{rs_3} + \frac{p_2}{r}e^{rs_3})\\ &+ \overline{F}_1(x_1e^{rs_3} + \frac{p_1}{r}e^{rs_3})\overline{F}_2(x_2e^{r(s_3+w)} + \frac{p_2}{r}e^{r(s_3+w)})\Big)d\lambda_3(s_3)d\lambda_3(w).\end{aligned}$$

By $F \in \mathcal{S} \subset \mathcal{L}$, we obtain that

$$\begin{aligned}\Psi_\vee(\vec{x},t) \gtrsim &\int_{0-}^t \overline{F}_1(x_1e^{rs_1})d\lambda_1(s_1)\int_{0-}^t \overline{F}_2(x_2e^{rs_2})d\lambda_2(s_2) + \int_{0-}^t \overline{F}_1(x_1e^{rs_1})d\lambda_1(s_1)\int_{0-}^t \overline{F}_2(x_2e^{rs_3})d\lambda_3(s_3)\\ &+ \int_{0-}^t \overline{F}_1(x_1e^{rs_3})d\lambda_3(s_3)\int_{0-}^t \overline{F}_2(x_2e^{rs_2})d\lambda_2(s_2) + \int_{0-}^t \overline{F}_1(x_1e^{rs_3})\overline{F}_2(x_2e^{rs_3})d\lambda_3(s_3)\\ &+ \iint_{\substack{s_3,w\ge 0\\ s_3+w\le t}} \left(\overline{F}_1(x_1e^{r(s_3+w)})\overline{F}_2(x_2e^{rs_3}) + \overline{F}_1(x_1e^{rs_3})\overline{F}_2(x_2e^{r(s_3+w)})\right)d\lambda_3(s_3)d\lambda_3(w).\end{aligned} \tag{34}$$

Combining (33) and (34), we complete the proof.

4.3. Proof of Theorem 3.2

For the lower bound, we have

$$\Psi_{max}(\vec{x},t) \geq \mathbb{P}\left(U_1(t)<0, U_2(t)<0\right) = \mathbb{P}\left(e^{-rt}U_1(t)<0, e^{-rt}U_2(t)<0\right).$$

Combining the relation (2), Lemma 4.1, and the proof of Theorem 3.1, it is easy to obtain the relation (6).

4.4. Proof of Theorem 3.3

To prove Theorem 3.3, we need the following relation

$$\begin{aligned}\Psi_{min}(\vec{x},t) = \mathbb{P}\left(T_{min}(\vec{x}\leq t)\right) =& \mathbb{P}\left(\left\{\inf_{0\leq s\leq t} U_1(s)<0\right\}\bigcup\left\{\inf_{0\leq s\leq t} U_2(s)<0\right\}\right)\\ =&\mathbb{P}\left(\inf_{0\leq s\leq t} U_1(s)<0\right)+\mathbb{P}\left(\inf_{0\leq s\leq t} U_2(s)<0\right)-\Psi_{\vee}(\vec{x},t).\end{aligned} \tag{35}$$

Following the proof of Theorem 2.1 in Li (2017) with slight modifications, we have

$$\sum_{i=1}^{2}\mathbb{P}\left(\inf_{0\leq s\leq t} U_i(s)<0\right) \sim \int_{0-}^{t}\overline{F}_1(x_1e^{rs_1})d\lambda_1(s_1)+\int_{0-}^{t}\overline{F}_2(x_2e^{rs_2})d\lambda_2(s_2)+\int_{0-}^{t}\left(\overline{F}_1(x_1e^{rs_3})+\overline{F}_2(x_2e^{rs_3})\right)d\lambda_3(s_3). \tag{36}$$

Since, we have

$$\Psi_{\vee}(\vec{x},t) = o\left(\int_{0-}^{t}\overline{F}_1(x_1e^{rs_1})d\lambda_1(s_1)+\int_{0-}^{t}\overline{F}_2(x_2e^{rs_2})d\lambda_2(s_2)+\int_{0-}^{t}\left(\overline{F}_1(x_1e^{rs_3})+\overline{F}_2(x_2e^{rs_3})\right)d\lambda_3(s_3)\right), \tag{37}$$

where $(x_1,x_2)\longrightarrow(\infty,\infty)$. According to (35), (36) and (37), we complete the proof.

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