



Universal Disposition is not a 3-Space Property

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Abstract. We prove that (almost) universal disposition and separable universal disposition of Banach spaces are not 3-space properties.

1. Introduction

Given a class \mathfrak{M} of Banach spaces, a Banach space X is said to be of (almost) universal disposition with respect to \mathfrak{M} if given $A, B \in \mathfrak{M}$, into isometries $u : A \rightarrow X$ and $\iota : A \rightarrow B$ (and $(\varepsilon > 0)$ there is an into isometry (an into $(1 + \varepsilon)$ -isometry) $u' : B \rightarrow X$ such that $u = u'\iota$. A Banach space of (almost) universal disposition with respect to the class \mathfrak{F} of finite dimensional spaces –respectively the class \mathfrak{S} of separable spaces– will be simply called a space of (almost) universal disposition –respectively of separable universal disposition–. The Gurariy space \mathcal{G} [19] is the only separable Banach space of almost universal disposition, up to isometries [24, 26]. Under CH, all spaces of separable universal disposition having density character at most \aleph_1 are isometric (see [1, 2]) and we will denote this unique specimen by \mathcal{F}_1 . However, there are different non isomorphic spaces of universal disposition [2, 14]; and also different spaces of separable universal disposition outside CH [2].

The Gurariy space has received extensive attention in the literature [17, 18, 20, 25, 26, 29, 31]. Gurariy conjectured the existence of Banach spaces of universal disposition, as well as of spaces of universal disposition with respect to the class \mathfrak{S} of separable spaces. This conjecture was proved to be true in [1], where a general method to construct spaces of universal disposition with respect to different classes was presented. In particular, it was shown that the space that Gurariy conjectured is isometric to the Fraïssé limit, in the category of separable Banach spaces and into isometries, constructed by Kubiś [22]. The papers [4, 6] extend the methods of [1, 22] to quasi-Banach and Fréchet spaces. Further aspects of spaces of universal disposition have been studied in [12, 14, 23]. The paper [11] introduces and studies the properties of (almost) universal complemented disposition and establishes the Kadets space [21] as a Gurariy space in a different category. The results of [11] are extended in [7] to cover the categories of p -Banach spaces but using a Fraïssé like approach in the spirit of [15].

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Three-space properties is, on the other hand, a well established topic (see [10]). Recall that a property \mathcal{P} of Banach spaces is said to be a 3-space property if whenever a subspace Y and the corresponding quotient X/Y of a given Banach space X have \mathcal{P} then also the space X must enjoy \mathcal{P} . Our purpose in this paper is to show that (almost) universal disposition and separable universal disposition are not a 3-space properties. Because of the uniqueness of the Gurariy space this in particular means that there exists a Banach space Ω not isomorphic to \mathcal{G} containing a subspace isomorphic to \mathcal{G} and such that $\Omega/\mathcal{G} \simeq \mathcal{G}$.

2. Background

2.1. Isomorphic properties of Banach spaces

A Banach space X has property (V) of Pełczyński if every operator from X into any other Banach space is either weakly compact or an isomorphism on some copy of c_0 . All $C(K)$ and Lindenstrauss spaces enjoy Pełczyński’s property (V) [28], which is not a 3-space property as it was shown in [9] –see also [10]– and later in [13].

2.2. Exact sequences

An exact sequence $0 \rightarrow Y \rightarrow X \rightarrow Z \rightarrow 0$ of Banach spaces is a diagram formed by Banach spaces and linear continuous operators in which the kernel of each operator coincides with the image of the preceding; the middle space X is also called a *twisted sum* of Y and Z . By the open mapping theorem this means that Y is isomorphic to a subspace of X and Z is isomorphic to the corresponding quotient. An exact sequence is said to split if the image of Y in X is complemented; i.e., it is equivalent to the trivial sequence $0 \rightarrow Y \rightarrow Y \oplus Z \rightarrow Z \rightarrow 0$. See [2] for details.

2.3. A device to construct spaces of (separable) universal disposition

Given an isometry $u : A \rightarrow B$ and an operator $t : A \rightarrow E$ there is an extension of t through u at the cost of embedding E in a larger space, called the *push-out space*, as it is showed in the diagram

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ t \downarrow & & \downarrow t' \\ E & \xrightarrow{u'} & \text{PO} \end{array}$$

where $t'u = u't$. It is important to realize that the operator u' is again an isometry and that t' is a contraction or an isometry if t is. Once a starting Banach space X has been fixed, the input data we need for our construction are:

- A class \mathfrak{M} of separable Banach spaces.
- The family \mathfrak{J} of all isometries acting between the elements of \mathfrak{M} .
- A family \mathfrak{L} of norm one X -valued operators defined on elements of \mathfrak{M} .

For any operator $s : A \rightarrow B$, we establish $\text{dom}(s) = A$ and $\text{cod}(s) = B$. Notice that the codomain of an operator is usually larger than its range, and that the unique codomain of the elements of \mathfrak{L} is X . Set $\Gamma = \{(u, t) \in \mathfrak{J} \times \mathfrak{L} : \text{dom } u = \text{dom } t\}$ and consider the Banach spaces of summable families $\ell_1(\Gamma, \text{dom } u)$ and $\ell_1(\Gamma, \text{cod } u)$. We have an obvious isometry

$$\oplus \mathfrak{J} : \ell_1(\Gamma, \text{dom } u) \longrightarrow \ell_1(\Gamma, \text{cod } u)$$

defined by $(x_{(u,t)})_{(u,t) \in \Gamma} \longmapsto (u(x_{(u,t)}))_{(u,t) \in \Gamma}$; and a contractive operator

$$\Sigma \mathfrak{L} : \ell_1(\Gamma, \text{dom } u) \longrightarrow X,$$

given by $(x_{(u,t)})_{(u,t) \in \Gamma} \mapsto \sum_{(u,t) \in \Gamma} t(x_{(u,t)})$.

Observe that the notation is slightly imprecise since both $\oplus \mathfrak{I}$ and $\Sigma \mathfrak{L}$ depend on Γ . We can form their push-out diagram

$$\begin{array}{ccc} \ell_1(\Gamma, \text{dom } u) & \xrightarrow{\oplus \mathfrak{I}} & \ell_1(\Gamma, \text{cod } u) \\ \Sigma \mathfrak{L} \downarrow & & \downarrow \\ E & \xrightarrow{t} & \text{PO} \end{array}$$

We obtain in this way an isometric enlargement of X such that for every $t : A \rightarrow X$ in \mathfrak{L} , the operator tt can be extended to an operator $t' : B \rightarrow \text{PO}$ through any embedding $u : A \rightarrow B$ in \mathfrak{I} provided $\text{dom } u = \text{dom } t = A$.

In the next step we keep the family \mathfrak{I} of isometries, replace the starting space X by PO and \mathfrak{L} by a family of norm one operators $\text{dom } u \rightarrow \text{PO}$, $u \in \mathfrak{I}$, and proceed again.

We start with $\mathcal{M}^0(X) = X$. The inductive step is as follows. Suppose we have constructed the directed system $(\mathcal{M}^\alpha(X))_{\alpha < \beta}$, including the corresponding linking maps $\iota_{(\alpha,\gamma)} : \mathcal{M}^\alpha(X) \rightarrow \mathcal{M}^\gamma(X)$ for $\alpha < \gamma < \beta$. To define $\mathcal{M}^\beta(X)$ and the maps $\iota_{(\alpha,\beta)} : \mathcal{M}^\alpha(X) \rightarrow \mathcal{M}^\beta(X)$ we consider separately two cases, as usual: if β is a limit ordinal, then we take $\mathcal{M}^\beta(X)$ as the direct limit of the system $(\mathcal{M}^\alpha(X))_{\alpha < \beta}$ and $\iota_{(\alpha,\beta)} : \mathcal{M}^\alpha(X) \rightarrow \mathcal{M}^\beta(X)$ the natural inclusion map. Otherwise $\beta = \alpha + 1$ is a successor ordinal and we construct $\mathcal{M}^\beta(X)$ applying the push-out construction as above with the following data: $\mathcal{M}^\alpha(X)$ is the starting space, \mathfrak{I} keeps being the set of all isometries acting between the elements of \mathfrak{M} and \mathfrak{L}_α is the family of all isometries $t : S \rightarrow \mathcal{M}^\alpha(X)$, where $S \in \mathfrak{M}$.

We then set $\Gamma_\alpha = \{(u, t) \in \mathfrak{I} \times \mathfrak{L}_\alpha : \text{dom } u = \text{dom } t\}$ and make the push-out

$$\begin{array}{ccc} \ell_1(\Gamma_\alpha, \text{dom } u) & \xrightarrow{\oplus \mathfrak{I}_\alpha} & \ell_1(\Gamma_\alpha, \text{cod } u) \\ \Sigma \mathfrak{L}_\alpha \downarrow & & \downarrow \\ \mathcal{M}^\alpha(X) & \longrightarrow & \text{PO} \end{array} \tag{1}$$

thus obtaining $\mathcal{M}^{\alpha+1}(X) = \text{PO}$. The embedding $\iota_{(\alpha,\beta)}$ is the lower arrow in the above diagram; by composition with $\iota_{(\alpha,\beta)}$ we get the embeddings $\iota_{(\gamma,\beta)} = \iota_{(\alpha,\beta)} \iota_{(\gamma,\alpha)}$, for all $\gamma < \alpha$.

Our construction will conclude at the first uncountable ordinal ω_1 providing a Banach space that we denote $\mathcal{M}^{\omega_1}(X)$.

The choice $\mathfrak{M} = \mathfrak{F}$ of finite dimensional spaces will provide the space of universal disposition $\mathcal{F}^{\omega_1}(X)$. The choice $\mathfrak{M} = \mathfrak{S}$ of separable spaces will provide the space of separable universal disposition $\mathcal{S}^{\omega_1}(X)$ (cf. [2, Chapter 3]).

Although spaces of universal disposition need not be isomorphic, it was proved in [2, Theorem 3.23] that all spaces $\mathcal{F}^{\omega_1}(X)$ are isometric for separable X and that all spaces $\mathcal{S}^{\omega_1}(X)$ are isometric for X having density character at most \aleph_1 . Consequently, it makes sense to use the following notation.

Definition 2.1.

- $\mathcal{F}(X)$ for $\mathcal{F}^{\omega_1}(X)$. We will simply write \mathcal{F}_0 to denote $\mathcal{F}(X)$ for separable X
- $\mathcal{G}(X)$ for $\mathcal{S}^{\omega_1}(X)$. We will simply write \mathcal{F}_1 to denote $\mathcal{G}(X)$ for X having density character at most \aleph_1 .

3. Universal disposition are not a 3-space properties

Observe that universal disposition, as well as all the other related properties, are geometrical properties; thus, one has that the space \mathcal{G} is not necessarily of almost universal disposition under an equivalent renorming. Therefore, to consider 3-space problems one need to think about the associated isomorphic properties: namely, to be isomorphic to a space of (almost) universal (separable) disposition. So we will do without this further explicit mention.

Theorem 3.1. *Universal and almost universal disposition are not 3-space properties.*

Proof. We get from [2] that a space of almost universal disposition is a Lindenstrauss space, and therefore it enjoys property (V). It follows from [5], see also [13], that there exist exact sequences

$$0 \longrightarrow C(\omega^\omega) \longrightarrow \Omega \longrightarrow c_0 \longrightarrow 0$$

in which Ω has not property (V). Since every separable Lindenstrauss space is 1-complemented in \mathcal{G} and property (V) is stable by products, multiplying adequately on the left and on the right one can thus obtain an exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \Omega' \longrightarrow \mathcal{G} \longrightarrow 0$$

in which Ω' cannot have property (V) and thus it cannot be isomorphic to \mathcal{G} . In particular, it cannot be a space of almost universal disposition.

To prove that universal disposition is not a 3-space property, we use [14, Proposition 2.1] and [14, Lemma 3.2] to get that both $C(\omega^\omega)$ and c_0 are complemented in the space of universal disposition $\mathcal{F}(C(\omega^\omega))$. Therefore, multiplying adequately on the left and on the right one can obtain an exact sequence

$$0 \longrightarrow \mathcal{F}(C(\omega^\omega)) \longrightarrow \Omega'' \longrightarrow \mathcal{F}(C(\omega^\omega)) \longrightarrow 0$$

namely

$$0 \longrightarrow \mathcal{F}_0 \longrightarrow \Omega'' \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

in which Ω'' contains Ω complemented and thus it cannot have property (V), which prevents it to be a space of universal disposition. \square

The strategy to show that separable universal disposition is not a 3-space property has to be different since spaces of separable universal disposition do not contain complemented separable subspaces.

Theorem 3.2. *Under CH, separable universal disposition is not a 3-space property.*

Proof. We need here the results of [1] asserting that a space of separable universal disposition enjoy a property called universal separable injectivity [2]. A Banach space X is universally separably injective if every operator $S \rightarrow X$ from a separable space S can be extended to any superspace. It is obvious that complemented subspaces of universally separably injective spaces enjoy the same property. Ultrapowers of \mathcal{L}_∞ spaces are also universally separably injective. The main result of [3, Theorem 1 (f)] asserts that, under CH, universal separable injectivity is not a 3-space property. More precisely, there exists an ultrapower $C[0, 1]_{\mathcal{U}}$ of $C[0, 1]$ and an exact sequence

$$0 \longrightarrow C[0, 1]_{\mathcal{U}} \longrightarrow \Psi \longrightarrow \ell_\infty \longrightarrow 0$$

in which the space Ψ is not universally separably injective. Now, since $C[0, 1]$ is complemented in \mathcal{G} then $C[0, 1]_{\mathcal{U}}$ is complemented in $\mathcal{G}_{\mathcal{U}}$.

On the other hand, $\mathcal{G}_{\mathcal{U}}$, as any ultrapower of an \mathcal{L}_∞ space, contains ℓ_∞ , and it must then contain it complemented. Thus, multiplying adequately on the left and on the right one gets an exact sequence

$$0 \longrightarrow \mathcal{G}_{\mathcal{U}} \longrightarrow \Psi \oplus \mathcal{G}_{\mathcal{U}} \longrightarrow \mathcal{G}_{\mathcal{U}} \longrightarrow 0$$

in which the twisted sum space $\Psi \oplus \mathcal{G}_{\mathcal{U}}$ cannot be of separable universal disposition since it is not even universally separably injective. \square

4. Stability by products

Since the properties of (almost) universal disposition and separable universal disposition are not 3-space properties, it makes sense to consider their (isomorphic) stability by products. We have been unable to determine whether the product of two spaces of (almost) universal disposition (or separable universal disposition) has to be isomorphic to a space with the same property. Part of the problem is that there is no available characterization for the property “to be isomorphic to a space of (almost) universal disposition”. One however has:

Proposition 4.1.

1. $\mathcal{G} \simeq \mathcal{G} \oplus \mathcal{G}$.
2. Under the diamond axiom \diamond (see [30]) there is a space \mathcal{S}_\diamond of density character \aleph_1 and of almost universal disposition such that \mathcal{S}_\diamond is not isomorphic to $\mathcal{S}_\diamond \oplus \mathcal{S}_\diamond$. However, any ultrapower $(\mathcal{S}_\diamond)_\mathcal{U}$ is a space of separable universal disposition such that $(\mathcal{S}_\diamond)_\mathcal{U} \simeq (\mathcal{S}_\diamond)_\mathcal{U} \oplus (\mathcal{S}_\diamond)_\mathcal{U}$.
3. Under CH, $\mathcal{F}_1 \oplus \mathcal{F}_1$ is isomorphic to a space of separable universal disposition.

Proof. (1) That $\mathcal{G} \oplus \mathcal{G} \simeq \mathcal{G}$ can be found in [2, p.133]: since $c_0(\mathcal{G})$ is a Lindenstrauss space, it is 1-complemented in \mathcal{G} ; and \mathcal{G} is obviously 1-complemented in $c_0(\mathcal{G})$. By Pełczyński’s decomposition method $c_0(\mathcal{G}) \simeq \mathcal{G}$ and, in particular, $\mathcal{G} \oplus \mathcal{G} \simeq \mathcal{G}$.

(2) The space \mathcal{S}_\diamond is the one constructed by Shelah [30, Theorem 5.1]. It has the property labeled (C) that every operator $\mathcal{S}_\diamond \rightarrow \mathcal{S}_\diamond$ has the form $\lambda I + S$ for some scalar λ and some separable range operator S . This makes impossible an isomorphism $\mathcal{S}_\diamond \oplus \mathcal{S}_\diamond \simeq \mathcal{S}_\diamond$.

The space \mathcal{S}_\diamond is of almost universal disposition by construction, and consequently $(\mathcal{S}_\diamond)_\mathcal{U}$ is of separable universal disposition. Since the diamond axiom \diamond implies CH, $\mathcal{F}_1 \simeq (\mathcal{S}_\diamond)_\mathcal{U}$, and we prove next that $\mathcal{F}_1 \oplus \mathcal{F}_1 \simeq \mathcal{F}_1$.

(3) Under CH, all spaces of separable universal disposition having density character at most \aleph_1 are isometric [2]. Therefore $\mathcal{F}_1 \simeq \mathcal{G}_\mathcal{U}$ for every countably incomplete ultrafilter on \mathbb{N} . Thus,

$$\mathcal{F}_1 \simeq \mathcal{G}_\mathcal{U} \simeq (\mathcal{G} \oplus \mathcal{G})_\mathcal{U} \simeq \mathcal{G}_\mathcal{U} \oplus \mathcal{G}_\mathcal{U} \simeq \mathcal{F}_1 \oplus \mathcal{F}_1$$

□

As we have already said, we have been unable to settle the following

Problem. *Is the product of spaces of universal disposition (isomorphic to) a space of universal disposition?*

Observe that, even under CH, there are at least three different non isomorphic spaces of universal disposition: $\mathcal{F}_1, \mathcal{F}_0$ (see [2]) and $\mathcal{F}(C(\Delta))$, where $C(\Delta)$ is a space of continuous functions on a compact that admits a representation

$$0 \longrightarrow c_0 \longrightarrow C(\Delta) \longrightarrow c_0(\aleph_1) \longrightarrow 0$$

(see [14]). The spaces \mathcal{F}_1 and \mathcal{F}_0 are very different since \mathcal{F}_1 is 1-separably injective while \mathcal{F}_0 is not even separably injective; \mathcal{F}_1 is a Grothendieck space while every copy of c_0 inside \mathcal{F}_0 is complemented.

The space $\mathcal{F}(C(\Delta))$ is different from the other two since it contains complemented and uncomplemented copies of c_0 . It is clear that $\mathcal{F}_1 \oplus \mathcal{F}_0$ cannot be isomorphic to either \mathcal{F}_1 (since it cannot be separably injective) or to \mathcal{F}_0 (since it contains uncomplemented copies of c_0). Is it a space of universal disposition?

We conjecture that it is not and that, consequently, the answer to the Problem is negative. Recall that it is also an open problem whether there exist a continuum of mutually non-isomorphic spaces of universal disposition. It is also likely that the spaces $\mathcal{F}(\mathcal{F}_0), \mathcal{F}(\mathcal{F}_1)$ and $\mathcal{F}(\mathcal{F}(C(\Delta)))$ are non isomorphic, which would yield an infinite sequence of non-isomorphic spaces of universal disposition.

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