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Some Inequalities Arising from Analytic Summability of Functions

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Abstract. Analytic summability of functions was introduced by the second author in 2016. He utilized Bernoulli numbers and polynomials for a holomorphic function to construct analytic summability. The analytic summand function f_σ (if exists) satisfies the difference functional equation $f_\sigma(z) = f(z) + f_\sigma(z-1)$. Moreover, some upper bounds for f_σ and several inequalities between f and f_σ were presented by him. In this paper, by using Alzer's improved upper bound for Bernoulli numbers, we improve those upper bounds and obtain some inequalities and new upper bounds. As some applications of the topic, we obtain several upper bounds for Bernoulli polynomials, sums of powers of natural numbers, (e.g., $1^p + 2^p + 3^p + ... + r^p \leq \frac{2p!}{\pi^{p+1}}(e^{\pi r} - 1)$) and several inequalities for exponential, hyperbolic and trigonometric functions.

1. Introduction and Preliminaries

The Bohr-Mollerup theorem states that the only logarithmic convex solution of the functional equation

$$f(x+1) = xf(x), \quad x > 0 \tag{1}$$

is of the form

$$f(x) = \lim_{n\to\infty} \frac{n!n^x}{(x+n)(x+n-1)\cdots(x+1)x} \text{ (for } \quad x > 0) \tag{2}$$

(see [3]). Taking logarithm of this formula (2) leads us to

$$\log f(x) + \log x = \lim_{n\to\infty} \left(x\log(n) + \sum_{k=1}^{n} \log k - \log(k+x) \right). \tag{3}$$

This is of course in a close relation with the difference functional equation

$$F(x+1) - F(x) = G(x), \tag{4}$$

for the special case $G(x) = x$.

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Inspired by this, Hooshmand introduced the concept of limit summability in [4]. A real or complex function f is called *limit summable* provided that the functional sequence

$$f_{\sigma_n}(x) := xf(n) + \sum_{k=1}^{n}(f(k) - f(k+x)). \tag{5}$$

is convergent. Later in [5], he noticed that some important elementary functions, such as polynomials of degree> 1 and trigonometric functions, are not limit summable. To deal with this inadequacy he introduced the concept of analytic summability in the preceding mentioned paper. In this regard the Bernoulli polynomials and numbers play a crucial role.

Given a complex number z the Bernoulli polynomials $B_0(z), B_1(z), \cdots$ are generated by the equation

$$\frac{te^{zt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(z)}{n!} t^n, \quad |t| < 2\pi.$$

The sequences $B_n := B_n(0)$, $b_n := B_n(1)$ are called first and second Bernoulli numbers respectively (see [2], Chapter 12). The following notation of [5] are used for the definition of the analytic summability

$$\sigma(z^n) = \frac{B_{n+1}(z+1) - b_{n+1}}{n+1}, \quad z \in \mathbb{C}, n \geq 0 \tag{6}$$

Notice that if $r \in \mathbb{N}$ then we have

$$\sigma(r^n) = 1^n + 2^n + 3^n + \cdots + r^n.$$

Also, putting

$$\beta_{nk} = \beta_{n,k} := \binom{n+1}{k} \frac{b_{n+1-k}}{n+1} = \frac{n!}{k!(n+1-k)!} b_{n+1-k} \tag{7}$$

we have

$$\sigma(z^n) = \sum_{k=1}^{n+1} \beta_{nk} z^k, \quad z \in \mathbb{C}, n \geq 0, \tag{8}$$

for all $n \geq 0$, $1 \leq k \leq n+1$ (see [5]). Recall from [5] the concepts of analytic summability and related definitions as follows.

Definition 1.1. *A complex or real analytic function defined on an open domain D of the form $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is called analytic summable (resp. absolutely analytic summable) at z_0 if the functional series*

$$f_{\sigma_{\mathcal{A}}}(z_0) = f_\sigma(z_0) = \sum_{n=0}^{\infty} c_n \sigma(z_0^n)$$

is convergent (resp. absolutely convergent). We call f analytic summable on $E \subseteq D$ if it is analytic summable at every point of E. The function $f_{\sigma_{\mathcal{A}}} = f_\sigma$ (with the largest possible domain) is called analytic summand function of f. If f is analytic summable on the whole $\mathbb{C}$, then we call f entire analytic summable.

Notice that if f is analytic summable on D, then it satisfies the functional difference equation

$$f_\sigma(z) = f(z) + f_\sigma(z-1), \;\; z \in D \cap (D+1). \tag{9}$$

In [5] there are given several criteria for analytic summability of holomorphic functions, in particular the upper bounds (10) for analytic summand functions are estimated.

Theorem 1.2. *Let $f(z)=\sum_{n=0}^{\infty}c_nz^n$ be an analytic function defined on an open domain D. If $\sum_{n=0}^{\infty}\frac{n!}{\pi^n}c_n$ is absolutely convergent (e.g., $\limsup_{n\to\infty}\sqrt[n]{n!|c_n|}<\pi$), then f is absolutely analytic summable on D. Moreover, by putting $\mathrm{Abs}_{!/\pi}(f):=\sum_{n=0}^{\infty}\frac{n!}{\pi^n}|c_n|$, $\mathrm{Abs}(f(z)):=\sum_{n=0}^{\infty}|c_n||z|^n$, We have the following upper bounds for $f_{\sigma_{\mathcal{A}}}$*

$$\begin{aligned}|f_\sigma(z)| &\le \frac{1}{\pi}\Big((\frac{\pi}{2}-1)\,\mathrm{Abs}(f(z))+(e^{\pi|z|}-1)\,\mathrm{Abs}_{!/\pi}(f)\Big) \\ &\le \frac{2}{\pi}(e^{\pi|z|}-1)\,\mathrm{Abs}_{!/\pi}(f)\end{aligned} \tag{10}$$

for every $z\in D$.

Proof. The proof exists in [5] Theorem 4.1. Here we only prove the inequality. To achieve (10) we define the functional sequence $\{g_n\}$ by

$$g_n(t):=\frac{e^{\pi t}-1}{\pi}\cdot\frac{n!}{\pi^n}+(\frac{1}{\pi}-\frac{1}{2})t^n,\quad t\ge 0.$$

Since $g_n(0)=0$, for all n, and

$$g_n'(t)=\frac{n!}{\pi^n}e^{\pi t}+(\frac{1}{\pi}-\frac{1}{2})n\cdot t^{n-1}\ge 0,\quad t\ge 0,\quad n\in\mathbb{N}$$

by putting $t=|z|$ one can conclude that $\sum_{n=0}^{\infty}|c_n|.g_n(|z|)\ge 0$ and consequently

$$\begin{aligned}&\frac{2}{\pi}(e^{\pi|z|}-1)Abs_{!/\pi}(f)-\frac{1}{\pi}\Big((\frac{\pi}{2}-1)Abs(f(z))+(e^{\pi|z|}-1)Abs_{!/\pi}(f)\Big)\\ &=\sum_{n=0}^{\infty}|c_n|\cdot g_n(|z|)\ge 0.\end{aligned}$$

□

Hooshmand used the following bound of the Bernoulli numbers from [6, p.575] in the proof of the preceding theorem

$$|B_n|=|b_n|<\frac{2n!}{(2\pi)^n}\cdot\frac{1}{1-2^{1-n}},\quad n=2,3,4,5,\cdots \tag{11}$$

and by using (11) the following upper bound for β_{nk} is obtained (see [5]):

$$|\beta_{nk}|\le\frac{2n!}{k!\pi^{n-k+1}}\ ;\ 1\le k\le n+1. \tag{12}$$

In this paper, we use Alzer's improvement (13) of (11) to improve the upper bounds of the analytic summand functions (see [1])

$$|B_{2n}|\le\frac{2(2n)!}{(2\pi)^{2n}}\cdot\frac{1}{1-2^{\beta-2n}},\ n\ge 1, \tag{13}$$

where

$$\beta=2+\frac{\log(1-\frac{6}{\pi^2})}{\log 2}=0.6491\cdots.$$

Also,we obtain some inequalities for many special functions by using the results.

2. Derived inequalities from new upper bounds

The upper bounds (12) yields the following upper bounds for $\sigma(z^n)$ (for more details see [5]):

$$|\sigma(z^n)| \leq \frac{\pi-2}{2\pi}|z|^n + \frac{n!}{\pi^{n+1}}\sum_{k=1}^{n+1}\frac{(\pi|z|)^k}{k!} \leq \frac{2n!}{\pi^{n+1}}(e^{\pi|z|}-1). \tag{14}$$

Now by using (13) and a proof similar to that of (14), we achieve the following improved upper bounds for β_{nk} and $\sigma(z^n)$:

$$\begin{aligned}|\beta_{nk}| &\leq \frac{n!}{k!(n-k+1)!}\cdot\frac{2(n-k+1)!}{(2\pi)^{n-k+1}}\cdot\frac{1}{1-2^{\beta-(n-k+1)}}\\ &= \frac{n!}{k!\pi^{n-k+1}}\cdot\frac{1}{1-2^{\beta-(n-k+1)}}\\ &\leq \frac{n!}{k!\pi^{n-k+1}}\cdot\frac{1}{2-2^{\beta-1}} \quad ; \qquad 1\leq k\leq n-1\end{aligned}$$

this leads us to

$$|\beta_{nk}| \leq \begin{cases} \frac{1}{2} & \text{if } k=n \\ 0 & \text{if } n+1-k \text{ is odd and } k<n \\ \mu\frac{n!}{k!\pi^{n+1-k}} & \text{if } n+1-k \text{ is even and } k<n+1 \end{cases} \tag{15}$$

where

$$\mu = \frac{1}{2-2^{\beta-1}} = \frac{1}{2-2^{1+\log_2(1-\frac{6}{\pi^2})}} = \frac{\pi^2}{12} = 0.822\cdots.$$

Also, we have

$$\begin{aligned}|\sigma(z^n)| &\leq \frac{|z|^{n+1}}{n+1} + \frac{|z|^n}{2} + \sum_{\substack{k=1\\ n-k \text{ is odd}}}^{n-1}|\beta_{nk}||z|^k\\ &= \begin{cases} \frac{|z|^n}{2} + \frac{|z|^{n+1}}{n+1} + \sum_{m=1}^{\frac{n}{2}}|\beta_{n,2m-1}||z|^{2m-1} & ;\text{ n is even}\\ \frac{|z|^n}{2} + \frac{|z|^{n+1}}{n+1} + \sum_{m=1}^{\frac{n-1}{2}}|\beta_{n,2m}||z|^{2m} & ;\text{ n is odd}\end{cases}\end{aligned} \tag{16}$$

We now are in a position to the previous relations in order to improve the upper bounds of the analytic summand functions. Moreover, several inequalities will be derived. Following Hooshmand's notation in [5], we use the next notation:

$$\mathrm{Abs}^e_{!/\pi}(f) = \sum_{\substack{n=0\\ \text{n is even}}}^{\infty}\frac{n!}{\pi^n}|c_n|, \quad \mathrm{Abs}^o_{!/\pi}(f) = \sum_{\substack{n=0\\ n \text{ is odd}}}^{\infty}\frac{n!}{\pi^n}|c_n|$$

$$F(z) = \sum_{n=0}^{\infty}\frac{c_n}{n+1}z^{n+1} \quad \text{(the primitive function of } f).$$

Theorem 2.1. *Let $f(z)=\sum_{n=0}^{\infty}c_nz^n$ be an analytic function defined on an open domain D. If $\sum_{n=0}^{\infty}\frac{n!}{\pi^n}c_n$ is absolutely convergent (e.g. $\limsup_{n\to\infty}\sqrt[n]{n!|c_n|}<\pi$), then f is analytic summable on D and*

$$\begin{aligned}|f_\sigma(z)| \leq{}& \frac{1}{2}\,\mathrm{Abs}(f(z)) + (1-\mu)\,\mathrm{Abs}(F(z)) + \frac{\mu}{\pi}\sinh(\pi|z|)\,\mathrm{Abs}^e_{!/\pi}(f)\\ &+ \frac{\mu}{\pi}(\cosh(\pi|z|)-1)\,\mathrm{Abs}^o_{!/\pi}(f),\ z\in D.\end{aligned} \tag{17}$$

Proof. Putting $f_{\sigma_N}(z) = f_{\sigma_{\mathcal{A}_N}} := \sum_{n=0}^{N} c_n\sigma(z^n)$, and using the inequalities (15), (16) and splitting the sigma into two summations (for even and odd terms), we obtain

$$
\begin{aligned}
|f_{\sigma_N}(z)| &\le \sum_{n=0}^{N} |c_n||\sigma(z^n)| \\
&= \sum_{\substack{n=0 \\ \text{n is even}}}^{N} |c_n|\left(\frac{|z|^n}{2} + \frac{|z|^{n+1}}{n+1} + \sum_{m=1}^{\frac{n}{2}} |\beta_{n,2m-1}||z|^{2m-1}\right) + \sum_{\substack{n=0 \\ n \text{ is odd}}}^{N} |c_n|\left(\frac{|z|^n}{2} + \frac{|z|^{n+1}}{n+1} + \sum_{m=1}^{\frac{n-1}{2}} |\beta_{n,2m}||z|^{2m}\right) \\
&\le \sum_{\substack{n=0 \\ \text{n is even}}}^{N} |c_n|\left(\frac{|z|^n}{2} + \frac{|z|^{n+1}}{n+1} + \sum_{m=1}^{\frac{n}{2}} \mu\frac{n!}{(2m-1)!\pi^{n-2m+2}}|z|^{2m-1}\right) \\
&\quad + \sum_{\substack{n=0 \\ n \text{ is odd}}}^{N} |c_n|\left(\frac{|z|^n}{2} + \frac{|z|^{n+1}}{n+1} + \sum_{m=1}^{\frac{n-1}{2}} \mu\frac{n!}{(2m)!\pi^{n-2m+1}}|z|^{2m}\right) \\
&= \frac{1}{2}\sum_{n=0}^{N} |c_n||z|^n + \sum_{\substack{n=0 \\ \text{n is even}}}^{N} \sum_{m=1}^{\frac{n}{2}+1} \mu|c_n|\frac{n!}{(2m-1)!\pi^{n-2m+2}}|z|^{2m-1} + (1-\mu)\sum_{\substack{n=0 \\ \text{n is even}}}^{N} \frac{|c_n|}{n+1}|z|^{n+1} \\
&\quad + \sum_{\substack{n=0 \\ n \text{ is odd}}}^{N} \sum_{m=1}^{\frac{n+1}{2}} \mu|c_n|\frac{n!}{(2m)!\pi^{n-2m+1}}|z|^{2m} + (1-\mu)\sum_{\substack{n=0 \\ n \text{ is odd}}}^{N} \frac{|c_n|}{n+1}|z|^{n+1} \\
&\le \frac{1}{2}\operatorname{Abs}_N(f(z)) + (1-\mu)\operatorname{Abs}_N(F(z)) + \frac{\mu}{\pi}\sinh(\pi|z|)\operatorname{Abs}^{e}_{!/\pi}(f) + \frac{\mu}{\pi}(\cosh(\pi|z|)-1)\operatorname{Abs}^{o}_{!/\pi}(f)
\end{aligned}
$$

Since $f_{\sigma_N}(z) \to f_\sigma(z)$ (by Theorem 1.2), by letting $N \to \infty$ we get (17). □

Corollary 2.2. *Under the conditions of the preceding theorem, we have*

$$
\begin{aligned}
|f_\sigma(z)| &\le \frac{1}{2}\operatorname{Abs}(f(z)) + (1-\mu)\operatorname{Abs}(F(z)) + \frac{\mu}{\pi}\sinh(\pi|z|)\operatorname{Abs}^{e}_{!/\pi}(f) + \frac{\mu}{\pi}(\cosh(\pi|z|)-1)\operatorname{Abs}^{o}_{!/\pi}(f) \qquad (18) \\
&\le \frac{1}{2}\operatorname{Abs}(f(z)) + (1-\mu)\operatorname{Abs}(F(z)) + \frac{\mu}{\pi}\sinh(\pi|z|)\operatorname{Abs}_{!/\pi}(f) \\
&\le \frac{1}{\pi}\left((\frac{\pi}{2}-1)\operatorname{Abs}(f(z)) + (e^{\pi|z|}-1)\operatorname{Abs}_{!/\pi}(f)\right) \\
&\le \frac{2}{\pi}(e^{\pi|z|}-1)\operatorname{Abs}_{!/\pi}(f)
\end{aligned}
$$

Proof. The last inequality of (18) is proved in Theorem 1.2, and the first and second inequalities of (18) are obtained from (17) together with the fact $\cosh(\pi|z|) - 1 \le \sinh(\pi|z|)$. What remains is to prove the third inequality. To do this, define the functional sequence $\{h_n\}$ by

$$h_n(t) := \frac{e^{\pi t} - 1 + \mu\sinh(\pi t)}{\pi} \cdot \frac{n!}{\pi^n} - \frac{t^n}{\pi} + \frac{\mu-1}{n+1}t^{n+1}, \quad t \ge 0$$

Since $h_n(0) = 0$, for all n, and

$$
\begin{aligned}
h'_n(t) &= (e^{\pi t} + \mu\cosh(\pi t)) \cdot \frac{n!}{\pi^n} - \frac{n}{\pi}t^{n-1} + (\mu-1)t^n \\
&= \frac{n!}{\pi^n}(e^{\pi t} - \frac{\pi^n}{n!}t^n - \frac{\pi^{n-1}}{(n-1)!}t^{n-1}) + \frac{\mu n!}{\pi^n}\cosh(\pi t) + \mu t^n \ge 0,
\end{aligned}
$$

for all n and $t \geq 0$, then we conclude that $h_n(t) \geq 0$. By putting $t = |z|$ we have $\sum_{n=0}^{\infty} |c_n|.h_n(|z|) \geq 0$ and so

$$\begin{aligned}
&\frac{1}{\pi}\left((\frac{\pi}{2} - 1)\operatorname{Abs}(f(z)) + (e^{\pi|z|} - 1)\operatorname{Abs}_{!/\pi}(f)\right) - \\
&\frac{1}{2}\operatorname{Abs}(f(z)) + (1-\mu)\operatorname{Abs}(F(z)) + \frac{\mu}{\pi}\sinh(\pi|z|)\operatorname{Abs}_{!/\pi}(f) \\
&= \sum_{n=0}^{\infty} |c_n| h_n(|z|) \geq 0,
\end{aligned}$$

this completes the proof. □

The following corollary presents an interesting inequality for the summation $f(1) + f(2) + \cdots + f(r)$ when f is an analytic function with conditions told before and r is a positive integer.

Corollary 2.3. *Let the assumptions in Theorem 2.1 hold. If r is a positive integer in D, then we have the following inequalities for the partial summation of the sequence $\{f_k := f(k)\}_{k=1}^{r}$.*

$$\begin{aligned}
\left|\sum_{k=1}^{r} f(k)\right| &\leq \frac{1}{2}\operatorname{Abs}(f(r)) + (1-\mu)\operatorname{Abs}(F(r)) + \frac{\mu}{\pi}\sinh(\pi r)\operatorname{Abs}^{e}_{!/\pi}(f) + \frac{\mu}{\pi}(\cosh(\pi r) - 1)\operatorname{Abs}^{o}_{!/\pi}(f) \qquad (19)\\
&\leq \frac{1}{2}\operatorname{Abs}(f(r)) + (1-\mu)\operatorname{Abs}(F(r)) + \frac{\mu}{\pi}\sinh(\pi r)\operatorname{Abs}_{!/\pi}(f) \\
&\leq \frac{1}{\pi}\{(\frac{\pi}{2} - 1)\operatorname{Abs}(f(r)) + (e^{\pi r} - 1)\operatorname{Abs}_{!/\pi}(f)\} \\
&\leq \frac{2}{\pi}(e^{\pi r} - 1)\operatorname{Abs}_{!/\pi}(f).
\end{aligned}$$

Proof. By using the difference functional equation (9) for $z = r$ and the point that when $r \in D$ so is $r - 1$, one can conclude

$$f_\sigma(r) = f(1) + f(2) + \cdots + f(r).$$

Now Corollary 2.2 completes the proof. □

Example 2.4. *According to [5] the natural exponential function* $\exp(z) = e^z$ *is entire analytic summable and*

$$\exp_\sigma(z) = \frac{e}{e-1}(e^z - 1), \; z \in \mathbb{C}.$$

By using (18) for the natural exponential function we have

$$\begin{aligned}
\left|\frac{e}{e-1}(e^z - 1)\right| &\leq \frac{1}{2}e^{|z|} + (1-\mu)(e^{|z|} - 1) + \frac{\mu\pi}{\pi^2 - 1}\sinh(\pi\,|z|) + \frac{\mu}{\pi^2 - 1}(\cosh(\pi\,|z|) - 1) \\
&\leq \frac{1}{2}e^{|z|} + (1-\mu)(e^{|z|} - 1) + \frac{\mu}{\pi - 1}\sinh(\pi\,|z|) \\
&\leq \frac{\pi - 2}{2\pi}e^{|z|} + \frac{1}{\pi - 1}(e^{\pi|z|} - 1) \\
&\leq \frac{2}{\pi - 1}(e^{\pi|z|} - 1)
\end{aligned}$$

which is stronger than the inequality obtained in Example 2.1 in [5].

3. Applications

In this section, we present several applications of the results of the preceding section. The main results are as follow: (1) finding upper bounds for Bernoulli polynomials, (2) obtaining upper bounds for sums of powers of natural numbers , (3) offering some inequalities for hyperbolic, trigonometric and the exponential functions.

Example 3.1. *Considering (6) we have:*

$$|B_n(z)| \leq n|\sigma(z-1)^{n-1}| + |b_n|.$$

Now by using (16) we conclude that

$$|B_n(z)| \leq \begin{cases} n\left(\frac{|z-1|^{n-1}}{2} + \frac{|z-1|^n}{n} + \sum_{m=1}^{\frac{n-1}{2}} \mu \frac{(n-1)!}{(2m-1)!\pi^{n-2m-2}} \| z-1|^{2m-1}\right) + |b_n|, \\ \qquad\qquad n \text{ is odd} \\ n\left(\frac{|z-1|^{n-1}}{2} + \frac{|z-1|^n}{n} + \sum_{m=1}^{\frac{n}{2}-1} \mu \frac{(n-1)!}{(2m)!\pi^{n-2m}} \| z-1|^{2m}\right) + |b_n|, \\ \qquad\qquad n \text{ is even} \end{cases}$$

where the constant μ is as in section 2.

Example 3.2. *Suppose p is an arbitrary positive integer and put $f(z) = z^p$. Since f is entire analytic summable, then by using Corollary 18 and 2.3 one can obtain the following upper bounds for "r sums of powers of natural numbers" as two cases:*

p is even:

$$\begin{aligned} 1^p + 2^p + 3^p + ... + r^p &\leq \frac{1}{2}r^p + (1-\mu)\frac{r^{p+1}}{p+1} + \frac{\mu p!}{\pi^{p+1}} \sinh(\pi r) \\ &\leq (\frac{1}{2} - \frac{1}{\pi})r^p + \frac{p!}{\pi^{p+1}}(e^{\pi r} - 1) \\ &\leq \frac{2p!}{\pi^{p+1}}(e^{\pi r} - 1). \end{aligned}$$

p is odd:

$$\begin{aligned} 1^p + 2^p + 3^p + ... + r^p &\leq \frac{1}{2}r^p + (1-\mu)\frac{r^{p+1}}{p+1} + \frac{\mu p!}{\pi^{p+1}}(\cosh(\pi r) - 1) \\ &\leq (\frac{1}{2} - \frac{1}{\pi})r^p + \frac{p!}{\pi^{p+1}}(e^{\pi r} - 1) \\ &\leq \frac{2p!}{\pi^{p+1}}(e^{\pi r} - 1). \end{aligned}$$

Hence, for every positive integer r, we have

$$1^p + 2^p + 3^p + ... + r^p \leq \frac{2p!}{\pi^{p+1}}(e^{\pi r} - 1).$$

Example 3.3. *Since* $(\sin)_\sigma(z) = \frac{\sin(z)+\sin(1)-\sin(z+1)}{2-2\cos(1)}$ *(see [5]), then*

$$\begin{aligned} \left|\frac{\sin(z) + \sin(1) - \sin(z+1)}{2 - 2\cos(1)}\right| &\leq \frac{1}{2}\sinh(|z|) + (1-\mu)(\cosh(|z|) - 1) + \frac{\mu}{\pi^2 - 1}(\cosh(\pi\,|z|) - 1) \\ &\leq \frac{1}{2}\sinh(|z|) + (1-\mu)(\cosh(|z|) - 1) + \frac{\mu}{\pi^2 - 1}(\sinh(\pi\,|z|) \\ &\leq \frac{1}{2} - \frac{1}{\pi}\sinh(|z|) + \frac{1}{\pi^2 - 1}(e^{\pi|z|} - 1) \\ &\leq \frac{2}{\pi^2 - 1}(e^{\pi|z|} - 1). \end{aligned}$$

In particular, if $z = r \in \mathbb{N}$ then

$$|\sin(1) + \sin(2) + \cdots + \sin(r)| \leq \frac{1}{2} - \frac{1}{\pi}\sinh(r) + \frac{e^{\pi r} - 1}{\pi^2 - 1} \leq \frac{2}{\pi^2 - 1}(e^{\pi r} - 1).$$

Remark 3.4. *In [5] Hooshmand showed that the complex function a^z is analytic summable for $|\ln a| < \pi$ and then*

$$\sigma_{\mathcal{A}}(a^z) = \frac{a}{a-1}(a^z - 1),\ |\ln a| < \pi,\ z \in \mathbb{C}. \tag{20}$$

Hence, for $e^{-z} = (\frac{1}{e})^z$, we have

$$\sigma_{\mathcal{A}}(e^{-z}) = \frac{1}{1-e}(e^{-z} - 1)$$

and so

$$\left|\frac{e^{-z} - 1}{1-e}\right| \leq \frac{1}{2}e^{|z|} + (1-\mu)(e^{|z|} - 1) + \frac{\mu\pi}{\pi^2 - 1}\sinh(\pi\,|z|) + \frac{\mu}{\pi^2 - 1}(\cosh(\pi\,|z|) - 1) \leq \frac{2}{\pi - 1}(e^{\pi|z|} - 1).$$

Also, we have

$$\sigma_{\mathcal{A}}(\cosh(z)) = \frac{1}{2}\sigma_{\mathcal{A}}(e^z) + \frac{1}{2}\sigma_{\mathcal{A}}(e^{-z}) = \frac{e^{z+1} + 1 - (e^{-z} + e)}{2(1-e)} = \frac{e^{z+1} - e^{-z}}{2(1-e)} + \frac{1}{2}$$

and we arrive at the inequalities

$$\left|\frac{e^{z+1} - e^{-z}}{2(1-e)} + \frac{1}{2}\right| \leq \frac{1}{2}\cosh(|z|) + (1-\mu)\sinh(|z|) + \frac{\mu\pi}{\pi^2 - 1}\sinh(\pi|z|) \leq (\frac{1}{2} - \frac{1}{\pi})\cosh(|z|) + \frac{\pi}{\pi^2 - 1}(e^{\pi|z|} - 1) \leq \frac{2\pi}{\pi^2 - 1}(e^{\pi|z|} - 1).$$

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